To the memory of Prof. A.A. Pervozvansky

Iterative Learning Control for Over-Determined, Under-Determined, and Ill Conditioned Systems

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Abstract

This paper studies iterative learning control for under-determined and over-determined systems. These are systems where the control action to produce the desired output is not unique, or systems for which it is not feasible to exactly track the desired output. We also study discrete time systems whose operators are ill conditioned. For all these cases we recommend to use the pseudoinverses or their approximations as a learning operator. In particular, we show that for over-determined systems, unless one knows the system exactly, the minimum possible output error is never achieved by a repetition invariant learning controller. Finally, the relation between several iterative learning control methods and Tikhonov-type regularization techniques is discussed.

1 Introduction and problem formulation

In this paper we study general iterative learning control (ILC) laws for over-determined and under-determined systems as well as the regularization technique for ill conditioned discrete time systems. Over-(under-) determined systems are systems where the desired output cannot be exactly achieved by any control, or systems in which several input control values can be chosen to obtain the desired output. There are many applications of iterative learning control: robotic manipulators, hard disk drives, chemical processing, etc. [1, 16, 19, 20, 22, 24, 27] Some of them, such as robotic manipulators, clearly can be either under-determined or over-determined systems. We note that in robotics such systems are also referred to as over-(under-)actuated systems. Iterative learning control is designed to improve the performance of cyclical systems and its basic idea is to use the information from the previous cycles to improve the system performance on the current cycle [1, 16, 20, 27].

As in [5, 6], let us consider a controlled system represented by the operator F acting on Hilbert spaces U and Y, that is $F : U \to Y$, where U is the control space and Y is the space of observations or system outputs. Unless noted explicitly, the vector norm is given by the scalar product ($||x|| = \sqrt{<x, x>}$) and the operator norm is induced by this scalar product norm. Since we consider a cyclical system, the system evolution is given by

$$y_k = F(u_k), \quad k = 0, 1, ... \tag{1}$$

where $u_k \in U$ and $y_k \in Y$ are the control input and the observed output of the system at the k-th cycle, respectively. Let y_d be the desired output of the system. Our aim is to find a control u_d which is a solution of the operator equation

$$y_d = F(u). \tag{2}$$

Of course, we suppose that we do not know exactly the operator F, otherwise the equation (2) can be solved by classical methods [11] provided the computation is not too ill conditioned. Since we are dealing with cyclically operating systems, iterative learning control can be used to solve the operator equation (2) *online*. Here we consider linear iterative learning control scheme [1, 16, 22, 23, 27]

$$u_{k+1} = u_k - Lz_k, \quad k = 0, 1, ... \tag{3}$$

where $z_k = y_k - y_d$ is the error at the k-th cycle and L is a linear *learning* operator.

Now let us give a strict definition of over-determined and under-determined systems. We say that a system is *under-determined* if for some outputs there is no unique control action and we say that a system is *over-determined* if there exist some values of the output that cannot be achieved, that is the range of operator F is a strict subset of the observation space Y. In the case of under-determined systems we are interested in the solution of equation (2) which minimizes the norm $||u||$. Whereas in the case of over-determined systems we are interested in the minimization of $||y_d - F(u)||$ [17]. Of course, in general the minimum of $||y_d - F(u)||$ is not zero. Moreover, we show that for over-determined systems one cannot achieve this minimum using the learning procedure (3). This is a sharp contrast with the systems which have one-to-one correspondence between the control actions and outputs. In the later case the minimum of $||y_d - F(u)||$ is equal to zero and can be achieved by the appropriate choice of the learning operator.

In the next section we study linear systems. The linear theory can be easily understood and it clearly demonstrates the essence of the problems as well as possible approaches to their solution. Then, in the third section, we discuss the regularization technique as well as the generalization to nonlinear systems. Finally, in the last section we show that the pseudoinverse method and regularization techniques can also be applied to discrete time systems with ill conditioned operators.

2 Linear systems

In this section we study the case of a general linear system described by bounded linear operator F. We assume that the range $\mathcal{R}(F)$ of the system operator is closed. This ensures the existence of pseudoinverse (or Moore-Penrose generalized inverse) operator $F^\dagger$ [12]. By straightforward calculations one can obtain the following recursion formula [16, 20, 27]

$$z_{k+1} = [I - FL]z_k, \quad k = 0, 1, ... \tag{4}$$

Note that if F is a one-to-one operator, then the condition

$$||I - FL|| < 1 \tag{5}$$

is satisfied for a large class of learning operators. In particular, this condition implies that the sequence $\{z_k\}$ converges to zero in norm.

Now let us consider over-determined systems. The recurrent equation (4) still holds, but the condition $||I - FL|| < 1$ cannot be satisfied by choosing any L. Let us show this in the case of finite-dimensional spaces U and Y. It is known [8] that if we take the Euclidean norm, i.e. $||A||_E = (\sum_{j=1}^n \sum_{i=1}^n |a_{ij}|^2)^{1/2}$, then $||I - FL||_E$ achieves its minimum at $L = F^\dagger$, where $F^\dagger$ is the pseudoinverse (or the Moore-Penrose generalized inverse) of the operator F. That is,

$$||I - FF^\dagger||_E \le ||I - FL||_E \tag{6}$$

for any operator L. Note that $FF^\dagger$ is an orthogonal projection mapping Y onto $\mathcal{R}(F)$. Therefore $I - FF^\dagger$ is equal to zero only for the systems described by a one-to-one operator or for under-determined systems. Next we consider over-determined systems for which $I - FF^\dagger \neq 0$. We note that $||A||_2 \le ||A||_E$, where $||\cdot||_2$ is the norm induced by the scalar product [11]. Next we recall that $I - FF^\dagger$ is an orthogonal projection and hence $||I - FF^\dagger||_2 = 1$. Thus,

$$1 = ||I - FF^\dagger||_2 \le ||I - FF^\dagger||_E \le ||I - FL||_E$$

and consequently the condition (5) cannot be satisfied for over-determined systems. Also we conclude from inequality (6) that probably the best strategy for the over-determined systems is to choose the learning operator L as close to the pseudoinverse $F^\dagger$ as possible.

An intuitive explanation of the fact that $||I - FL|| \ge 1$ for over-determined systems is clear: There exists such values of the output which cannot be achieved by the system using any control action and hence the sequence $\{z_k\}$ cannot converge to zero with any L. However, if $y_d \notin \mathcal{R}(F)$ we are interested in the minimization of $||y_d - Fu||$. That is, ideally we want to choose such a learning operator that the sequence $\{z_k\}$ converges to $z_d := Fu_d - y_d \neq 0$, where $u_d = argmin||y_d - Fu||$. Recall that $u_d = F^\dagger y_d$ provides a solution to the optimization problem $\min_u ||y_d - Fu||$ [8, 9]. Thus,

$$z_d = Fu_d - y_d = FF^\dagger y_d - y_d = [FF^\dagger - I]y_d. \tag{7}$$

In the next theorem we give convergence conditions of the general iterative learning procedure (3) for linear over-determined systems. In particular, we will show that unless one knows the system exactly, the limit of the sequence $\{z_k\}$ does not equal z_d. Namely, by using the learning algorithm (3) we are not able to minimize $||y_d - Fu||$. Nethertheless, as the theorem shows, we are able to find a good approximation of the optimal solution.

Theorem 1 *Let the learning operator satisfy the following condition*

$$||F^\dagger - L|| < \frac{1}{||F||}. \tag{8}$$

Then the iterative learning procedure (3) converges, that is

$$\lim_{k\to\infty} z_k = z_\infty = [I - F(F^\dagger - L)]^{-1} z_d, \tag{9}$$

where z_d is given by (7).

PROOF: Let us consider the recurrent formula (4) and rewrite the operator $I - FL$ as follows:

$$\begin{aligned} I - FL &= I - FF^{\dagger} + FF^{\dagger} - FL \\ &= I - FF^{\dagger} + F(F^{\dagger} - L). \end{aligned}$$

Next we consider the product

$$[I - FF^{\dagger}]F(F^{\dagger} - L) = [F - FF^{\dagger}F](F^{\dagger} - L).$$

Since the pseudoinverse operator $F^{\dagger}$ satisfies the equation $FF^{\dagger}F = F$ [9], we have that

$$[I - FF^{\dagger}]F(F^{\dagger} - L) = 0.$$

Next, using the above property and the fact that $I - FF^{\dagger}$ is an orthogonal projection, we can give the following expression of the powers of $I - FL$

$$\begin{aligned} [I - FL]^k &= [(I - FF^{\dagger}) + F(F^{\dagger} - L)]^k \\ &= (I - FF^{\dagger}) + F(F^{\dagger} - L)(I - FF^{\dagger}) \\ &\quad + ... + [F(F^{\dagger} - L)]^{k-1}(I - FF^{\dagger}) \\ &\quad + [F(F^{\dagger} - L)]^k \\ &= (I + F(F^{\dagger} - L) \\ &\quad + ... + [F(F^{\dagger} - L)]^{k-1})(I - FF^{\dagger}) \\ &\quad + [F(F^{\dagger} - L)]^k \end{aligned}$$

From condition (8) we conclude that the series $I + F(F^{\dagger} - L) + [F(F^{\dagger} - L)]^2 + ...$ is absolutely convergent, and consequently

$$[I - FL]^k \to [I - F(F^{\dagger} - L)]^{-1}(I - FF^{\dagger})z_0,$$

as $k \to \infty$. Finally, we show that $(I - FF^{\dagger})z_0 = z_d$

$$\begin{aligned} (I - FF^{\dagger})z_0 &= (I - FF^{\dagger})(Fu_0 - y_d) \\ &= (F - FF^{\dagger}F)u_0 + [FF^{\dagger} - I]y_d \\ &= [FF^{\dagger} - I]y_d = z_d. \end{aligned}$$

This completes the proof.

□

The following straightforward norm estimation of the limiting error can be immediately obtained.

Corollary 1 *The upper bound on the norm of limiting error in the case of over-determined system is given by*

$$||z_{\infty}|| \leq \frac{1}{1 - ||F||||F^{\dagger} - L||}||z_d||.$$

There are several useful conclusions that we can draw from the results of Theorem 1.

First, the expression (9) for the limiting error suggests that the learning operator L should be chosen as close to the pseudoinverse $F^{\dagger}$ of the system operator as possible. The later guarantees that we find a good approximation of the solution to the minimization problem $\min ||y_d - Fu||$.

Second, the convergence of (3) takes place if condition (8) is satisfied. This condition can be easily satisfied in the case when the operators L and $F^{\dagger}$ act on the same subspaces. However, it is not the case in the presence of singular perturbations [3, 4, 25]. The updating ILC procedures discussed below could be a solution to this problem.

Finally, one can see that, unless one knows exactly the system operator F, it is not possible to achieve the minimum of $||y_d - Fu||$ using the learning procedure (3) with any choice of L. Therefore, the ILC procedures with *updating* learning operator [6, 7, 17] can be very useful for over-determined systems. Namely, the updating methods [6, 7, 17] construct the sequence of learning operators $\{L_k\}$ which converge to $F^{\dagger}$, thus one may expect that the updating learning procedure will provide an exact solution to the optimization problem $\min_u ||y_d - Fu||$.

As was mentioned at the beginning of the section, for systems described by a one-to-one operator the learning procedure (3) converges (under certain conditions on L) to a solution of the operator equation (2). The next theorem demonstrates that the application of the learning procedure (3) to under-determined systems produces similar results.

Theorem 2 *Let the controlled system be under-determined, that is $\mathcal{R}(F) = Y$. And let the following condition be satisfied*

$$||F^{\dagger} - L|| < \frac{1}{||F||}.$$

Then the learning procedure (3) *converges to a solution of* (2) *and the next norm estimation holds*

$$||z_{k+1}|| \leq ||F||||F^{\dagger} - L||||z_k|| \quad k = 0, 1, ... \tag{10}$$

PROOF: Recall that $FF^{\dagger}$ is an orthogonal projection onto $\mathcal{R}(F)$, the range of the system operator. Since the system is over-determined, $\mathcal{R}(F) = Y$ and hence $FF^{\dagger} = I$. Thus, we can write

$$I - FL = FF^{\dagger} - FL = F[F^{\dagger} - L].$$

The latter immediately implies (10). Next the norm inequality (10) ensures convergence if $||F^{\dagger} - L|| < 1/||F||$.

□

Note that the convergence condition in Theorem 2 is the same as the condition (8) in Theorem 1. This shows that in both cases of over-determined and under-determined systems, one should try to choose the learning operator as close to the pseudoinverse $F^{\dagger}$ as possible.

3 Regularization methods and extension to nonlinear models

As was pointed out in the previous section, if the system operator is not invertible, we should take a good approximation of the pseudoinverse operator of the system as the learning operator. Namely, let $\tilde{F}$ be some approximation (or known part) of the controlled system. Then, in theory we must take $L = \tilde{F}^{\dagger}$. However, in practice the computation of $\tilde{F}^{\dagger}$ is quite demanding and often numerically unstable [9]. Especially, this is the case for singularly perturbed systems [3, 4, 25]. To overcome this problem, we suggest to use Tikhonov regularization [8, 28]. Namely, it is known that the pseudoinverse operator can be expressed as the following limit [8, 9, 28]

$$F^{\dagger} = \lim_{\alpha \to 0} (F^* F + \alpha^2 I)^{-1} F^*,$$

where F^* is the adjoint operator to F. Thus, in practice one has to solve the next system for $\Delta_{k+1} u = u_{k+1} - u_k$

$$(\tilde{F}^* \tilde{F} + \alpha^2 I) \Delta_{k+1} u = -\tilde{F}^* z_k, \qquad (11)$$

where the parameter α is chosen in such a way that the matrix $\tilde{F}^* \tilde{F} + \alpha^2 I$ is well-conditioned. A regularization for ILC similar to (11) was first proposed in the paper of Pervozvansky [23]. Note that (11) is the stationary point equation for the minimization problem [8, 28]

$$\min_{\Delta_{k+1} u} \{ ||z_k + \tilde{F} \Delta_{k+1} u||^2 + \alpha^2 ||\Delta_{k+1} u||^2 \}. \qquad (12)$$

The latter formulation allows us to generalize the method for the case of nonlinear systems. That is, the control for the next iteration is calculated by $u_{k+1} = u_k + \Delta_{k+1} u$, where $\Delta_{k+1} u$ is a solution of the following optimization problem

$$\min_{\Delta_{k+1} u} \{ ||z_k + D\tilde{F}(u_k) \Delta_{k+1} u||^2 + \alpha^2 ||\Delta_{k+1} u||^2 \}.$$

where $D\tilde{F}(u_k)$ is a Freche derivative at $u = u_k$ of the system operator $F(u)$. Also one may construct a learning procedure which solves *online* the following nonlinear (and not quadratic) optimization problem

$$\min_{u} \{ ||y_d - F(u)||^2 + \alpha^2 ||u||^2 \}$$

instead of the operator equation (2). The convergence of these *regularization* learning procedures for nonlinear systems will be investigated in future publications.

4 Application of the regularization techniques to discrete time systems

This section applies some of the concepts developed above to the standard ILC problem that is neither over-determined nor underdetermined, but instead has a one-to-one mapping between input histories and output histories. Systems governed by linear discrete time models are considered. Many of the same issues arise in this problem as in the previous problems because of the ill conditioning of the matrix describing the relationship between the input and output time histories. The emphasis of the developments is on the ability to handle the ill conditioning problem, but toward the end of the section comments are made about how to creat robustness of the learning to errors in our model of the system F.

Ill conditioning of inverse control in discrete time systems: When a system governed by a linear differential equation is fed by a zero order hold as in typical digital control, it is possible to describe without approximation, the input-output relationship by a difference equation. When the pole excess of the continuous time transfer function is greater than one, this discretization process will generically introduce enough zeros in the z-transfer function that it will have a pole excess of 1. Asymptotically as the sample time gets large, when one new zero is introduced it will appear at -1, when two zeros are introduced one will be inside the unit circle and the other will be outside, when three zeros are introduced one will be inside, one outside and the third at -1, etc. [2]. This means that in the majority of practical applications, there are zeros in the discrete time model that are outside the unit circle. If one wants to invert the system to determine the control needed to produce a chosen output, one substitutes the desired solution into the output terms of the difference equation, and then must solve for the input that can made the sum of the input terms match the sum of output terms. But with a zero outside the unit circle, this is the solution of an unstable difference equation, which usually precludes the use of inverse control. In a matrix formulation, the same effect is manifested as ill conditioning of a matrix inverse computation. The linear operator F in this discrete time problem is a matrix of Markov parameters giving the output history of the system as the product of F times the input history (plus an initial condition term that is the same every repetition). It is normally full rank - all that is required is to have the first Markov parameter be non zero (or in the multi-input, multi-output case full rank). Then all singular values of F are non zero. However, every unstable zero of the system z-transfer function is associated with a small singular value that makes the matrix ill conditioned. For typical trajectory lengths with typical sample rates, the matrix F is large and sufficiently ill conditioned that computation of inverse control become impossible. The following paragraphs discuss possible approaches to this problem.

An approximate inverse control: Suppose that we can find the singular value decomposition of F, $F = U\Sigma V^* = [U_1\ U_2]diag(\Sigma_1, \Sigma_2)[V_1\ V_2]^*$, where $diag$ represents the square block diagonal matrix with the diagonal blocks as indicated. The Σ_1, Σ_2 are diagonal matrices of singular values, with Σ_2 containing the small singular values that cause the ill conditioning. An approximation to F is given by $\underline{F} = [U_1\ U_2]diag(\Sigma_1, 0)[V_1\ V_2]^* = U_1\Sigma_1 V_1^*$, obtained by setting Σ_2 to zero. Setting it to zero eliminates the ill conditioning, creating instead a singular matrix. Suppose we pick the learning gain matrix L to be the Moore-Penrose pseudo inverse of $\underline{F}$, i.e. $L = V_1\Sigma_1^{-1}U_1^*$ ([21] uses this approach to create local learning strategies). Then using equation (3) and $\Delta_{k+1}y = F\Delta_{k+1}u$ produces equation (4) which can be written

$$[U_1\ U_2]^* z_{k+1} = diag(0, I)[U_1\ U_2]^* z_k, \qquad (13)$$

where y_k, u_k, z_k are column matrices of the output, input, and error histories. This equation shows that provided the model is perfect, the iterations converge in one repetition, making the error zero in the space spanned by the orthonormal columns of U_1 and leaving the error in the space spanned by U_2 unaltered. The steady state error is $z_\infty = U_2U_2^* z_0$. Because the learning control gain matrix L is singular, this falls into the category of an over-determined system. To make Theorem 1 apply to this problem, replace the $\tilde{F}$ by $\underline{\tilde{F}}$ in equations (7), (8) and (9). This approach is a reasonable way to eliminate the ill conditioning problem, and produce a substantial decrease in tracking error. However, it would be good if we could create a method that can also eliminate the error in the space spanned by U_2, as in the following paragraphs.

ILC based on a modified inverse: Now let us investigate what happens if we handle the ill conditioning by simply increasing the small singular values by an amount that makes the matrix nicely invertible. Create a modified F as $\overline{F} = U diag(\Sigma_1, \Sigma_2 + diag(\alpha_i))V^*$ which adds an α_i to each σ_i on the diagonal of Σ_2, picking the α_i such that the matrix is no longer ill conditioned. Then choose the learning matrix as $L = \overline{F}^{-1}$. Paralleling the developments of the previous paragraph produces

$$U^* z_{k+1} = diag(0, \Sigma_2(\Sigma_2 + diag(\alpha_i))^{-1})U^* z_k \qquad (14)$$

Again the components of the error in the space spanned by U_1 are eliminated in the first repetition. The amplitude of the error for the i-th column of U, a column appearing in U_2, is multiplied by $\alpha_i/(\sigma_i + \alpha_i)$ every repetition. Hence, provided the model is correct, the learning control law converges to zero tracking error when the α_i are chosen to satisfy

$$\left|\frac{\alpha_i}{\sigma_i + \alpha_i}\right| < 1. \qquad (15)$$

This is satisfied by all positive α_i. Note what has been accomplished here. The learning law immediately eliminates the error in that part of the space for which we can obtain an inverse. Then the iterations of the learning control process converge to the inverse of the rest of the matrix, eventually eliminating all error. Equation (2) is finally solved, in spite of the fact that the matrix F is too ill conditioned to invert.

Regularization and an enhanced regularization: The learning law (11) based on regularization is closely related to the law of the previous paragraph. Substitute F for $\tilde{F}$ presuming we have an accurate model, and use a single α rather than a different one for each singular value. Then the learning law (11) gives an L in the form $L = V(\Sigma^2 + \alpha^2 I)^{-1}\Sigma U^*$ and the error as a function of repetitions satisfies

$$\begin{aligned} U^* z_{k+1} &= [I - \Sigma(\Sigma^2 + \alpha^2 I)^{-1}\Sigma]U^* z_k \\ &= diag\left(\frac{\alpha^2}{\sigma_i^2 + \alpha^2}\right)U^* z_k \qquad (16) \end{aligned}$$

This converges to zero tracking error provided $|\alpha^2/(\sigma_i^2 + \alpha^2)| < 1$ for all i. We can create a more sophisticated regularization law by using a different weight α_i for each singular value of F. In place of (11) use

$$(F^*F + V diag(\alpha_i^2)V^*)\Delta_{k+1}u = -F^* z_k \qquad (17)$$

which produces learning law $L = V[\Sigma^2 + diag(\alpha_i^2)]^{-1}\Sigma U^*$. Then the error as a function of repetitions satisfies

$$U^* z_{k+1} = diag(\alpha_i^2/(\sigma_i^2 + \alpha_i^2))U^* z_k \qquad (18)$$

The extra freedom of having many α_i to choose is helpful. The first singular values are normally well determined and there is little need for the regularization with regard to this part of the system. Use of a nonzero α_i for these singular values unnecessarily slows down the convergence of this part of the error. For later singular values, the regularization becomes important and nonzero values can easily be very helpful for convergence.

Relation to the contraction mapping ILC law: Reference [14] suggests the use of a learning law $L = sF^*$. This law can be interpreted as performing at each repetition a step along the steepest descent direction for minimizing the sum of the squares of the errors for all time steps after the current one. If F is known exactly, this produces monotonic decay of the tracking error. Note that as the scalar α becomes large, and makes Σ become negligible in comparison, the Tikhonov based learning law above equation (16) approaches this contraction mapping law with the learning gain $s = \alpha^{-2}$. Hence, the contraction mapping law is a special case of the Tikhanov regularization and hence a special case of quadratic cost ILC as discussed below.

Relationship to and enhancements of quadratic cost based ILC: Note that the $z_k + F\Delta_{k+1}u$ in equation (12) is equal to z_{k+1}, so that (12) can be rewritten as

$$\min_{\Delta_{k+1}u} \{||z_{k+1}||^2 + \alpha^2||\Delta_{k+1}u||^2\} \qquad (19)$$

This is the ILC quadratic cost usually introduced to control the learning transients, but with the usual state and control weighting matrices, Q and R, chosen as the identity and α^2 times the identity [13]. We conclude that the Tikhonov regularization for this problem produced a special case of the quadratic cost ILC control problem. By penalizing the difference of controls in the last two repetitions, this optimization criterion keeps the changes in control from one repetition or cycle to the next from being too large and producing poor learning transients. This quadratic cost (19) is distinct from a quadratic cost ILC problem which aims to learn the quadratic cost control action [10, 17]. This latter problem addresses the issue of desired trajectories that are not feasible due to actuator saturation constraints. By contrast, the non feasibility addressed by the over-determined problem in the previous sections can be classified as a geometric non feasibility.

Now consider the more sophisticated regularization of equation (17). This is associated with a quadratic cost of the form

$$\min_{\Delta_{k+1}u} \{z_{k+1}^* z_{k+1} + \Delta_{k+1}u^*[V diag(\alpha_i^2)V^T]\Delta_{k+1}u\} \qquad (20)$$

The weighting matrix Q is still the identity, although using such quadratic cost thinking we can modify it to emphasize parts of the trajectory where good tracking early in the learning process is important. This time the control weighting matrix has a very specialized structure making use of the knowledge of the system. One of the weakest aspects of LQ theory is that one has little guidance as to how to pick the Q and R matrices, other than trial and error. Hence, this more sophisticated Tikhonov regularization has given guidance to LQ to address this problem, which can be a significant contribution to the use of quadratic cost ILC.

Relation to frequency response, frequency weighted cost functionals, singular perturbations, and learning robustness: Reference [15] discusses the relationship between the singular values of F and the magnitude of the frequency response of the system at discrete frequencies. As the trajectory becomes long the singular values converge to the magnitude response. Stated another way, (20) is a finite time, time domain version of a frequency weighted cost functional, with the α_i^2 being weights for different frequencies. There is a natural connection between quadratic cost in the time domain and the frequency domain using Parseval's theorem.

An important issue in designing ILC laws is producing robustness to singular perturbations [3, 4]. Unlike normal feedback control, parasitic poles at high frequencies, far above the bandwidth of the controller can easily destabilize a learning control system. In [3, 4, 16, 18, 25] stability and good transient robustness is obtained for such unmodeled high frequency dynamics by cutting off the learning above the frequency for which one no longer has confidence in the model (this cutoff can also be tuned based on observing the frequency content of the response of the learning process).

In the quadratic cost Tikhonov regularization of (20) this kind of robustness can be accomplished as a limiting case by letting the α_i weights associated with singular values above the cutoff frequency tend to infinity. Then no control action is taken in this part of the space. To get to the limit in this process, one can transform the control variables to new coordinates $\bar{u} = V^*u$, then delete those control variables associated with singular values that relate to frequencies above the cutoff, and then formulate the quadratic cost for the remaining control coordinates. Truncation of singular values is a finite time version of the cliff filtering in [26]. (For related developments for continuous time systems see also [3, 4, 25].) This limiting case produces an over-determined problem.

In terms of the new control variables we can create a new rectangular F matrix, and then the learning law, for example the one above equation (18), becomes $L = V[\Sigma^T\Sigma + diag(\alpha_i^2)]^{-1}\Sigma^*U^*$. This L generates the changes in the control action $\bar{u}$ from that of the previous repetition, making use of the singular value decomposition of this new rectangular F. A procedure of this kind is also appropriate when the dimension of an F for which one can numerically obtain a singular value decomposition is less than the full dimension of the problem.

As a regularization procedure, the Tikhonov regularization aims to produce robustness. Tikhonov regularization naturally produces robustness to errors in the knowledge of the frequency magnitude response of the system, i.e. the singular values, and addresses the issues of ill conditioning. But no statements are made here relative to producing robustness to errors in one's knowledge of the singular vectors in U and V. These vectors contain the phase information, and stability of the ILC iterations is sensitive to phase errors. As the trajectory gets long compared to the time constants of the system, the column vectors in U and V start to look rather like sine and cosine waves, or two sines with different phases. The $V^* z_k$ finds the projection of the error onto these sinusoids. Then the sinular values puts in the amplitude change, and multiplying by the columns of U puts in the output sinusoid with the appropriate phase change included for the associated discrete frequency. It is important to have robustness to errors in the phase information in the model F.

Hence, it is recommended here that the enhanced regularization or LQ ILC law be used, equations (17) and (20), and that it be combined with a singular value cutoff as described above. This approach to learning control benefits from use of a system model, it involves regularization that solves the ill conditioned model inversion problem by iterations, the regularization produces robustness to singular value or magnitude frequency response errors of the model, and then the truncation produces robustness to errors in phase information at high frequency.

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