Inversion of analytic matrix functions that are singular at the origin *

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Abstract

In this paper we study the inversion of an analytic matrix valued function $A(z)$. This problem can also be viewed as an analytic perturbation of the matrix $A_0 = A(0)$. We are mainly interested in the case where A_0 is singular but $A(z)$ has an inverse in some punctured disc around $z = 0$. It is known that $A^{-1}(z)$ can be expanded as a Laurent series at the origin. The main purpose of this paper is to provide efficient computational procedures for the coefficients of this series. We demonstrate that the proposed algorithms are computationally superior to symbolic algebra when the order of the pole is small.

Key words. matrix inversion, matrix valued functions, analytic perturbation, Laurent series.

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Abbreviated title. Inversion of analytic matrix functions.

1 Introduction

Let $\{A_k\}_{k=0,1,\ldots} \subseteq \mathbb{R}^{n\times n}$ be a sequence of matrices that defines the analytic matrix valued function

$$A(z) = A_0 + zA_1 + z^2A_2 + \cdots \ . \tag{1}$$

The above series is assumed to converge in some non-empty neighbourhood of $z = 0$. We will also say that $A(z)$ is an *analytic perturbation* of the matrix $A_0 = A(0)$. Assume the inverse matrices $A^{-1}(z)$ exist in some (possibly punctured) disc centred at $z = 0$. In particular, we are primarily interested in the case where A_0 is singular. In this case it is known that $A^{-1}(z)$ can be expanded as a Laurent series in the form

$$A^{-1}(z) = \frac{1}{z^s}(X_0 + zX_1 + \cdots), \tag{2}$$

where $X_0 \neq 0$ and s is a natural number, known as the order of the pole at $z = 0$. The main purpose of this paper is to provide efficient computational procedures for the Laurent series coefficients X_k, $k \geq 0$. As one can see from the following literature review, few computational methods have been considered in the past.

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The inversion of nearly singular operator valued functions was probably first studied in the paper by Keldysh [22]. In that paper he studied the case of a polynomial perturbation

$$A(z) = A_0 + zA_1 + \cdots + z^m A_m, \tag{3}$$

where A_k, $1 \le k \le$ are compact operators on Hilbert space. In particular, he showed that the principal part of the Laurent series expansion for the inverse operator $A^{-1}(z)$ can be given in terms of generalized Jordan chains. The generalized Jordan chains were initially developed in the context of matrix and operator polynomials (see [13, 26, 30] and numerous references therein). However, the concept can be easily generalized to the case of an analytic perturbation (1).

Following Gohberg and Sigal [15] and Gohberg and Rodman [14], we say that the vectors $\varphi_0, \ldots, \varphi_{r-1}$ form a Jordan chain of the perturbed matrix $A(z)$ at $z = 0$, if $\varphi_0 \neq 0$ and if

$$\sum_{i=0}^{k} A_i \varphi_{k-i} = 0,$$

for each $0 \le k \le r - 1$. Note that φ_0 is an eigenvector of the unperturbed matrix A_0 corresponding to the zero eigenvalue. The number r is called the length of the Jordan chain and φ_0 is the initial vector. Let $\{\varphi_0^{(j)}\}_{j=1}^{p}$ be a system of linearly independent eigenvectors, which span the null space of A_0. Then one can construct Jordan chains initializing at each of the eigenvectors $\varphi_0^{(j)}$. This generalized Jordan set plays a crucial role in the analysis of analytic matrix valued functions $A(z)$.

Gantmacher [11] analysed the polynomial matrix (3) by using the canonical Smith form. Vishik and Lyusternik [37] studied the case of a linear perturbation $A(z) = A_0 + zA_1$ and showed that one can express $A^{-1}(z)$ as a Laurent series as long as $A(z)$ is invertible in some punctured neighbourhood of the origin. In addition, an undetermined coefficient method for the calculation of Laurent series terms was given in [37]. Langenhop [25] showed that the coefficients of the regular part of the Laurent series for the inverse of a linear perturbation form a geometric sequence. The proof of this fact was refined later in Schweitzer [33, 34] and Schweitzer and Stewart [35]. In particular, the paper [35] proposed a method for computing the Laurent series coefficients. However the method of [35] cannot be applied (at least immediately) to the general case of an analytic perturbation. Many authors have obtained existence results for operator valued analytic and meromorphic functions [3, 15, 23, 27, 29, 36]. In particular, Gohberg and Sigal [15], used a local Smith form to elaborate on the structure of the principal part of the Laurent series in terms of generalized Jordan chains. Recently, Gohberg, Kaashoek and Van Schagen [12] have refined the results of [15]. Furthermore, Bart, Kaashoek and Lay [5] used their results on the stability of the null and range spaces [4] to prove the existence of meromorphic relative inverses of finite meromorphic operator valued functions. The ordinary inverse operator is a particular case of the relative inverse. For the applications of the inversion of analytic matrix functions see for example [8, 9, 20, 23, 24, 28, 31, 32, 36].

Howlett [20] provided a computational procedure for the Laurent series coefficients based on a sequence of row and column operations on the coefficients of the original power series (1). Howlett used the rank test of Sain and Massey [32] to determine s, the order of the pole. He also showed that the coefficients of the Laurent series satisfy a finite linear recurrence relation in the case of a polynomial perturbation. The method of [20] can be considered as a starting point for our research. The algebraic reduction technique which is used in the present paper was introduced by Haviv and Ritov [17, 18] in the special case of stochastic matrices. Haviv, Ritov and Rothblum [19] also applied this approach to the perturbation analysis of semi-simple eigenvalues.

In this paper we provide three related methods for computing the coefficients of the Laurent series (2). The first method uses generalized inverse matrices to solve a set of linear equations and extends the work in [17] and [20]. The other two methods use results that appear in [2, 17, 18, 19] and are based on a reduction technique [6, 10, 21, 23]. All three methods depend in a fundamental way on equating coefficients for various powers of z. By substituting the series (1) and (2) into the identity $A(z)A^{-1}(z) = I$ and collecting coefficients of the same power of z, one obtains the following system which we will refer to as the fundamental equations:

$$
\begin{aligned}
A_0X_0 &= 0 && (4.0)\\
A_0X_1 + A_1X_0 &= 0 && (4.1)\\
&\vdots && \vdots\\
A_0X_s + \cdots + A_sX_0 &= I && (4.s)\\
A_0X_{s+1} + \cdots + A_{s+1}X_0 &= 0 && (4.s+1)\\
&\vdots && \vdots
\end{aligned}
$$

A similar system can written when considering the identity $A^{-1}(z)A(z) = I$ but of course the set of fundamental equations $(4.0), (4.1), \ldots$ is sufficient. Finally, for matrix operators each infinite system of linear equations uniquely determines the coefficients of the Laurent series (2). This fact has been noted in [3, 20, 23, 37, 36].

2 Main results

Let us define the following augmented matrix $\mathcal{A}^{(t)} \in \mathbb{R}^{(t+1)n\times(t+1)n}$

$$
\mathcal{A}^{(t)} = \begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 \\ A_1 & A_0 & 0 & \cdots & 0 \\ A_2 & A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_t & A_{t-1} & \cdots & A_1 & A_0 \end{bmatrix}
$$

and prove the following basic lemma.

Lemma 1 *Let s be the order of the pole at the origin for the inverse function $A^{-1}(z)$. Any eigenvector $\Phi \in \mathbb{R}^{(s+1)n}$ of $\mathcal{A}^{(s)}$ corresponding to the zero eigenvalue possesses the property that its first n elements are zero.*

Proof: Suppose on the contrary that there exists an eigenvector $\Phi \in \mathbb{R}^{(s+1)n}$ such that

$$\mathcal{A}^{(s)}\Phi = 0 \tag{5}$$

and not all of its first n entries are zero. Then, partition the vector Φ into $s+1$ blocks and rewrite (5) in the form

$$
\begin{gathered}
A_0\varphi_0 = 0\\
A_0\varphi_1 + A_1\varphi_0 = 0\\
\vdots\\
A_0\varphi_s + \cdots + A_s\varphi_0 = 0
\end{gathered}
$$

with $\varphi_0 \neq 0$. This means that we have found a generalized Jordan chain of length $s+1$. However, from the results of Gohberg and Sigal [15], we conclude that the maximal length of a generalized Jordan chain of $A(z)$ at $z = 0$ is s. Hence, we came to a contradiction and, consequently, $\varphi_0 = 0$.

□

Remark 1 *A direct proof of Lemma 1 is given in Appendix 1.*

Remark 2 *All vectors $\Phi \in \mathbb{R}^{(s+j+1)n}$ in the null space of the augmented matrix $\mathcal{A}^{(s+j)}$, $j \geq 0$, possess the property that the first $(j+1)n$ elements are zero.*

The following theorem provides a theoretical basis for the recursive solution of the infinite system of fundamental equations (4).

Theorem 1 *Each coefficient X_k, $k \geq 0$ is uniquely determined by the previous coefficients $X_0, \cdots, X_{k-1}$ and the set of s fundamental equations $(4.k) - (4.k+s)$.*

PROOF: It is obvious that the sequence of Laurent series coefficients $\{X_i\}_{i=0}^{\infty}$ is a solution to the fundamental equations (4). Suppose the coefficients X_i, $0 \leq i \leq k-1$, have been determined. Next we show that the set of fundamental equations (4.k)-(4.k+s) uniquely determines the next coefficient X_k. Indeed, suppose there exists another solution $\tilde{X}_k$. Since X_k and $\tilde{X}_k$ are both solutions, we can write

$$\mathcal{A}^{(s)} \begin{bmatrix} \tilde{X}_k \\ \vdots \\ \tilde{X}_{k+s} \end{bmatrix} = \begin{bmatrix} J_k - \sum_{i=1}^{k} A_i X_{k-i} \\ \vdots \\ J_{k+s} - \sum_{i=1}^{k} A_{i+s} X_{k-i} \end{bmatrix} \tag{6}$$

and

$$\mathcal{A}^{(s)} \begin{bmatrix} X_k \\ \vdots \\ X_{k+s} \end{bmatrix} = \begin{bmatrix} J_k - \sum_{i=1}^{k} A_i X_{k-i} \\ \vdots \\ J_{k+s} - \sum_{i=1}^{k} A_{i+s} X_{k-i} \end{bmatrix}, \tag{7}$$

where the matrix J_i is defined as follows:

$$J_i = \begin{cases} I, & i = s, \\ 0, & \text{otherwise} \end{cases}$$

and where $\tilde{X}_{k+1}, \ldots, \tilde{X}_{k+s}$ are any particular solutions of the nonhomogenous linear system (4.k)-(4.k+s). Note that (6) and (7) have identical righthand sides. Of course, the difference between these two righthand sides, $[\tilde{X}_k - X_k \cdots \tilde{X}_{k+s} - X_{k+s}]^T$, is in the right null space of $\mathcal{A}^{(s)}$. Invoking Lemma 1, the first n rows of $[\tilde{X}_k - X_k, \cdots, \tilde{X}_{k+s} - X_{k+s}]^T$ are hence zero. In other words, $\tilde{X}_k - X_k = 0$ which proves the theorem.

□

Using the above theoretical background, in the next section we provide three recursive computational schemes which are based on the generalized inverses and on a reduction technique. The reduction technique is based on the following result. A weaker version of this result was utilized in [17] and in [19].

Theorem 2 *Let $\{C_k\}_{k=0}^{t} \subseteq \mathbb{R}^{m \times m}$ and $\{R_k\}_{k=0}^{t} \subseteq \mathbb{R}^{m \times n}$, with $m \leq n$, and suppose that the system of $t+1$ matrix equations*

$$\begin{aligned} C_0 V_0 &= R_0 && (8.0) \\ C_0 V_1 + C_1 V_0 &= R_1 && (8.1) \\ &\vdots && \vdots \\ C_0 V_t + \cdots + C_t V_0 &= R_t && (8.t) \end{aligned}$$

is feasible. Then the general solution is given by

$$V_k = C_0^\dagger(R_k - \sum_{i=1}^{k} C_i V_{k-i}) + QW_k \tag{9}$$

where $C_0^\dagger$ is the Moore-Penrose generalized inverse of C_0 and $Q \in \mathbb{R}^{m\times p}$ is any matrix whose columns form a basis for the right null space of C_0. Furthermore, the sequence of matrices W_k, $0 \le k \le t-1$ solves a reduced finite set of t matrix equations

$$\begin{aligned} D_0W_0 &= S_0 && (10.0)\\ D_0W_1 + D_1W_0 &= S_1 && (10.1)\\ &\vdots && \vdots\\ D_0W_{t-1} + \cdots + D_tW_0 &= S_{t-1} && (10.t-1) \end{aligned}$$

where the matrices $D_k \in \mathbb{R}^{p\times p}$ and $S_k \in \mathbb{R}^{p\times n}$, $0 \le k \le t-1$, are computed by the following recursion. Set $U_0 = C_1$ and calculate

$$U_k = C_{k+1} - \sum_{i=1}^{k} C_i C_0^\dagger U_{k-i}, \quad k = 1, \ldots, t-1 \tag{11}$$

Then,

$$D_k = MU_kQ \quad \text{and} \quad S_k = M(R_{k+1} - \sum_{i=0}^{k} U_i C_0^\dagger R_{k-i}) \tag{12}$$

where $M \in \mathbb{R}^{p\times m}$ is any matrix whose rows form a basis for the left null space of C_0.

PROOF: The general solution to the matrix equation (7.0) can be written in the form

$$V_0 = C_0^\dagger R_0 + QW_0, \tag{13}$$

where $W_0 \in \mathbb{R}^{p\times n}$ is some arbitrary matrix.

In order for the equation

$$C_0V_1 = R_1 - C_1V_0$$

to be feasible, we need that the right hand side $R_1 - C_1V_0$ belongs to $R(C_0) = N^\perp(C_0^T)$, that is

$$M(R_1 - A_1V_0) = 0,$$

where the rows of M form a basis for $N(C_0^T)$. Substituting expression (13) for the general solution V_0 into the above feasibility condition, one finds that W_0 satisfies the equation

$$M(R_1 - C_1(C_0^\dagger R_0 + QW_0)) = 0$$

which can be rewritten as

$$MC_1QW_0 = M(R_1 - C_1C_0^\dagger R_0).$$

Thus we have obtained the first reduced fundamental equation (9.0) with

$$D_0 := MU_0Q \quad \text{and} \quad S_0 := M(R_1 - U_0C_0^\dagger R_0).$$

where $U_0 = C_1$. Next we observe that the general solution of equation (7.1) is represented by the formula

$$V_1 = C_0^\dagger(R_1 - C_1V_0) + QW_1 \tag{14}$$

with $W_1 \in \mathbb{R}^{p\times n}$. Moving on and applying the feasibility condition to equation (7.2), we obtain

$$M(R_2 - (C_1V_1 + C_2V_0)) = 0$$

and again the substitution of expressions (13) and (14) into the above condition yields

$$MC_1(C_0^\dagger(R_1 - C_1[C_0^\dagger R_0 + QW_0]) + QW_1) + MC_2(C_0^\dagger R_0 + QW_0) = MR_2$$

which is rearranged to give

$$MC_1QW_1 + M(C_2 - C_1C_0^\dagger C_1)QW_0 = M(R_2 - C_1C_0^\dagger R_1 - (C_2 - C_1C_0^\dagger C_1)C_0^\dagger R_0).$$

The last equation is the reduced equation (9.1) with

$$D_1 := MU_1Q \quad \text{and} \quad S_1 := M(R_2 - U_0C_0^\dagger R_1 - U_1C_0^\dagger R_0)$$

where $U_1 = C_2 - C_1C_0^\dagger U_0$. Note that this equation imposes restrictions on W_1 as well as on W_0. By proceeding in the same way, we eventually obtain the complete system of equations (9) with coefficients given by formulas (11) and (12) each of which can be proved by induction in a straightforward way.

□

Remark 3 *In the above theorem it is important to observe that the reduced system has the same form as the original but the number of matrix equations is decreased by one and the coefficients are reduced in size to matrices in $\mathbb{R}^{p\times p}$, where p is the dimension of $N(C_0)$ or, equivalently, the number of redundant equations defined by the coefficient C_0.*

In the next section we use this reduction process to solve the system of fundamental equations. Note that the reduction process can be employed to solve any appropriate finite subset of the fundamental equations.

3 Solution methods

In this section we discuss three methods for solving the fundamental equations. The first method is based on the direct application of Moore-Penrose generalized inverses. The second method involves the replacement of the original system of the fundamental equations by a system of equations with a reduced dimension. In the third method we show that the reduction process can be applied recursively to reduce the problem to a non-singular system. Since all methods depend to some extent on the prior knowledge of s, we begin by discussing a procedure for the determination of s. A special procedure for determining this order for the case where the matrices $A(z)$ are stochastic and the perturbation series is finite is given in [16]. It is based on combinatorial properties (actually, network representation) of the processes and hence it is a stable procedure. However, as will be seen in Section 3.4, it is possible to use the third method without prior knowledge of s. Actually, the full reduction version of our procedure determines s as well. Of course, as in any computational method which is used to determine indices which have discrete values, using our procedures in order to compute the order of singularity might lack stability.

3.1 The determination of the order of the pole

The rank test on the matrix $\mathcal{A}^{(t)}$ proposed by Sain and Massey in [32] is likely to be the most effective procedure for determining the value of s. The calculation of rank is essentially equivalent to the reduction of $\mathcal{A}^{(t)}$ to a row echelon normal form and it can be argued that row operations can be used successively in order to calculate the rank of $\mathcal{A}^{(0)}$,$\mathcal{A}^{(1)}$,$\mathcal{A}^{(2)}$,... and find the minimum value of t for which rank$\mathcal{A}^{(t-1)}+n$. This minimum value of t equals s, the order of the pole. Note that previous row operations for reducing $\mathcal{A}^{(t-1)}$ to row echelon form are replicated in the reduction of $\mathcal{A}^{(t)}$ and do not need to be repeated. For example, if a certain combination of row operations reduces A_0 to row echelon form, then the same operations are used again as part of the reduction of

$$\begin{bmatrix} A_0 & 0 \\ A_1 & A_0 \end{bmatrix}$$

to row echelon form.

3.2 Basic generalized inverse method

In this section we obtain a recursive formula for the Laurent series coefficients X_k, $k \geq 0$ by using the Moore-Penrose generalized inverse of the augmented matrix $\mathcal{A}^{(s)}$.

Let $\mathcal{G}^{(s)} \stackrel{def}{=} [\mathcal{A}^{(s)}]^\dagger$ be the Moore-Penrose generalized inverse of $\mathcal{A}^{(s)}$ and define the matrices $G_{ij}^{(s)} \in \mathrm{IR}^{n\times n}$ for $0 \leq i,j \leq t$ by

$$\mathcal{G}^{(s)} = \begin{bmatrix} G_{00}^{(s)} & \cdots & G_{0s}^{(s)} \\ \vdots & \ddots & \vdots \\ G_{s0}^{(s)} & \cdots & G_{ss}^{(s)} \end{bmatrix}$$

Furthemore, we would like to note that in fact we use only the first n rows of the generalized inverse $\mathcal{G}^{(s)}$, namely, $[G_{00}^{(s)} \ \cdots \ G_{0s}^{(s)}]$.

Proposition 1 *The coefficients of the Laurent series* (2) *can be calculated by the following recursive formula*

$$X_k = \sum_{j=0}^{s} G_{0j}^{(s)}(J_{j+k} - \sum_{i=1}^{k} A_{i+j}X_{k-i}) \quad , \quad k \geq 1, \tag{15}$$

where $X_0 = G_{0s}^{(s)}$ *and the matrix* J_i *is defined by*

$$J_i = \begin{cases} I, & i = s \\ 0, & otherwise \end{cases}$$

PROOF: According to Theorem 1, once the coefficients X_i, $0 \leq i \leq k-1$ are determined, the next coefficient X_k can be obtained from the (4.k)-(4.k+s) fundamental equations.

$$\mathcal{A}^{(s)} \begin{bmatrix} X_k \\ \vdots \\ X_{k+s} \end{bmatrix} = \begin{bmatrix} J_k - \sum_{i=1}^k A_i X_{k-i} \\ \vdots \\ J_{k+s} - \sum_{i=1}^k A_{i+s} X_{k-i} \end{bmatrix}$$

The general solution to the above system is given in the form

$$\begin{bmatrix} X_k \\ \tilde{X}_{k+1} \\ \vdots \\ \tilde{X}_{k+s} \end{bmatrix} = \begin{bmatrix} G_{00}^{(s)} & \cdots & G_{0s}^{(s)} \\ G_{10}^{(s)} & \cdots & G_{1s}^{(s)} \\ \vdots & \ddots & \vdots \\ G_{s0}^{(s)} & \cdots & G_{ss}^{(s)} \end{bmatrix} \begin{bmatrix} J_k - \sum_{i=1}^k A_i X_{k-i} \\ J_{k+1} - \sum_{i=1}^k A_{i+1} X_{k-i} \\ \vdots \\ J_{k+s} - \sum_{i=1}^k A_{i+s} X_{k-i} \end{bmatrix} + \begin{bmatrix} 0 \\ \Phi_1 \\ \vdots \\ \Phi_s \end{bmatrix},$$

where the first block of matrix Φ is equal to zero according to Lemma 1. Thus, we immediately obtain the recursive expression (15). In particular, applying the same arguments as above to the first $s+1$ fundamental equations, we obtain that $X_0 = G_{0s}^{(s)}$.

□

Note that the matrices J_{j+k} in the expression (15) disappear when the regular coefficients are computed.

Remark 4 *The formula* (15) *is a generalization of the recursive formula for the case where* A_0 *is invertible. In this case,*

$$X_k = -A_0^{-1} \sum_{i=1}^{k} A_i X_{k-i}, \quad k \geq 1$$

while initializing with $X_0 = A_0^{-1}$.

Remark 5 *Probably from the computational point of view it is better not to compute the generalized inverse* $\mathcal{G}^{(s)}$ *beforehand, but rather to find the SVD or LU decomposition of* $\mathcal{A}^{(s)}$ *and then use these decompositions for solving the fundamental equations* (3.k)–(3.$k+s$). *This is the standard approach for solving linear systems with various righthand sides.*

3.3 The one step reduction process

In this section we describe an alternative scheme that can be used in the case where it is relatively easy to compute the bases for the right and for the left null spaces of A_0. Specifically, let $p = n - r(A_0)$ be the dimension of the null space of A_0, let $Q \in \mathrm{IR}^{n \times p}$ be a matrix whose p columns form a basis for the right null space of A_0 and let $M \in \mathrm{IR}^{p \times n}$ be a matrix whose p rows form a basis for the left null space of A_0. Of course, although $p = 0$ and hence $s = 0$ is possible, we are interested in the singular case where $p \geq 1$.

Again, as before, we suppose that the coefficients X_i, $0 \leq i \leq k-1$, have already been determined. Then, by Theorem 1, the next coefficient X_k is the unique solution to the subsystem of fundamental equations

$$\begin{array}{rcl} A_0 X_k & = & J_k - \sum_{i=1}^{k} A_i X_{k-i} \\ A_0 X_{k+1} + A_1 X_k & = & J_{k+1} - \sum_{i=1}^{k} A_{i+1} X_{k-i} \\ & \vdots & \\ A_0 X_{k+s} + \cdots + A_s X_k & = & J_{k+s} - \sum_{i=1}^{k} A_{i+s} X_{k-i}. \end{array} \tag{16}$$

The above system is like the one given in (9) with $C_i = A_i$, $0 \leq i \leq s$ and with $R_j = J_{k+j} - \sum_{i=1}^{k} A_{i+j} X_{k-i}$, $0 \leq j \leq s$. Therefore, we can apply the reduction process described in Theorem 2. This results in the system

$$\begin{array}{rcl} D_0 W_0 & = & S_0 \\ D_0 W_1 + D_1 W_0 & = & S_1 \\ & \vdots & \\ D_0 W_{s-1} + \cdots + D_{s-1} W_0 & = & S_{s-1}, \end{array} \tag{17}$$

where the coefficients D_i and S_i, $i = 0, \ldots, s-1$ can be calculated by the recursive formulae (11) and (12).

Remark 6 *Note that in many practical applications* p *is much less than* n *and hence the above system (17) with* $D_i \in \mathrm{IR}^{p \times p}$ *is much smaller than the original system (16).*

Now we have two options. We can either apply the reduction technique again (see the next subsection for more details) or we can solve the reduced system directly by using the generalized inverse approach. In the latter case, we define

$$\mathcal{D}^{(t)} \stackrel{def}{=} \begin{bmatrix} D_0 & 0 & 0 & \cdots & 0 \\ D_1 & D_0 & 0 & \cdots & 0 \\ D_2 & D_1 & D_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ D_t & D_{t-1} & \cdots & D_1 & D_0 \end{bmatrix}$$

and

$$\mathcal{H}^{(t)} = \begin{bmatrix} H_{00}^{(t)} & \cdots & H_{0t}^{(t)} \\ \vdots & \ddots & \vdots \\ H_{t0}^{(t)} & \cdots & H_{tt}^{(t)} \end{bmatrix} \stackrel{def}{=} [\mathcal{D}^{(t)}]^{\dagger}.$$

Then, by carrying out a similar computation to the one presented in the proof of Proposition 1, we obtain

$$W_0 = \sum_{i=0}^{s-1} H_{0i}^{(s-1)} S_i.$$

Once W_0 is determined it is possible to obtain X_k from the formula

$$X_k = A_0^{\dagger} R_0 + Q W_0 = A_0^{\dagger} R_0 + Q \sum_{i=0}^{s-1} H_{0i}^{(s-1)} S_i.$$

Furthermore, substituting for S_i, $0 \le i \le s-1$, from (12) and changing the order of summation gives

$$X_k = \left(A_0^{\dagger} - \sum_{i=0}^{s-1} Q H_{0i}^{(s-1)} M U_i A_0^{\dagger} \right) R_0 + \sum_{j=1}^{s} \left(Q H_{0j-1}^{(s-1)} M - \sum_{i=j}^{s-1} Q H_{0i}^{(s-1)} M U_{i-j} A_0^{\dagger} \right) R_j. \tag{18}$$

Note that by convention the sum disappears when the lower limit is greater than the upper limit. Now, substituting $R_j = J_{k+j} - \sum_{i=1}^{k} A_{i+j} X_{k-i}$, $0 \le j \le s$ into the expression (18), we obtain the explicit recursive formula for the Laurent series coefficients

$$\begin{aligned} X_k \quad = \quad & \left(A_0^{\dagger} - \sum_{i=0}^{s-1} Q H_{0i}^{(s-1)} M U_i A_0^{\dagger} \right) (J_k - \sum_{i=1}^{k} A_i X_{k-i}) \\ & + \sum_{j=1}^{s} \left(Q H_{0j-1}^{(s-1)} M - \sum_{i=j}^{s-1} Q H_{0i}^{(s-1)} M U_{i-j} A_0^{\dagger} \right) (J_{k+j} - \sum_{i=1}^{k} A_{i+j} X_{k-i}) \end{aligned} \tag{19}$$

for all $k \ge 1$. In particular, the coefficient of the first singular term in (2) can be given by the formula

$$X_0 = Q H_{0s-1}^{(s-1)} M. \tag{20}$$

3.4 The complete reduction process

As was pointed out in the previous section, the reduced system has essentially the same structure as the original one and hence one can apply again the reduction step described in Theorem 2. Note that each time the reduction step is carried out, the number of matrix equations is reduced by one. Therefore one can perform up to s reduction steps. We now outline how these steps can be executed. We start by introducing the sequence of reduced systems. The fundamental matrix equations for the l-th reduction step are

$$\begin{aligned} A_0^{(l)}X_0^{(l)} &= R_0^{(l)} && (21.0)\\ A_0^{(l)}X_1^{(l)} + A_1^{(l)}X_0^{(l)} &= R_1^{(l)} && (21.1)\\ &\vdots && \vdots\\ A_0^{(l)}X_{s-l}^{(l)} + \cdots + A_{s-l}^{(l)}X_0^{(l)} &= R_{s-l}^{(l)} && (21.s-l) \end{aligned}$$

With $l = 0$, one gets the original system of fundamental equations and with $l = 1$ one gets the reduced system for the first reduction step described in the previous subsection. Initializing with $R_i^{(0)} = 0$, $0 \le i \le s-1$, and $R_s^{(0)} = I$ and with $A_i^{(0)} = A_i$, $0 \le i \le s$, the matrices $A_j^{(l)}$ and $R_j^{(l)}$, $0 \le j \le s-l$, for each reduction step $1 \le l \le s$, can be computed successively by a recursion similar to (11) and (12). In general we have

$$U_0^{(l)} = A_1^{(l-1)} \quad , \quad U_j^{(l)} = A_{j+1}^{(l-1)} - \sum_{i=1}^{j} A_i^{(l-1)} A_0^{(l-1)\dagger} U_{j-i}^{(l)}, \quad j = 1, \ldots, s-l$$

$$A_j^{(l)} = M^{(l)} U_j^{(l)} Q^{(l)} \quad j = 0, \ldots, s-l,$$

$$R_j^{(l)} = M^{(l)} \left(-\sum_{i=0}^{j} U_{j-i}^{(l)} A_0^{(l-1)\dagger} R_i^{(l-1)} + R_{j+1}^{(l-1)} \right) \quad j = 0, \ldots, s-l$$

where $Q^{(l)}$ and $M^{(l)}$ are the basis matrices for the right and left null spaces respectively of the matrix $A_0^{(l-1)}$ and where $A_0^{(l-1)\dagger}$ is the Moore-Penrose generalized inverse of $A_0^{(l-1)}$. After s reduction steps, one gets the final system of reduced equations

$$A_0^{(s)} X_0^{(s)} = R_0^{(s)}. \tag{22}$$

Since X_0 is a unique solution to the subsystem of fundamental equations (4.0)-(4.s) and Theorem 2 states the equivalence of the l-th and $(l+1)$-st systems of reduced equations, the system (22) possesses a unique solution, and hence matrix $A_0^{(s)}$ is invertible. Thus,

$$X_0^{(s)} = [A_0^{(s)}]^{-1} R_0^{(s)}. \tag{23}$$

The original solution $X_0 = X_0^{(0)}$ can be now retrieved by the backwards recursive relationship

$$X_0^{(l-1)} = A_0^{(l-1)\dagger} R_0^{(l-1)} + Q^{(l)} X_0^{(l)} \quad l = s, \ldots, 1 \tag{24}$$

Now by taking $R_j^{(0)} = J_{k+j} - \sum_{i=1}^{k} A_{i+j} X_{k-i}$, $0 \le j \le s$, one gets the algorithm for computing the Laurent series coefficients $X_k, k \ge 1$. Of course, recursive formulae similar to (15) and (19) can be obtained, but they are quite complicated in the general case.

The order s of the pole can also be obtained from the reduction process by continuing the process until $A_0^{(l)}$ becomes non-singular. The number of reduction steps equals the order of the pole. Note also that the sequence of matrices $A_0^{(l)}$, $l \ge 0$, can be computed irrespectively of the right hand sides. Once s is determined, one can compute $R_j^{(l)}$, $1 \le l \le s$, $0 \le j \le s-l$.

4 Computational complexity and comparison with symbolic algebra

In this section we compare the computational complexity of the one-step-reduction process when applied to compute X_0 with the complexity of symbolic algebra. In particular, we show that the former comes with a reduced complexity in the case where the pole has a relatively small order. The computational complexity of the other two procedures can be determined similarly.

To compute the coefficients D_i, $0 \leq i \leq s-1$, of the reduced fundamental system (17), one needs to perform $O(s^2n^3)$ operations. The total number of reduced equations is sp (recall that p is the dimension of the null space of A_0). Hence, the computational complexity for determining X_0 by the one-step-reduction process is $O(\max\{s^2n^3, s^3p^3\})$. The Laurent series (2) in general, and the coefficient X_0 in particular, can also be computed by using symbolic algebra. This for example can be executed by MATLAB symbolic toolbox and is done as follows. Since X_0 is uniquely determined by the first $s+1$ fundamental equations $(4.0), \cdots, (4.s)$, all one needs to do in order to compute X_0 is it to invert symbolically the following matrix polynomial

$$\hat{A}(z) = A_0 + zA_1 + \cdots + z^s A_s. \tag{25}$$

Symbolic computations here mean performing operations, such as multiplication and division, over the field of rational functions (and not over the field of the reals). In particular, if the degrees of numerators and of denominators of rational functions do not exceed q, then each operation (multiplication or division) which is performed in the field of rational functions translates into $qlog(q)$ operations in the field of real numbers [1]. Note that during the symbolic inversion of the polynomial matrix (25), the degree of rational functions does not exceed sn. The latter fact follows from Cramer's rule. Thus, the complexity of the symbolic inversion of (25) equals $O(n^3) \times O(snlog(sn)) = O(sn^4 log(sn))$. As a result, one gets a matrix $\hat{A}^{-1}(z)$ whose elements are rational functions of z. The elements of the matrix X_0 can then be immediately calculated by dividing the leading coeficients of the numerator and denominator. Finally, one can see that if $s << n$ and $p << n$, which is typically the case, then our method comes with a reduced computational burden.

5 Concluding remarks

In this paper we have shown that the Laurent series for the inversion of an analytic matrix valued function can be computed by solving a system of fundamental linear equations. Furthermore, we demonstrated that the system of fundamental equations can be solved recursively. In particular, the coefficient X_k is determined by the previous coefficients $X_0, \dots, X_{k-1}$ and the next $s+1$ fundamental equations, where s is the order of the pole. We suggest three basic methods, one without any reduction (see (15)), one with a single reduction step (see (19) and (20)), and one using a complete reduction process with s steps (see (23) and (24)). Of course, an intermediate process with the number of reductions between 1 and s could be used too. We note that when the complete reduction process is used the order of the pole can be determined through the execution of the algorithm. When $s << n$ and $p << n$, the proposed algorithms by far outperform the method based on symbolic algebra.

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Apendix 1: Another proof of Lemma 1

A direct proof of Lemma 1 can be carried out using augmented matrices. Specifically, define

$$\mathcal{X}^{(t)} = \begin{bmatrix} X_0 & 0 & 0 & \cdots & 0 \\ X_1 & X_0 & 0 & \cdots & 0 \\ X_2 & X_1 & X_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_t & X_{t-1} & \cdots & X_1 & X_0 \end{bmatrix}$$

where X_k, $0 \leq k \leq t$ are the coefficients of the Laurent series (2). Then it follows from the fundamental systems (4) and (5) that the augmented matrices $\mathcal{A}^{(t)}$ and $\mathcal{X}^{(t)}$ satisfy the relationship

$$\mathcal{A}^{(t)}\mathcal{X}^{(t)} = \mathcal{X}^{(t)}\mathcal{A}^{(t)} = \mathcal{E}^{(t)} \tag{26}$$

where the augmented matrix $\mathcal{E}^{(t)} \in \mathbb{R}^{(t+1)n \times (t+1)n}$ is defined by setting $\mathcal{E}^{(t)} = [E_{pq}]_{p,q=0}^{t}$ where $E_{pq} \in \mathbb{R}^{n\times n}$ and

$$E_{pq} = \begin{cases} I & \text{for} \quad p-q=s, \\ 0 & \text{for} \quad p-q \neq s. \end{cases}$$

Now, as before, let $\Phi \in \mathbb{R}^{(s+1)n}$ satisfy the equation

$$\mathcal{A}^{(s)}\Phi = 0. \tag{27}$$

If we multiply equation (27) from the left by $\mathcal{X}^{(s)}$, then it reduces to

$$\mathcal{E}^{(s)}\Phi = 0.$$

The vector $\mathcal{E}^{(s)}\Phi$ has φ_0 as the $(s+1)$-st block, which gives the required result.

Apendix 2: A Numerical example

Let us consider the matrix valued function

$$A(z) = A_0 + zA_1 = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} + z \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix},$$

where $\mathrm{rank}(A_0) = 2$. Construct the augmented matrices

$$\mathcal{A}^{(0)} = A_0 \quad \text{and} \quad \mathcal{A}^{(1)} = \begin{bmatrix} A_0 & 0 \\ A_1 & A_0 \end{bmatrix}.$$

and note that $\mathrm{rank}(\mathcal{A}^{(1)}) - \mathrm{rank}(\mathcal{A}^{(0)}) = 5 - 2 = 3$, which is the dimension of the original coefficients A_0 and A_1. Therefore, according to the test of Sain and Massey [32], the Laurent expansion for $A^{-1}(z)$ has a simple pole. Alternatively, we can compute a basis for $N(\mathcal{A}^{(1)})$, which in this particular example consists of only one vector

$$q^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & -3 \end{bmatrix}^T.$$

The first three zero elements in $q^{(1)}$ confirm that the Laurent series has a simple pole. Next we compute the generalized inverse of $\mathcal{A}^{(1)}$ given by

$$\mathcal{A}^{(1)\dagger} = \mathcal{G}^{(1)} = \begin{bmatrix} G_{00}^{(1)} & G_{01}^{(1)} \\ G_{10}^{(1)} & G_{11}^{(1)} \end{bmatrix} = \begin{bmatrix} 1/3 & -5/12 & -1/12 & 1/8 & 1/8 & -1/8 \\ 0 & 1/4 & 1/4 & 1/8 & 1/8 & -1/8 \\ 1/3 & -5/12 & -1/12 & -3/8 & -3/8 & 3/8 \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}.$$

Consequently,

$$X_0 = G_{01}^{(1)} = \frac{1}{8}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -3 & -3 & 3 \end{bmatrix}. \tag{28}$$

Alternatively, we know that X_0 is uniquely determined by the fundamental equations

$$\begin{aligned} A_0 X_0 &= 0, \\ A_0 X_1 + A_1 X_0 &= I. \end{aligned}$$

After one reduction step these equations reduce to

$$M A_1 Q W_0 = M,$$

where

$$M = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & 1 & -3 \end{bmatrix}^T.$$

Hence,

$$W_0 = (M A_1 Q)^{-1} M = \frac{1}{8}\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$$

and

$$X_0 = Q W_0 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \frac{1}{8}\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -3 & -3 & 3 \end{bmatrix}.$$

The latter expression is identical with (28) and coincides with the one computed by expanding $A^{-1}(z)$ with the help of the MATLAB symbolic toolbox. Note that even for this three dimensional example the direct symbolic calculation of the Laurent series takes a relatively long time.

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