Perturbation of null spaces with application to the eigenvalue problem and generalized inverses

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Abstract

We consider properties of a null space of an analytically perturbed matrix. In particular, we obtain Taylor expansions for the eigenvectors which constitute a basis for the perturbed null space. Furthermore, we apply these results to the calculation of Puiseux expansion of the perturbed eigenvectors in the case of general eigenvalue problem as well as to the calculation of Laurent series expansions for the perturbed group inverse and pseudoinverse matrices.

1 Introduction

The primary goal of this paper is to analyse the null space of an analytically perturbed matrix

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots, \tag{1}$$

with $A_k \in R^{n\times n}, k = 0, 1, ...,$ when the above series converges in a region $0 \le |\varepsilon| \le \varepsilon_{\max}$ for some positive $\varepsilon_{\max}$. Then, we present two applications of our results. Firstly, the results on the perturbation of null spaces can be immediately applied to calculate the Puiseux series expansion of the general perturbed eigenvalue problem [5, 7, 8, 17, 19, 20, 21, 22, 23]

$$A(\varepsilon)x(\varepsilon) = \lambda(\varepsilon)x(\varepsilon).$$

Secondly, we show how our analysis coupled with techniques used for the inversion of singular perturbed matrices [3] can be used in order to compute the Laurent series expansion for the group inverse of an analytically perturbed martrix (in the case it exists). We would like to note that an additional use of our results lies in an immediate application they have in computing series expansions for the Moore-Penrose generalized inverse of non-invertible matrices. Namely, as it was pointed out in [2], the calculation of $A^\dagger$, the generalizied inverse of a matrix A, can be reduced to the calculation of the group inverse of another symmetric matrix as

$$A^\dagger = (A^T A)^g A^T.$$

Consequently, the perturbation analysis of the generalized inverse of an arbitrary matrix can be carried out via the perturbation analysis of the group invese of a symmetric matrix (which is known to exist). [1]

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[1] Note that $A^\dagger$ stands for the Moore-Penrose generalized inverse of A. In the case that A is symmetric $A^\dagger$ can be replaced by $A^\#$ which is the group inverse of A. See [6] for more on these matrices.

We assume that the unperturbed matrix A_0 has eigenvalue zero with multiplicity $m \geq 1$ and that the perturbed matrices $A(\varepsilon)$ also have eigenvalue zero with multiplicity $\bar{m}$ for ε sufficiently small but different from zero. We emphasize that the dimension of the perturbed null space does not depend on ε in some small punctured neighbourhood around $\varepsilon = 0$ [11, 12]. When the perturbation parameter ε deviates from zero, the zero eigenvalues of the unperturbed matrix can split into zero and non-zero eigenvalues [5, 17]. This fact implies that $\bar{m} \leq m$. More detailed discussion on the stability properties of null spaces can be found in [4] and the references therein. We assume that $\bar{m} \geq 1$ and (for computational purposes) that the value of $\bar{m}$ should be known in advance. The case where $\bar{m} = 0$ and hence $A(\varepsilon)$ is invertible for $\varepsilon \neq 0$ and sufficiently small, was dealt with in [3]. A perturbation is said to be *rank-preserving* if $\bar{m} = m$, and it is said to be *non-rank-preserving* if $\bar{m} < m$. The following examples clarify the distinction between these two types of perturbation.

Example 1 *First we consider an example of the rank-preserving perturbation. Namely, let the perturbed matrix be given by*

$$A(\varepsilon) = A_0 + \varepsilon A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The null spaces of A_0 and $A(\varepsilon)$ are both one-dimensional and they are spanned, respectively, by

$$\tilde{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad , \quad v(\varepsilon) = \begin{bmatrix} 1 \\ -\varepsilon \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

We can see that $v(\varepsilon)$ is holomorphic and converges to v as ε goes to zero.

Example 2 *The following is an example of a non-rank-preserving perturbation with $\bar{m} \geq 1$.*

$$A(\varepsilon) = A_0 + \varepsilon A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The null space of A_0 is two-dimensional and it is spanned by

$$\tilde{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \quad \tilde{v}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \quad .$$

The null space of $A(\varepsilon)$ is one-dimensional and it is spanned by the holomorphic vector-valued function

$$v(\varepsilon) = \begin{bmatrix} 1 \\ -\varepsilon \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}. \tag{2}$$

Thus, we can see that as $\varepsilon \to 0$, $v(\varepsilon)$ converges to a vector which belongs to the unperturbed null space of matrix A_0, but there is a gap in the dimensions between the perturbed and unperturbed null spaces.

We denote by $\tilde{v}_i, i = 1, \ldots, m$, m orthonormal eigenvectors of A_0 corresponding to the eigenvalue zero and form the matrix $\tilde{V} := [\tilde{v}_1, \ldots, \tilde{v}_m]$. This matrix satisfies the following equations

$$A_0 \tilde{V} = 0, \tag{3}$$

$$\tilde{V}^T \tilde{V} = I_m. \tag{4}$$

Similarly, let $v_i(\varepsilon), i = 1, ..., \bar{m}$ be linearly independent eigenvectors of the perturbed matrix $A(\varepsilon)$ corresponding to the eigenvalue zero. Again one can form the matrix $V(\varepsilon) := [v_1(\varepsilon), \dots, v_{\bar{m}}(\varepsilon)]$, which satisfies the equation

$$A(\varepsilon)V(\varepsilon) = 0. \tag{5}$$

From Chapter S6 of [11] and the results of [12], we know that there exists a holomorphic family of vector-valued functions $v_i(\varepsilon)$ which constitute a basis for the null space of $A(\varepsilon)$ for $\varepsilon \neq 0$. Therefore, $V(\varepsilon)$ can be expressed as a power series in some neighbourhood of zero

$$V(\varepsilon) = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + \cdots. \tag{6}$$

Of course, there exists an orthonormal family of perturbed eigenvectors $v_i(\varepsilon), i = 1, \dots, \bar{m}$. However, it is more convenient to construct a "quasi-orthonormal" family of eigenvectors described by the following condition

$$V_0^T V(\varepsilon) = I_{\bar{m}}, \tag{7}$$

where V_0 is the first coefficient of the power series expansion (6)(rather than $V^T(\varepsilon)V(\varepsilon) = I_{\bar{m}}$). The above normalized conditions were used in [7, 8] for the case of rank-preserving perturbations. Note that even though this family of eigenvectors is not orthonormal for $\varepsilon \neq 0$, it is linearly independent when ε is sufficiently small [7, 8]. Also note that (7) was introduced in order to make $V(\varepsilon)$ unique once the leading term V_0 is determined. As we show later, there is some freedom in selecting V_0. Furthermore, one can always obtain an orthonormal basis from a "quasi-orthonormal" basis by applying a Gram-Schmidt-like procedure over the vectors with elements as power series expansions. This procedure will be discussed in more detail in Section 5. As mentioned above, we distinguish between two cases, the rank-preserving case where $\bar{m} = m$ and the non-rank-preserving case where $1 \leq \bar{m} < m$. Note that only in the rank-preserving case it is possible to set $V_0 = V$. Moreover, as will be demonstrated in Section 5 in the case of non-rank-preserving perturbation only, the group inverse of the perturbed matrix (when it exists) has a singularity, that is, a pole at $\varepsilon = 0$. Later on we also show that the group inverse of $A(\varepsilon)$, denoted next by $A^g(\varepsilon)$, can be expanded as a Laurent series:

$$A^g(\varepsilon) = \frac{1}{\varepsilon^s}B_{-s} + \cdots + \frac{1}{\varepsilon}B_{-1} + B_0 + \varepsilon B_1 + \cdots \tag{8}$$

with $B_{-s} \neq 0$, i.e., where s is the order of the pole at the singularity.

One of the main purposes of this paper is to obtain an efficient recursive algorithm for the computation of coefficients $V_k, k = 0, 1, \dots$ as well as for the computation of $B_l, l = -s, -s+1, \cdots$, determining the Laurent expansion (8). We would like to point out that, at least with our procedure, the latter task cannot be accomplished without prior completion of the former task. The algorithm for computing $V_k, k = 0, 1, \dots$ is based on recursively solving a system of *fundamental equations*. The fundamental equations named $F0, F1, \dots$, are obtained by substituting (1) and (6) into (5) to give

$$\begin{aligned}
A_0 V_0 &= 0 && (F0)\\
A_0 V_1 + A_1 V_0 &= 0 && (F1)\\
A_0 V_2 + A_1 V_1 + A_2 V_0 &= 0 && (F2)\\
&\vdots && \vdots\\
A_0 V_k + A_1 V_{k-1} + \cdots + A_{k-1} V_1 + A_k V_0 &= 0 && (Fk)\\
&\vdots && \vdots
\end{aligned}$$

The normalization condition (7) gives rise to another system of equations named $N0, N1, \dots$ and given by

$$
\begin{array}{rcl}
V_0^T V_0 & = & I_{\bar{m}} \qquad (N0)\\
V_0^T V_1 & = & 0 \qquad (N1)\\
& \vdots & \qquad \vdots\\
V_0^T V_k & = & 0 \qquad (Nk)\\
& \vdots & \qquad \vdots
\end{array}
$$

We will refer to the latter system as the system of *normalization equations.* We treat the cases of rank-preserving and non-rank-preserving perturbations separately. In Section 2 we provide an algorithm for computing the coefficients V_k, $k \geq 0$, in the rank-preserving case. This algorithm is based on a straightforward recursive procedure. The non-rank-preserving case is treated in Section 3, where we suggest three algorithms for computing $\{V_i\}_{i=0}^{\infty}$. The first is based on defining an augmented matrix which leads to the solution. The second algorithm is based on reducing the dimension of the equations to a set of equations whose type coincides with the rank-preserving case. The third algorithm is a combination of the previous two algorithms and is based on an early abortion of the reduction process and then solving the resulting system with the help of a generalized inverse. Finally, in Section 4 we show how our results can be applied to a perturbation analysis of the general eigenvalue problem and in Section 5 we present a second application of our results to the computation of the Laurent series expansion for the perturbed group inverse.

2 Rank-Preserving perturbations

Before proceeding further, we would like to state a well-know lemma. This lemma was shown to be useful in solving simultaneously and recursively systems of the type of (F) and (N) in [13, 15].

Lemma 1 *The system of linear equations $Ax = b$ is feasible if and only if for any row-vector u^T, $u^T b = 0$ whenever $u^T A = 0$. Moreover, in the case that the system is feasible $x = A^\dagger b + v$ is a solution if and only if $Av = 0$.*

The following lemma states a necessary condition for a perturbation to be rank preserving. This condition, of course, can be check in practice only in case of polynomial pertirbation.

Lemma 2 *If the perturbation is rank-preserving then the sequence of matrices $\{A_k\}_{k=0}^{\infty}$ satisfies the following conditions:*

$$
\tilde{U}^T \left(\sum_{p=1}^{k+1} (-1)^{p-1} \sum_{\nu_1 + \cdots + \nu_p = k+1} A_{\nu_1} A_0^\dagger A_{\nu_2} \cdots A_0^\dagger A_{\nu_p} \right) \tilde{V} = 0, \quad k = 0, 1, \cdots , \tag{9}
$$

where $\nu_i \geq 1$, and where $\tilde{V}$ and $\tilde{U}$ are bases for the right and left null spaces of matrix A_0, respectively.

PROOF: From equation (F0) we conclude that

$$
V_0 = \tilde{V} C_0 \tag{10}
$$

where C_0 is some coefficient matrix. Since we consider the case of a rank-preserving perturbation, the rank of V_0 equals m. This in turn implies that $C_0 \in R^{m \times m}$ and that it is a full rank matrix.

Since $\tilde{U}^T A_0 = 0$, we obtain by Lemma 1, the following feasibility condition for equation (F1):

$$
\tilde{U}^T A_1 V_0 = 0
$$

Upon substituting (10) into the above expression, we get that

$$\tilde{U}^T A_1 \tilde{V} C_0 = 0.$$

Moreover, since C_0 is a full rank matrix, we conclude that

$$\tilde{U}^T A_1 \tilde{V} = 0, \tag{11}$$

which is the first feasibility condition of (9).

Since the perturbation is rank-preserving, there exists a holomorphic basis $V(\varepsilon)$ for the perturbed null space such that $V(0) = \tilde{V}$. The coefficients $V_k, k = 0, 1, \ldots$ of the power series (6) satisfy the fundamental equations (F0),(F1),... Hence the feasibility condition for equation (F1) is satisfied and we can write its general solution in the form

$$V_1 = \tilde{V} C_1 - A_0^\dagger A_1 \tilde{V} C_0 \tag{12}$$

where $C_1 \in R^{m \times m}$ is some matrix.

Define

$$D_k = \sum_{p=1}^{k+1} (-1)^{p-1} \sum_{\nu_1 + \cdots + \nu_p = k+1} A_{\nu_1} A_0^\dagger A_{\nu_2} \cdots A_0^\dagger A_{\nu_p}.$$

Note that the above formula can be rewritten in the recursive form

$$D_k = A_{k+1} - \sum_{i=1}^{k} A_i A_0^\dagger D_{k-i}. \tag{13}$$

Next we prove by induction that

$$\tilde{U}^T D_k \tilde{V} = 0, \tag{14}$$

and that

$$V_{k+1} = \tilde{V} C_{k+1} - A_0^\dagger \sum_{i=0}^{k} D_i V C_{k-i}, \tag{15}$$

where $C_i, i = 0, \ldots, k$ are some coefficient matrices. We recall that (14) is condition (9). We assume that relation (14) and formula (15) hold for $k = 0, \ldots, l$ and then we show that they also hold for $k = l + 1$. Note that we have already proven the induction base.

According to Lemma 1, the following feasibility condition for the $(l + 2)$-th fundamental equation is satisfied

$$\tilde{U}^T (A_1 V_{l+1} + A_2 V_l + \cdots + A_{l+2} V_0) = 0.$$

Substituting formula (15) for each $V_{k+1}, k = 0, \ldots, l$ and rearranging the terms, we get

$$\tilde{U}^T A_1 \tilde{V} C_{l+1} + \tilde{U}^T (A_2 - A_1 A_0^\dagger D_0) \tilde{V} C_l + \ldots + \tilde{U}^T (A_{l+2} - \sum_{i=1}^{l+1} A_i A_0^\dagger D_{l+1-i}) \tilde{V} C_0 = 0.$$

By induction hypothesis all terms of the above equation vanish except for the last one. Hence, we have

$$\tilde{U}^T (A_{l+2} - \sum_{i=1}^{l+1} A_i A_0^\dagger D_{l+1-i}) \tilde{V} C_0 = 0.$$

Using the recursive formula (13) and the fact that C_0 is a full rank matrix, we conclude that $\tilde{U}^T D_{l+1} V = 0$.

Next we show that formula (15) also holds for $k = l+1$. The general solution for the $(l+2)$-th fundamental equation is given by

$$V_{l+2} = \tilde{V} C_{l+2} - A_0^\dagger (A_1 V_{l+1} + \cdots + A_{l+2} V_0),$$

where C_{l+2} is some coefficient matrix. Substituting (15) for $V_{k+1}, k = 0, \ldots, l$ into the above equation and rearranging terms yield the formula (15) for $k = l+1$. Thus, by induction, relation (14) and formula (15) hold for any integer k.

□

The next theorem provides a recursive formula for the computation of the coefficients $V_k, k = 0, 1, \ldots$.

Theorem 1 *Let the matrix $A(\varepsilon)$ be a rank-preserving perturbation of A_0. Then there exists a holomorphic family of eigenvectors $V(\varepsilon)$ corresponding to the zero eigenvalue and satisfying the normalization condition (7). Moreover, the coefficients of the power series for $V(\varepsilon)$ can be calculated recursively by the formula*

$$V_k = -A_0^\dagger \sum_{j=1}^{k} A_j V_{k-j} \ , \quad k = 1, 2, \ldots, \tag{16}$$

where $V_0 = \tilde{V}$.

PROOF: It follows from the proof of Lemma 2 that the general solution of the fundamental equations is

$$V_k = V C_k - A_0^\dagger \sum_{j=1}^{k} A_j V_{k-j} \ , \quad k = 1, 2, \ldots$$

with $V_0 = \tilde{V} C_0$. By choosing $C_0 = I_m$, we get $V_0 = \tilde{V}$, which satisfies the normalization condition (N0).

Now the coefficients $C_k, k = 1, 2, \ldots$, are uniquely determined by the normalization conditions (Nk). Namely, we have

$$V_0^T V_k = V_0^T (\tilde{V} C_k - A_0^\dagger \sum_{j=1}^{k} A_j V_{k-j}) = 0,$$

or, equivalently,

$$\tilde{V}^T \tilde{V} C_k - \tilde{V}^T A_0^\dagger \sum_{j=1}^{k} A_j V_{k-j} = 0,$$

since $V_0 = \tilde{V}$ when $C_0 = I_m$. Recalling that $\tilde{V}^T \tilde{V} = I_m$ and $\tilde{V}^T A_0^\dagger = 0$, since $N(A) = R(A^\dagger)^\perp$ [6], we obtain that $C_k = 0, k = 1, 2, \ldots$ as required.

□

Next, we would like to address the issue of radius of convergence. First, above we have implicitly assumed that the series $\{A_i\}_{i=0}^{\infty}$ has a positive radius of convergence. This is, for example, the case when $\| A_i \| \leq a r^i$ for some constants a and r and where $\| \ \|$ is some matrix norm. In this case the radius of convergence is (at least) r^{-1}. The next theorem (which follows a technique used in [16]) gives a bound on the radius of convergence of the series given in (6) with coefficients as in (16).

Theorem 2 *Suppose $||A_i|| \leq a r^i$ for some positive constants a and r, then the radius of convergence of the series $V(\varepsilon) = V_0 + \varepsilon V_1 + \cdots$, where V_k is computed by (16), is at least $(1 + a||A_0^\dagger||)^{-1} r^{-1}$.*

PROOF: First, we prove by induction the following inequality

$$||V_k|| \leq ||V_0||(1 + a||A_0^\dagger||)^k r^k, \tag{17}$$

which trivially holds when $k = 0$. Now suppose that inequality (17) holds for the coefficients $V_0, .., V_{k-1}$. From (16), we obtain

$$||V_k|| \leq ||A_0^\dagger|| \sum_{j=1}^{k} ||A_j|| ||V_{k-j}|| \leq a||A_0^\dagger|| \sum_{j=1}^{k} r^j ||V_{k-j}||.$$

Now using inequality (17) for $j = 0, 1, \ldots, k-1$, (which is the inductive hypothesis) we get

$$\begin{aligned} ||V_k|| &\leq a||A_0^\dagger|| \sum_{j=1}^{k} r^j ||V_0|| (1 + a||A_0^\dagger||)^{k-j} r^{k-j} \\ &\leq a||A_0^\dagger|| ||V_0|| r^k \sum_{j=1}^{k} (1 + a||A_0^\dagger||)^{k-j}. \end{aligned}$$

Note that

$$\sum_{j=1}^{k} (1 + a||A_0^\dagger||)^{k-j} = \frac{(1 + a||A_0^\dagger||)^k - 1}{1 + a||A_0^\dagger|| - 1} = \frac{(1 + a||A_0^\dagger||)^k - 1}{a||A_0^\dagger||}.$$

Thus,

$$\begin{aligned} ||V_k|| &\leq ||V_0|| r^k [(1 + a||A_0^\dagger||)^k - 1] \\ &\leq ||V_0|| r^k (1 + a||A_0^\dagger||)^k \end{aligned}$$

as required. Consequently, the radius of convergence for the power series $V(\varepsilon) = V_0 + \varepsilon V_1 + \cdots$ is at least $(1 + a||A_0^\dagger||)^{-1} r^{-1}$.

□

Example 1 (cntd) First we check that conditions (9) indeed hold for Example 1. For $k = 0$, we have

$$\tilde{U}^T A_1 \tilde{V} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0.$$

Since $A_k = 0, k \geq 2$, matrices $D_k, k = 1, 2, \ldots$ satisfy the following recursive relationship

$$D_k = -A_1 A_0^\dagger D_{k-1},$$

with $D_0 = A_1$. Next, we calculate

$$A_1 A_0^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, $D_k = 0, k = 1, 2, \ldots$, and hence, conditions (9) are indeed satisfied. As the perturbation is rank-preserving, one can take $V_0 = \tilde{V}$. Using the recursive formula (16), we compute the terms $V_k, k = 1, 2, \ldots$ by

$$V_k = -A_0^\dagger A_1 V_{k-1} = -\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} V_{k-1}$$

This results in

$$V_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad \text{and} \quad V_k = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad k = 2, 3, \ldots$$

3 Non-Rank-Preserving Perturbations

In this subsection we deal with the case of non-rank-preserving perturbations, namely when the dimension of the perturbed null space $N(A(\varepsilon)), 0 < \varepsilon \leq \varepsilon\max$ is strictly less than the dimension of $N(A_0)$. Next we propose two algorithms. The first is based on generalized inverses applied to to-be-defined augmented matrices, whereas the second is based on a reduction technique. Both methods have their own merits. Finally, we also suggest a way to combine these two approaches. Our analysis generalizes this of [13] which deals with linear perturbation of stochastic matrices and applies the reduction step only once.

3.1 An approach based on augmented matrices

Some definitions are required prior to the introduction of our analysis for the case of non-rank-preserving perturbations, that is, the case where $\bar{m} < m$. First, for any integer t, $t \geq 0$, we define the augmented matrix $\mathcal{A}^{(t)} \in R^{n(t+1)\times n(t+1)}$ [3, 12, 13, 16]:

$$\mathcal{A}^{(t)} = \begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 \\ A_1 & A_0 & 0 & \cdots & 0 \\ A_2 & A_1 & A_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_t & A_{t-1} & \cdots & A_1 & A_0 \end{bmatrix}$$

Second, we partition the generalized inverse $\mathcal{G}^{(t)} := [\mathcal{A}^{(t)}]^\dagger$ into a block structure which corresponds to the structure of the augmented matrix $\mathcal{A}^{(t)}$. Namely,

$$\mathcal{G}^{(t)} = \begin{bmatrix} G_{00}^{(t)} & \cdots & G_{0t}^{(t)} \\ \vdots & \ddots & \vdots \\ G_{t0}^{(t)} & \cdots & G_{tt}^{(t)} \end{bmatrix},$$

where $G_{ij}^{(t)} \in R^{n\times n}$ for $0 \leq i, j \leq t$.

Third, let $M_t \subseteq R^n$ be the linear subspace of vectors w such that for some vector $v \in N(\mathcal{A}^{(t)}) \subseteq R^{n(t+1)}$, the first n entries in v coincide with w. Since $\bar{v} \in N(\mathcal{A}^{(t+1)})$ implies that the first $n(t+1)$ entries of $\bar{v}$ form a vector $v \in N(\mathcal{A}^{(t)})$, $M_{t+1} \subseteq M_t$ for any $t \geq 0$ and hence $dim(M_t)$ is non-increasing with t. Finally, let $\tau = \arg\min_t\{dim(M_t)\}$. In other words, τ is the smallest value of t where the minimum of $dim(M_t)$ is attained.

Theorem 3 *For any $V_0 \in M_\tau$, there exists a sequence $\{V_i\}_{i=0}^\infty$ which solves*

$$\mathcal{A}^{(t)} \begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ V_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{18}$$

for any $t \geq 0$. In particular, $\bar{m} = dim(M_\tau)$.

PROOF: A necessary (but not sufficient) condition for V_0 to be a leading term in such a series is that $A_0V_0 = 0$, i.e., that $V_0 \in M_0$. But what is further required is that for this V_0, there exists a V_1 such that $A_0V_1 + A_1V_0 = 0$, i.e., that $V_0 \in M_1$. Conversely, any $V_0 \in M_1$ (coupled with an appropriate V_1), solves (18) for $t = 1$. Similarly, one can see that $V_0 \in M_2$ (coupled with the corresponding V_1 and V_2 which exist by the definition of M_2) if and only if (18) holds for $t = 2$. By induction, we conclude that V_0 leads to a solution for (18) for any $t \geq 0$, if and only if $V_0 \in M_t$

for any $t \geq 0$, i.e., if and only if $V_0 \in M_\tau$. The equality $\bar{m} = dim(M_\tau)$ follows from the fact, that for each $V_0 \in M_\tau$ one can construct an analyticly perturbed eigenvector $V(\varepsilon) = V_0 + \varepsilon V_1 + \cdots$. Thus, the dimension of M_τ coincides with the dimension of the perturbed null space.

□

The equations (F0)–(Fτ) are

$$\mathcal{A}^{(\tau)} \begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ V_\tau \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{19}$$

Above we argued that any vector in M_τ will lead to a solution of (19). Imposing the normalization condition (N0) is now equivalent to requiring that V_0 be an orthonormal basis. Finally, any such orthonormal basis will be appropriate for our purposes.

Once V_0 is determined, the next goal is the determination of the corresponding V_1. Using (F1)–(F$(\tau+1)$) we have,

$$\mathcal{A}^{(\tau)} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{\tau+1} \end{bmatrix} = \begin{bmatrix} -A_1 V_0 \\ -A_2 V_0 \\ \vdots \\ -A_{\tau+1} V_0 \end{bmatrix} \tag{20}$$

which is as (18) with $t = \tau$ but with a different righthand side. Note that by definition of τ and by the fact that $V_0 \in M_\tau$, that (20) is solvable. Hence, by Lemma 1,

$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{\tau+1} \end{bmatrix} = [\mathcal{A}^{(\tau)}]^\dagger \begin{bmatrix} -A_1 V_0 \\ -A_2 V_0 \\ \vdots \\ -A_{\tau+1} V_0 \end{bmatrix} + y$$

for *some* $y \in Null(\mathcal{A}^{(\tau)})$. However, note that not any $y \in Null(\mathcal{A}^{(\tau)})$ will do since in (20) we have not considered all fundamental equations. However, for any $w \in M_\tau$ there exists such a y with w being its first n entries. Moreover, any such w leads to a vector V_1 such that coupled with V_0, they are the leading two terms in a series expansion for $V(\varepsilon)$. The reason is that whatever was true for V_0 is now true for V_1 since in the latter case one gets the same set of equations but with a different righthand side. The normalization condition (N1), coupled with the fact that V_0 is chosen, imply a unique value for the matrix V_1.

Above we have shown how the value of V_0 leads to the value of V_1. Next, we show that this is the case in general. Specifically, once $V_0, \cdots, V_k$ are determined, one can compute V_{k+1} by the recursive formula provided in the next theorem.

Theorem 4 *The solution of the system of fundamental equations (F) coupled with the normalization conditions (N) is given by the following recursive formula*

$$V_{k+1} = -(I_n - V_0 V_0^T) \sum_{j=0}^{\tau} G_{0j}^{(\tau)} \sum_{i=1}^{k+1} A_{i+j} V_{k+1-i}, \tag{21}$$

where V_0 is any orthogonal basis of the linear subspace M_τ.

PROOF: Consider the set of fundamental equations (F.k+1)-(F.k+1+τ). Since they are feasible, the general solution is of the form

$$\begin{bmatrix} V_{k+1} \\ \vdots \\ V_{k+1+\tau} \end{bmatrix} = \mathcal{G}^{(\tau)} \begin{bmatrix} -\sum_{i=1}^{k+1} A_i V_{k+1-i} \\ \vdots \\ -\sum_{i=1}^{k+1} A_{i+\tau} V_{k+1-i} \end{bmatrix} + y$$

where $y \in Null(\mathcal{A}^{(\tau)})$. Since the first n entries of y constitute a vector w in M_τ and V_0 is an orthogonal basis of M_τ, the general solution for V_{k+1} can be written as follows

$$V_{k+1} = -\sum_{j=0}^{\tau} G_{0j}^{(\tau)} \sum_{i=1}^{k+\tau} A_{i+\tau} V_{k+1-i} + V_0 C_{k+1}, \tag{22}$$

where C_{k+1} is some matrix coefficient that can be determined from the normalization condition (N.k+1). Specifically,

$$-V_0^T \sum_{j=0}^{\tau} G_{0j}^{(\tau)} \sum_{i=1}^{k+\tau} A_{i+\tau} V_{k+1-i} + C_{k+1} = 0,$$

and hence

$$C_{k+1} = V_0^T \sum_{j=0}^{\tau} G_{0j}^{(\tau)} \sum_{i=1}^{k+\tau} A_{i+\tau} V_{k+1-i}.$$

Substituting the above expression for the coefficient C_{k+1} into the formula (22) results in the recursive formula (21). This completes the proof.

□

Remark 1 *We would like to point out that although above we call for $[A^{(\tau)}]^\dagger$, only its first $\bar{m}$ rows are required in order to carry out the desired computations.*

Example 2 (cntd) It is easy to check that in this example the subspace M_1 is one dimensional and is spanned by the vector $[c\ 0\ c]^T$, where $c \in 0$ is an arbitrary constant. Hence, $\tau = 1$ and the first term of power series (6) is given by

$$V_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Then, to compute the terms $V_k, k = 1, 2, \ldots$, we use the recursive formula (21) which has the following form for this particular example

$$V_{k+1} = -(I - V_0 V_0^T) G_{00} A_1 V_k, \quad k = 0, 1, \ldots$$

Also,

$$I - V_0 V_0^T = \begin{bmatrix} 0.5 & 0 & -0.5 \\ 0 & 1 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}, \quad G_{00} A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0 & 0 \end{bmatrix}.$$

Consequently,

$$V_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad V_k = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ k \geq 2.$$

Note that in both Examples 1 and 2, we obtained finite expansions for $V(\varepsilon)$ instead of infinite series. Of course, this is due to the simplicity of the examples. However, if one calculates "orthonormal" bases instead of "quasi-orthonormal" bases, one will have to deal with *infinite* series even in the case of these simple examples. This fact demonstrates an advantage of using "quasi-orthonormal" bases instead of "orthonormal".

3.2 An algorithm based on a reduction process

Next, we show that by using a reduction process one can transform the system of fundamental equations (F) to another system with coefficient matrices of reduced dimensions. The latter system can be solved by the algorithm proposed in Section 2 for the regular case. Thus, we reduce the non-rank-preserving problem to the regular one. The next theorem is a key to the reduction process.

Theorem 5 *A solution of the fundamental equations (F) together with the normalization conditions (N) is given by the following recursive formula*

$$V_k = \tilde{V} W_k - A_0^{\dagger} \sum_{j=1}^{k} A_j V_{k-j}, \quad k = 1, 2, \ldots \tag{23}$$

with $V_0 = \tilde{V} W_0$, and where the sequence of auxiliary matrices W_k, $k \geq 0$, is a solution to the next system of reduced fundamental equations (RF):

$$\begin{array}{rcll} B_0 W_0 & = & 0 & \qquad (RF0) \\ B_0 W_1 + B_1 W_0 & = & 0 & \qquad (RF1) \\ B_0 W_2 + B_1 W_1 + B_2 W_0 & = & 0 & \qquad (RF2) \\ & \vdots & & \qquad \vdots \\ B_0 W_k + B_1 W_{k-1} + \cdots + B_{k-1} W_1 + B_k W_0 & = & 0 & \qquad (RFk) \\ & \vdots & & \qquad \vdots \end{array}$$

and reduced normalization conditions (RN):

$$\begin{array}{rcll} W_0^T W_0 & = & I_{\bar{m}} & \qquad (RN0) \\ W_0^T W_k & = & 0 \quad k = 1, 2, \ldots . & \qquad (RNk) \end{array}$$

with the coefficient matrices B_k, $k \geq 0$, are given by the formula

$$B_k = \tilde{U}^T \left(\sum_{p=1}^{k+1} (-1)^{p-1} \sum_{\nu_1 + \cdots + \nu_p = k+1, \nu_i \geq 1} A_{\nu_1} A_0^{\dagger} A_{\nu_2} A_0^{\dagger} \cdots A_0^{\dagger} A_{\nu_p} \right) \tilde{V} \quad . \tag{24}$$

Proof: From the fundamental equation (F0) we conclude that V_0 belongs to the null space of A_0, that is

$$V_0 = \tilde{V} W_0, \tag{25}$$

where $W_0 \in R^{m \times m_1}$ is some coefficient matrix, and where m_1 is a number to be determined with $\bar{m} \leq m_1 \leq m$. By Lemma 1 the equation (F1) is feasible if and only if

$$\tilde{U}^T A_1 V_0 = 0 \ .$$

Substituting the expression (25) for V_0, one gets that

$$\tilde{U}^T A_1 V W_0 = 0 \ .$$

This is the first equation of the reduced system (RF) with $B_0 = \tilde{U}^T A_1 \tilde{V}$. Note that m_1 above is the dimension of the null space of B_0. Next we consider the fundamental equations (F1). Their solution in the general form is

$$V_1 = \tilde{V} W_1 - A_0^{\dagger} A_1 V_0, \tag{26}$$

where $W_1 \in R^{m \times m_1}$ is some coefficient matrix, which describes the general solution of the corresponding homogeneous system and where $-A_0^{\dagger} A_1 V_0$ is a particular solution of (F1). The coefficient matrices W_0 and W_1 have to be chosen so that they satisfy the feasibility condition for the next fundamental equation (F2)

$$\tilde{U}^T(A_1V_1 + A_2V_0) = 0.$$

Upon substitution of V_0 (see (25)) and V_1 (see (26)) into the above condition, one obtains

$$\tilde{U}^T A_1 \tilde{V} W_1 + \tilde{U}^T(A_2 - A_1 A_0^\dagger A_1)\tilde{V} W_0 = 0$$

which is the reduced fundamental equation (RF1) with $B_1 = \tilde{U}^T(A_2 - A_1 A_0^\dagger A_1)\tilde{V}$.

Note that the recursive formula (23) is just the general form of the solution of the fundamental equation (Fk). The reduced system of equations (RF) is the set of feasibility conditions for $W_k, k = 0, 1, \ldots$, which are obtained in a way similar to the above considerations. The general formula (24) for the coefficients can now be established by induction similar to the proof of Lemma 2.

Next, we show that the new normalization conditions (RN) also hold. First, consider the normalization condition for W_0. Substituting the expression (25) into (N0), we get

$$(\tilde{V} W_0)^T \tilde{V} W_0 = I_{\bar{m}}$$

or

$$W_0^T \tilde{V}^T \tilde{V} W_0 = I_{\bar{m}} \quad .$$

Recall that we have chosen the basis $\tilde{V}$ for the null space of A_0 such that $\tilde{V}^T \tilde{V} = I_m$. The latter implies that

$$W_0^T W_0 = I_{\bar{m}}.$$

Thus, we have obtained the normalization condition (RN0). Next we show that the normalization conditions (RNk), $k \geq 1$, hold as well. To this end, substitute the recursive expression (23) into (Nk) to obtain

$$V_0^T \tilde{V} W_k - V_0^T A_0^\dagger (\sum_{j=1}^{k} A_j V_{k-j}) = 0.$$

Note that since V_0 belongs to the null space of A_0 and since $N(A) = R(A^\dagger)^\perp$ [6], $V_0^T A_0^\dagger = 0$. Thus,

$$V_0^T \tilde{V} W_k = 0.$$

By substituting V_0 from (25) and taking into account that $\tilde{V}^T \tilde{V} = I_m$, we get that

$$W_0^T \tilde{V}^T \tilde{V} W_k = W_0^T W_k = 0,$$

which is the normalization condition (RNk). This completes the proof.

□

Remark 2 *Note that the computation of the coefficient matrices B_k, $k \geq 0$, by (24) will be tedious. Therefore, similarly to [1] we propose to compute these coefficients recursively. Specifically, define the sequence of matrices $\{D_k\}_{k=0}^\infty$ as follows:*

$$D_k = \sum_{p=1}^{k+1} (-1)^{p-1} \sum_{\nu_1 + \cdots + \nu_p = k+1} A_{\nu_1} A_0^\dagger A_{\nu_2} A_0^\dagger \cdots A_{\nu_p}, \quad k = 0, 1, \ldots$$

These auxiliary matrices can be computed by the following recursion

$$D_k = A_{k+1} - \sum_{i=1}^{k} A_i A_0^\dagger D_{k-i}, \quad k = 1, 2, \ldots \tag{27}$$

initialising with $D_0 = A_1$. Then the coefficient matrices B_k, $k \geq 0$, are simply given by

$$B_k = U^T D_k V.$$

We would like to point out that the reduced system of equations (RF) together with the normalization condition (RN) have exactly the same structure as the initial system of fundamental equations (F) with the normalization conditions (N). Thus, one has two options as how to proceed from here. The first is to solve it using the procedure described in Subsection 3.1, that is, to use the augmented matrix approach. The second is to apply one more reduction step, this time the system composed of (RF) and (RN). If the latter option is pursued, then once again one may face the same alternative and so on. At first sight, it might seem that one may end up carrying out an infinite number of reduction steps. However, as it turns out, termination is guaranteed after a finite number of steps. The next theorem addresses this issue.

Theorem 6 *Suppose that $\{B_k^{(l)}\}_{k=0}^{\infty}, l = 1, 2, \ldots$ are the coefficients of the reduced system obtained at the l-th reduction step ($B_k^{(1)} = B_k$). Also, let m_l be the dimension of the null space of $B_0^{(l)}$. Then, the reduction process terminates after a finite number of steps with $m_l = \bar{m}$, where $\bar{m}$ is the dimension of the null space of the perturbed matrices $A(\varepsilon)$, $0 < |\varepsilon| < \varepsilon_{\max}$. Furthermore, the final system of reduced equations (namely, the system of reduced fundamental equations derived at the last reduction step) can be solved by the recursive procedure which was proposed for the case of rank-preserving perturbations described in Section 2 (see formula (16)).*

Proof: Note that after each reduction step the dimension of the null space of $B_0^{(l)}$ does not increase. Since we deal with a finite-dimensional problem and since the sequence m_l, $l \geq 1$, is of integers, we conclude that the sequence of m_l achieves its limit, say m_*, in a finite number of steps. Next we argue that this limit m_* equals $\bar{m}$ and once it is reached there is no need to make any further reduction steps. Note also that the solution to the final system of reduced equations (the reduction process terminates when the null space of $B_0^{(l)}$ has dimension m_*) can be obtained by the recursive algorithm proposed in Section 2.

The latter means that a basis for the null space of the perturbed matrix $A(\varepsilon)$ is constructed and this basis is holomorphic with the parameter ε. This basis is formed by m_* linearly independent vectors. However, according to our assumptions the dimension of the null space of $A(\varepsilon)$ is $\bar{m}$. This implies that the limit m_* equals $\bar{m}$.

□

Finally, we would like to suggest a practical implementation of the above scheme. If one applies the reduction process as described above to calculate V_k, then one needs to compute $B_i, i = 0, ..., k + \bar{m}$, $B_i^{(2)}, i = 0, ..., k + \bar{m} - 1$ and so on. This could result in a large amount of calculations even when the recursive formula given in Remark 2 is used. Fortunately, it is possible to develop a recursion which uses only a finite number of precomputed $B_i^{(l)}$. Kostya: This 'finite' statement is not clear to me. Namely, suppose that we have already obtained $V_0, \ldots, V_k, k > r$, where r denotes the number of reduction steps needed to obtain the final system of reduced equations. Then we can rewrite the fundamental system Fk,Fk+1,..., as follows:

$$
\begin{aligned}
A_0 V_{k+1} &= -(A_1 V_k + \cdots + A_{k+1} V_0) \\
A_0 V_{k+2} + A_1 V_{k+1} &= -(A_2 V_k + \cdots + A_{k+2} V_0) \\
&\vdots \\
A_0 V_{k+r+1} + \cdots + A_r V_{k+1} &= -(A_{r+1} V_k + \cdots + A_{k+r+1} V_0)
\end{aligned}
$$

This system of equations can be effectively solved by the same reduction technique. Moreover, note that the auxiliary matrices such as $B_i^{(l)}$ can be stored and used afterwards to compute the next terms $V_{k+2}, V_{k+3}, \cdots$

Remark 3 *It is worth pointing out that in most applications the number of reduction steps r is small and hence we need to compute and to store only a small number of coefficients $B_i^{(l)}$.*

Remark 4 *If it is needed, the estimation of the convergence radius can be also obtained for the non-rank-preserving case. This can be done by applying recursively the arguments of Theorem 2.*

4 Perturbation of general eigenvalue problem

The results on the perturbation of null spaces can be immediately applied to the general perturbed eigenvalue problem [5, 7, 8, 17, 19, 20, 21, 22, 23]

$$A(\varepsilon)x(\varepsilon) = \lambda(\varepsilon)x(\varepsilon). \tag{28}$$

Recall that the perturbed eigenvalue $\lambda(\varepsilon)$ satisfies the secular equation

$$det(A(\varepsilon) - \lambda(\varepsilon)I) = 0.$$

This is a polynomial equation

$$(-1)^n\lambda^n + a_{n-1}(\varepsilon)\lambda^{n-1} + \cdots + a_1(\varepsilon)\lambda + a_0(\varepsilon) = 0,$$

where the coefficients $a_i(\varepsilon)$ are analytic functions. Using the method of the Newton diagram [5, 20, 22], one can obtain the Puiseux expansion of the perturbed eigenvalue

$$\lambda(\varepsilon) = \lambda_0 + \varepsilon^{1/p}\lambda_1 + \varepsilon^{2/p}\lambda_2 + \cdots$$

where p is a positive integer. Now let us introduce the new variable $\eta := \varepsilon^{1/p}$. Note that the perturbed eigenvalue depends analytically on this new variable. Consequently, the system of equations for the perturbed eigenvectors can be written in the form

$$[A(\eta^p) - \lambda(\eta)I]x(\eta) = 0.$$

Hence, we have reduced the general perturbed eigenvalue problem to the problem of analytic perturbation of the null space, which can be effectively solved by the methods described in Sections 2 and 3.

5 Perturbations of group inverse

Next we discuss the applications of the above analysis for the designing of several methods for calculating the Laurent series expansion (8) of the perturbed group inverse. We assume that the perturbed group inverse $A^g(\varepsilon)$ exists for $\varepsilon \in (0, \bar{\varepsilon})$. In the previous subsections we have developed the power series expansion for the eigenvectors corresponding to the zero eigenvalue of $A(\varepsilon)$. The matrix $V(\varepsilon)$ forms a basis for the null space of $A(\varepsilon)$, though it is not necessarily orthogonal. To proceed further we need to construct the orthogonal projection $P(\varepsilon)$ on the null space of $A(\varepsilon)$. There are (at least) two possible directions to pursue here. The first is based on the direct application of the formula

$$P(\varepsilon) = V(\varepsilon)(V^T(\varepsilon)V(\varepsilon))^{-1}V^T(\varepsilon), \quad 0 < |\varepsilon| < \varepsilon_{\max}. \tag{29}$$

Since $\lim_{\varepsilon\to 0} V^T(\varepsilon)V(\varepsilon) = V_0^T V_0 = I_{\bar{m}}$, the matrix $(V^T(\varepsilon)V(\varepsilon))^{-1}$ is the "well-behaved" inverse of the perturbed identity matrix $I_{\bar{m}}$. In this case we can apply directly the Neumann series [17]. The expansion is of the form

$$(V^T(\varepsilon)V(\varepsilon))^{-1} = I - \varepsilon(V_0^T V_1 + V_1^T V_0) + \cdots$$

Multiplying the above expression by $V(\varepsilon)$ from the left and by $V^T(\varepsilon)$ from the right, we obtain the power series for the orthogonal projection $P(\varepsilon)$.

The second method is based on a Gram-Schmidt-like orthogonalization process. Firstly, we perform the Gram-Schmidt procedure (without normalization) over the analytic vector-valued functions $v_i(\varepsilon), i = 1, ..., \bar{m}$, which constitute the "quasi-orthogonal" basis $V(\varepsilon)$. Note that summation, multiplication and division operations which are used in the orthogonalization procedure can be carried out on power series (and not only on real numbers). This results in an orthogonal basis for the perturbed null space. Each new basis element is a vector-valued function analytic in the punctured disc: $0 < |\varepsilon| < \varepsilon_{\max}$. Next we show that the normalization procedure leads to a basis whose elements are analytic vector-valued functions at $\varepsilon = 0$. Indeed, consider a vector-valued function $a(\varepsilon)$ which is analytic in $0 < |\varepsilon| < \varepsilon_{\max}$. It can be expanded as a Laurent series. And let $a_i(\varepsilon) = \varepsilon^m a_{im} + \varepsilon^{m+1} a_{im+1} + \cdots$ be the largest element (in absolute value and for sufficiently small ε) of the vector $a(\varepsilon)$. Then, clearly

$$||a(\varepsilon)|| = \sqrt{a_1^2(\varepsilon) + \cdots + a_2^2(\varepsilon)} = \varepsilon^m(\nu_0 + \varepsilon\nu_1 + \cdots), \quad \nu_0 \neq 0.$$

The latter implies that the normalized vector $a(\varepsilon)/||a(\varepsilon)||$ can be expanded as a series with positive powers of ε and with a non-zero leading coefficient. As a result of the above procedure we obtain an orthonormal basis, say $\bar{V}(\varepsilon)$. Then the orthogonal projection is given by

$$P(\varepsilon) = \bar{V}^T(\varepsilon)\bar{V}(\varepsilon).$$

Next we suggest some methods for finding the Laurent series expansion for the group inverse $A^g(\varepsilon)$ which are based on results given [3]. In particular, we assume that the Laurent series expansion for the inverse of a singular perturbed martix is available, for example by any method suggested in [3]. Moreover, the power series for $P(\varepsilon)$ is assumed to have been developed already.

The first method is based on a straightforward application of the inversion of singular perturbed matrices. It utilizes the formula [9, 18]

$$A^g(\varepsilon) = [A(\varepsilon) + P(\varepsilon)]^{-1} - P(\varepsilon),$$

where the projection $P(\varepsilon)$ is given by (29).

The second method is based on the solution of the following system of equations [17]

$$A(\varepsilon)A^g(\varepsilon) = I - P(\varepsilon), \tag{30}$$

$$P(\varepsilon)A^g(\varepsilon) = 0, \tag{31}$$

where $P(\varepsilon)$ is the same projection as before. Note that equation (31) can be written in the equivalent form,

$$V^T(\varepsilon)A^g(\varepsilon) = 0, \tag{32}$$

which might be more useful for practical computations.

Finally, the third method is based on the construction of the matrix $C(\varepsilon) = [\tilde{A}(\varepsilon), V(\varepsilon)]$, where the matrix $\tilde{A}(\varepsilon) \in R^{n\times(n-m)}$ consists of columns of $A(\varepsilon)$ which span its range. The matrix $C(\varepsilon)$ is invertible for ε sufficiently small but distinct from zero, and the perturbed group inverse $A^g(\varepsilon)$ can be immediately retrieved by [10]

$$A^g(\varepsilon) = [I - P(\varepsilon)]C^{-1}(\varepsilon).$$

We would like to discuss in more detail the second method based on the equations (30) and (31). This approach is a generalization of the results given in [1, 13, 14]. By substituting the power series (1),(8) and the power series for the projection $P(\varepsilon)$ into equations (30), (31) and equating terms with the same power of ε, we obtain two new systems of fundamental equations

$$\begin{array}{rcll} A_0 B_{-s} &=& 0 & (G0)\\ A_0 B_{-s+1} + A_1 B_{-s} &=& 0 & (G1)\\ &\vdots& & \vdots\\ A_0 B_0 + A_1 B_{-1} + \cdots + A_{s-1} B_{-s+1} + A_s B_{-s} &=& I - P_0 & (Gs)\\ A_0 B_1 + A_1 B_0 + \cdots + A_s B_{-s+1} + A_{s+1} B_{-s} &=& -P_1 & (Gs+1)\\ &\vdots& & \vdots \end{array}$$

and

$$\begin{array}{rcll} P_0 B_{-s} &=& 0 & (H0)\\ P_0 B_{-s+1} + P_1 B_{-s} &=& 0 & (H1)\\ &\vdots& & \vdots \end{array}$$

The next result provides a way to solve recursively the above infinite systems of equations. It generalizes a corresponding result for stochastic matrices and linear perturbation which appears in [14].

Theorem 7 *Equations* $(G0)-(Gs)$ *together with the equation* $(H0)$ *constitute the minimal set of fundamental equations that uniquely determine the first singular coefficient* B_{-s} *of the Laurent series* (8).

PROOF: It is easy to see that one needs to take into account at least the first $s+1$ fundamental equations from the set (G) and the first equation from the set (H0).

Now we prove that this set of equations is sufficient to determine uniquely the first Laurent series coefficient B_{-s}. Suppose on the contrary that there exists another solution $C_{-s}, C_{-s+1}, ..., C_0 \in R^{n\times n}$ to (G0)-(Gs) and (H0) such that $C_{-s} \neq B_{-s}$. Consider the sequence $\{D_i\}_{i=-s}^{\infty}$ defined by $D_i = B_i - C_i$ for $-s \leq i \leq 0$ and $D_i = B_i$ for $i > 0$. Clearly, for sufficiently small ε we have by (30)

$$A(\varepsilon)[D(\varepsilon) + C(\varepsilon)] = I - P(\varepsilon)$$

and hence

$$A(\varepsilon)D(\varepsilon) = I - P(\varepsilon) - A(\varepsilon)C(\varepsilon).$$

Using fundamental equations (G0)-(Gs), we conclude that

$$A(\varepsilon)D(\varepsilon) = O(\varepsilon) \quad .$$

Now we multiply the above equation by $A^g(\varepsilon)$ from the left and get that

$$[I - P(\varepsilon)]D(\varepsilon) = O(\frac{1}{\varepsilon^{s-1}})$$

Since $P_0B_0 - P_0C_0 = 0 - 0 = 0$ and consequently the lefthand side above is of order $O(1/\varepsilon^s)$, we have reached a contradiction.

□

The above theoretical result points to a numerical procedure. First, solve the set of equations (G0)-(Gs) and (H0) to find B_{-s}. Note that, since this is a linear system, a number of efficient methods are available for its solution. For example, one can use methods similar to those described in [3] and in Sections 2 and 3. Then, use equations (G1)-(Gs+1) and (H1) to find B_{-s+1} and so on. We would like to emphasize that the system of equations (G1)-(Gs+1) and (H1) has exactly the same lefthand sides as the system of equations (G0)-(Gs) and (H0). Thus, the auxiliary results which were obtained during the solution of the system (G0)-(Gs) and (H0) can be applied again. Moreover, one can use the quantities obtained from the reduction process of the fundamental system (F) in order to solve the system (G).

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