Analytic Perturbation of Generalized Inverses

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Abstract

We investigate the analytic perturbation of generalized inverses. Firstly we analyze the analytic perturbation of the Drazin generalized inverse (also known as reduced resolvent in operator theory). Our approach is based on spectral theory of linear operators as well as on a new notion of group reduced resolvent. It allows us to treat regular and singular perturbations in a unified framework. We provide an algorithm for computing the coefficients of the Laurent series of the perturbed Drazin generalized inverse. In particular, the regular part coefficients can be efficiently calculated by recursive formulae. Finally we apply the obtained results to the perturbation analysis of the Moore-Penrose generalized inverse in the real domain.

1 Introduction

We study generalized inverses of analytically perturbed matrices:

$$A(z) = A_0 + zA_1 + z^2 A_2 + \dots \, . \tag{1}$$

First, we provide the perturbation analysis for the *Drazin generalized inverse.* In such a case we assume that the matrices $A_k, k = 0, 1, \dots$, are square of dimension n with complex entries. Furthermore, since we are interested in the perturbation analysis of the generalized inverse, we assume that the null space of the perturbed matrix $A(z)$ is non-trivial.

Here we distinguish between regular and singular perturbations. The perturbation is said to be *regular* if it does not change the dimension of null space. Otherwise, the perturbation is said to be *singular*. The regular perturbation is also called *rank-preserving* perturbation. One of the main advantages of the complex analytic approach employed in the present work is that it allows us to treat both regular and singular perturbations in a unified framework.

If the coefficient matrices $A_k, k = 0, 1, \dots$, are real and we restrict ourselves to real z, the perturbation analysis of the Drazin generalized inverse can be applied to the perturbation analysis of the *Moore-Penrose generalized inverse.*

The main goals of the present work is to prove the existence of the Laurent series expansion for the perturbed Drazin generalized inverse

$$A^{\#}(z) = \sum_{j=-s}^{+\infty} z^j H_j, \tag{2}$$

and to provide a method for efficient computation of coefficients $H_j, j = -s, -s+1, \dots$.

We derive recursive formulae for the (matrix) coefficients of the regular part of the Laurent series expansion (2). Besides their theoretical interest, the recursive formulae are particularly

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useful when one needs to compute a significant number of terms in the Laurent series expansion. These formulae require knowledge of the singular part coefficients and the latter are obtained via a *reduction technique* based on the complex analytic approach. In particular, this reduction technique uses a new notion of *group reduced resolvent.* The order of the pole of (2) is obtained as a by-product.

Last but not least, the limit matrix in the Taylor series expansion of the 0-eigenprojection matrix, has a simple exression in terms of (a) the original unperturbed 0-eigenprojection, (b) the perturbation matrices and (c), the singular part of the Laurent series (2). This provides some insight on how the perturbed 0-eigenvectors relate to the original 0-eigenvectors.

Let us also mention an important issue. In the course of the procedure that we propose, one has to determine the multiplicity of zero eigenvalues and verify whether they are semi-simple. Of course, in the general situation neither task is easy in the presence of rounding errors. Nevertheless, we note that in many particular applications this issue can be effectively resolved. For instance, for the perturbation of the Moore-Penrose generalized inverse the semi-simplicity assumption is satisfied automatically, as we transform the problem into an equivalent problem with symmetric matrices.

To our best knowledge, Stewart [27] was the first to carry out the perturbation analysis of the Moore-Penrose generalized inverse. Even though he has mostly dealt with the regular perturbation case, he has pointed out that the error bounds grow infinitely as the perturbation vanishes in the singular case. He has provided error bounds and a first order approximation for the general componentwise perturbation of the Moore-Penrose generalized inverse. In fact, up to the present most of the work on perturbation analysis of generalized inverses is concentrated on regular componentwise perturbation. An interested reader can find results on regular componentwise perturbation of generalized inverses in [10, 11, 30] and references therein. The analytic perturbation is more restrictive than the general componentwise perturbation. However, it allows us to perform a deeper analysis of the singular case. The case of singular perturbations is interesting for it provides information on the lack of continuity of solutions to perturbed linear systems. The work on singular perturbation of the generalized inverse is very limited. Deif [13] proposed an algorithm for computing the Laurent series for the generalized inverse of an analytically perturbed matrix. However, to implement his algorithm one has to obtain power series expansions for *all* the quantities in the singular value decomposition, namely for all the singular eigenvectors and eigenvalues. This can be very costly, especially for matrices of high dimension. In contrast, we propose an algorithm, which works in some *reduced* subspaces of small dimension.

We note that if matrix valued function $A(z)$ is invertible in some punctured neibourhood around $z = 0$, then its generalized Drazin inverse is just the ordinary inverse. Therefore, our results are applicable to the classical problem of the inversion of nearly singular operators [2, 6, 15, 20, 21, 22, 24, 26, 28, 29], a problem with many important practical applications. In [3] the perturbation analysis of the group inverse has been provided. Since the group inverse is a particular case of the Drazin inverse when the space can be decomposed into a direct sum of the null space and the range space, the results of the present work can be applied to the perturbation analysis of the group inverses. However, since the Drazin inverse is a more general notion than the group inverse, the results of [3] cannot be applied to the perturbation analysis of the Drazin inverse.

The results of the present paper can be immediately applied to singular perturbations of Markov chains [5, 14, 17, 25]. For example, the deviation matrix of a Markov chain is just (with minus sign) the reduced resolvent of the Markov chain generator. To demonstrate this application and verify the validity of our theoretical results, a simple illustrative example is provided at the end of the paper. Another application is concerned with perturbations of spectral inverses arising in the chemical networks, as treated in the work of Bohl and

Lancaster [8]. Yet another interesting potential application of our results is the perturbation analysis of least square problems [13, 27].

Some preliminary results of the present work have appeared in the research report [4]. However, there are very important differences between the contents of the research report [4] and the present article. In the research report the proofs were carried out under the assumption that coefficient matrices of the analytic perturbation are real (even though the perturbation parameter z is complex). Here we have extended the results, whenever possible, to the case of complex coefficient matrices. Also the presentation of the proofs has been significantly improved. There is a new, very important, observation, that in general the Moore-Penrose generalized matrix of the complex perturbation is not analytic. This was absent in the research report.

The paper is organized as follows. In the next Section 2, we provide preliminaries on the complex analytic approach and spectral theory. In Section 3, we prove the existence of a Laurent series expansion for the Drazin generalized inverse of a perturbed matrix. In Section 4, we derive a recursive formula for the coefficients of the regular part of the Laurent series. For clarity of exposition, the proof of the main result is postponed to the Appendix. In Section 5, we propose a reduction process that permits to compute the coefficients of the singular part. In Section 6 we apply the obtained results to the perturbation analysis of the Moore-Penrose and group generalized inverses. Finally, in Section 7, we derive a simple expression for the limiting 0-eigenprojection matrix as the perturbation vanishes.

2 Preliminaries on complex analytic approach and spectral theory

Let us recall some facts from complex analysis and spectral theory. The book of Kato [19] is an excellent source of the material on the subject.

Any matrix $A \in \mathbb{C}^{n\times n}$ possesses the following spectral representation

$$A = \sum_{i=0}^{p} (\lambda_i P_i + D_i), \tag{3}$$

where $p+1$ is the number of distinct eigenvalues of A, P_i is the eigenprojection and D_i is the nilpotent operator corresponding to the eigenvalue λ_i. By convention, λ_0 is the zero eigenvalue of A, that is, $\lambda_0 = 0$. In the case when there is no zero eigenvalue, the eigenvalues are enumerated from $i = 1$. The *resolvent* is another very important object in spectral theory.

Definition 1 *The following operator-valued function of the complex parameter ζ is called the resolvent of the operator $A \in \mathbb{C}^{n\times n}$*

$$R(\zeta) = (A - \zeta I)^{-1}.$$

The resolvent satisfies the *resolvent identity*:

$$R(\zeta_1) - R(\zeta_2) = (\zeta_1 - \zeta_2) R(\zeta_1) R(\zeta_2), \tag{4}$$

for all $\zeta_1, \zeta_2 \in \mathbb{C}$. The resolvent has singularities at the points $\zeta = \lambda_k$, where λ_k are the eigenvalues of A. In a neighbourhood of each singular point λ_k the resolvent can be expanded as a Laurent series

$$R(\zeta) = -\sum_{n=1}^{m_k-1} \frac{1}{(\zeta-\lambda_k)^{n+1}} D_k^n - \frac{1}{\zeta-\lambda_k} P_k + \sum_{n=0}^{\infty} (\zeta-\lambda_k)^n S_k^{n+1}, \tag{5}$$

where S_k is the *reduced resolvent* corresponding to the eigenvalue λ_k with geometric multiplicity m_k. In fact, S_k is the Drazin generalized inverse of $(A - \lambda_k I)$. And, in particular, we have $S_0 = A^{\#}$.

The Drazin generalized inverse has the following basic properties

$$AA^{\#} = I - P_0, \tag{6}$$

$$P_0 A^{\#} = 0. \tag{7}$$

The above equations show that $A^{\#}$ is the "inverse" of A in the complementary subspace to the generalized null space of A. Here by generalized null space we mean a subspace which is spanned by all eigenvectors and generalized (Jordan) eigenvectors corresponding to the zero eigenvalue. Note that P_0 is a projection onto this generalized null space.

Moreover, if the underlying space admits a decomposition into the *direct* sum of the null space and the range of the operator A (recall from [9] that this is a necessary and sufficient condition for the existence of the group inverse), then the Drazin inverse and the group inverse coincide, and the following Laurent expansion holds

$$R(\zeta) = -\frac{1}{\zeta} P_0 + \sum_{n=0}^{\infty} \zeta^n (A^{\#})^{n+1}. \tag{8}$$

Since the Drazin generalized inverse is the constant term in the Laurent series (5) at $z = \lambda_0$, it can be calculated via the following Cauchy integral formula

$$A^{\#} = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{1}{\zeta} R(\zeta)\, d\zeta, \tag{9}$$

where Γ_0 is a closed positively-oriented contour in the complex plane, enclosing 0 but no other eigenvalue of A [1]. The above formula will play a crucial role in the sequel.

The Drazin inverse also has a simple expression in terms of eigenprojections, eigenvalues and nilpotent operators of the original operator A. Namely,

$$A^{\#} = \sum_{i=1}^{p} \left\{ \frac{1}{\lambda_i} P_i + \sum_{j=1}^{m_i - 1} (-1)^j \frac{1}{\lambda_i^{j+1}} D_i^j \right\}. \tag{10}$$

We emphasize that the above sum is taken over all indecies corresponding to non-zero eigenvalues. This expression again demonstrates that the Drazin generalized inverse is the inverse operator in the complementary subspace to the generalized null space. Moreover, this expression exactly represents the inverse operator A^{-1} whenever A has no zero eigenvalue.

3 Existence of the Laurent series expansion

In this section we prove the existence of a Laurent series expansion (2) for the Drazin generalized inverse $A^{\#}(z)$ of the analytically perturbed matrix $A(z)$.

First let us consider the resolvent $R(\zeta, z) := (A(z) - \zeta I)^{-1}$ of the perturbed $A(z)$. One can expand $R(\zeta, z)$ in a power series with respect to the complex variable z near $z = z_0$, as follows (see e.g. [19])

$$R(\zeta, z) \;=\; R(\zeta, z_0) + \sum_{n=1}^{\infty} (z - z_0)^n R^{(n)}(\zeta, z_0), \tag{11}$$

where

$$R^{(n)}(\zeta, z_0) := \sum_{\nu_1+\ldots+\nu_p=n} (-1)^p R(\zeta, z_0) A_{\nu_1} R(\zeta, z_0) A_{\nu_2} \cdots R(\zeta, z_0) A_{\nu_p} R(\zeta, z_0),$$

where A_{ν_k} are the coefficients of $A(z)$ and $\nu_k \geq 1$. The above expansion is called the second Neumann series for the resolvent. It is uniformly convergent for z sufficiently close to z_0 and $\zeta \in D$, where D is a compact subset of the complex plane which does not contain the eigenvalues of $A(z_0)$ [19].

Theorem 1 *Let $A(z)$ be the analytic perturbation of the matrix A_0 given by* (1)*. Then, the Drazin genralized inverse $A^{\#}(z)$ of the perturbed operator $A(z)$ can be expanded as a Laurent series* (2)*.*

Proof: We first show that there exists a domain $0 < |z| < z_{max}$ such that $A^{\#}(z)$ can be expanded in Taylor series at any point z_0 in this domain. For a fixed, arbitrary $z > 0$, (9) becomes

$$A^{\#}(z) = \frac{1}{2\pi i} \int_{\Gamma_0(z)} \frac{1}{\zeta} R(\zeta, z) d\zeta, \tag{12}$$

where $\Gamma_0(z)$ is a closed positively-oriented curve enclosing the origin but no other eigenvalue of $A(z)$.

With z_{max} less than the modulus of any non-zero eigenvalue of A_0, expand the perturbed resolvent in the power series (11), around the point z_0 (with $0 < |z_0| < z_{max}$). Then, the substitution of that series in the integral formula (12), yields

$$A^{\#}(z) = \frac{1}{2\pi i} \int_{\Gamma_0(z_0)} \frac{1}{\zeta} \left[R(\zeta, z_0) + \sum_{n=1}^{\infty} (z - z_0)^n R^{(n)}(\zeta, z_0) \right] d\zeta.$$

Since the power series for $R(\zeta, z)$ is uniformly convergent for z sufficiently close to z_0, we can integrate the above series term by term,

$$\begin{aligned} A^{\#}(z) &= \frac{1}{2\pi i} \int_{\Gamma_0(z_0)} \frac{1}{\zeta} R(\zeta, z_0)\, d\zeta + \sum_{n=1}^{\infty} (z - z_0)^n \frac{1}{2\pi i} \int_{\Gamma_0(z_0)} \frac{1}{\zeta} [R^{(n)}(\zeta, z_0)]\, d\zeta \\ &= A^{\#}(z_0) + \sum_{n=1}^{\infty} (z - z_0)^n H_n(z_0), \end{aligned} \tag{13}$$

where the coefficients are defined by

$$H_n(z_0) := \frac{1}{2\pi i} \int_{\Gamma_0(z_0)} \frac{1}{\zeta} [R^{(n)}(\zeta, z_0)]\, d\zeta.$$

The convergence of power series (13) in some non-empty domain $0 < |z| < z_{max}$ can be shown by using the bounds for the contour integrals (for very similar development see for example [19, Ch.2, Sec.3]). From the power series (13), we can see that $A^{\#}(z)$ is holomorphic in the domain $0 < |z| < z_{max}$. Consequently, by Laurent's theorem (see, e.g., [23, v. II, p. 7]), we conclude that $A^{\#}(z)$ possesses a Laurent series expansion at $z = 0$ (with radius of convergence z_{max}), i.e.,

$$A^{\#}(z) = \sum_{n=-\infty}^{+\infty} z^n H_n. \tag{14}$$

We next show that the pole at $z = 0$ can be at most of finite order. Consider the spectral representation (10) for the reduced resolvent of the perturbed operator $A(z)$

$$A^{\#}(z) = \sum_{i=1}^{p} \left\{ \frac{1}{\lambda_i(z)} P_i(z) + \sum_{j=1}^{m_i-1} (-1)^j \frac{1}{\lambda_i^{j+1}(z)} D_i^j(z) \right\}.$$

From Kato [19], we know that the perturbed eigenvalues $\lambda_i(z)$ are bounded in $|z| \le z_{max}$ and they have at most algebraic singularities. Furthermore, the eigenprojections $P_i(z)$ and nilpotents $D_i(z)$ can also have only algebraic singularities and poles of finite order. Therefore, none of the functions $\lambda_i(z)$, $P_i(z)$ and $D_i(z)$ can have an essential singularity. This latter fact implies that their finite sums, products or divisions as in $A^{\#}(z)$, do not have an essential singularity as well and, consequently, the order of pole in (14) is finite. This completes the proof.

□

4 Recursive formula for the regular part coefficients

Here we derive recursive formulae for the coefficients of the regular part of the Laurent series (2). We use an analytic technique based on Cauchy contour integrals and resolvent-like identities.

First, observe that the structure of the perturbed Drazin inverse $(A_0 + zA_1 + z^2A_2 + ...)^{\#}$ is similar to the structure of the classical resolvent $(A_0 - zI)^{-1}$. Moreover, $A^{\#}(z)$ becomes precisely the resolvent if $A_1 = -I$ and $A_k = 0$ for $k \ge 2$. Therefore, one can expect that these two mathematical objects have some similar features. It turns out that the Drazin inverse of an analytically perturbed matrix $A(z)$ satisfies an identity similar to the resolvent identity (4).

Lemma 1 *The reduced resolvent $A^{\#}(z)$ of the analytically perturbed operator $A(z) = \sum_{k=0}^{\infty} z^k A_k$ satisfies the resolvent-like identity:*

$$A^{\#}(z_1) - A^{\#}(z_2) = \sum_{k=1}^{\infty} (z_2^k - z_1^k) A^{\#}(z_1) A_k A^{\#}(z_2) + A^{\#}(z_1) P_0(z_2) - P_0(z_1) A^{\#}(z_2), \quad (15)$$

where $P_0(z)$ is the eigenprojection matrix corresponding to the zero eigenvalue.

Proof: Consider the following expression

$$A(z_2) - A(z_1) \; = \; \sum_{k=1}^{\infty} (z_2^k - z_1^k) A_k.$$

Premultiplying by $A^{\#}(z_1)$ and postmultiplying by $A^{\#}(z_2)$, yields

$$A^{\#}(z_1) A(z_2) A^{\#}(z_2) - A^{\#}(z_1) A(z_1) A^{\#}(z_2) \; = \; \sum_{k=1}^{\infty} (z_2^k - z_1^k) A^{\#}(z_1) A_k A^{\#}(z_2).$$

Then, using (6), we get

$$A^{\#}(z_1)[I - P_0(z_2)] - [I - P_0(z_1)] A^{\#}(z_2) \; = \; \sum_{k=1}^{\infty} (z_2^k - z_1^k) A^{\#}(z_1) A_k A^{\#}(z_2).$$

Equivalently,

$$A^{\#}(z_1) - A^{\#}(z_2) = \sum_{k=1}^{\infty}(z_2^k - z_1^k)A^{\#}(z_1)A_k A^{\#}(z_2) + A^{\#}(z_1)P_0(z_2) - P_0(z_1)A^{\#}(z_2),$$

which is the desired identity (15).

□

In the next theorem, we obtain a general relation between the coefficients of the Laurent series (2).

Theorem 2 *Let $H_k, k = -s, -s+1, ...$ be the coefficients of the Laurent series* (2) *and $P_0(z) = \sum_{k=0}^{\infty} z^k P_{0k}$ be a power series for the eigenprojection corresponding to the zero eigenvalue of the perturbed operator. Then the coefficients $H_k, k = -s, -s+1, ...$ satisfy the following relation*

$$\sum_{k=1}^{\infty}\sum_{i=0}^{k-1} H_{n-i}A_k H_{m+i-k+1} = -(\eta_n + \eta_m - 1)H_{n+m+1}$$

$$-\begin{cases} 0, & m < 0, \\ \frac{1}{2\pi i}\int\limits_{\Gamma_1} z_1^{-n-1}A^{\#}(z_1)[P_{0m+1} + z_1 P_{0m+2} + ...]dz_1, & m \geq 0 \end{cases}$$

$$-\begin{cases} 0, & n < 0 \\ \frac{1}{2\pi i}\int\limits_{\Gamma_2} z_2^{-m-1}[P_{0n+1} + z_2 P_{0n+2} + ...]A^{\#}(z_2)dz_2, & n \geq 0, \end{cases} \tag{16}$$

where

$$\eta_m := \begin{cases} 1, & m \geq 0, \\ 0, & m < 0. \end{cases}$$

For the sake of clarity of presentation, the detailed proof is postponed to the Appendix. Now the recursive formula for the coefficients of the regular part of the Laurent series (2) becomes a corollary of the above general result.

Corollary 1 *Suppose that the coefficients $H_k, k = -s, ..., -1, 0$ and $P_{0k}, k = 0, 1, ...$ are given. Then, the coefficients of the regular part of the Laurent expansion* (2) *can be computed by the following recursive formula:*

$$\begin{aligned} H_{m+1} \;\; = \;\; & -\sum_{i=0}^{m+s}\left(\sum_{j=0}^{s} H_{-j}A_{i+j+1}\right)H_{m-i} - \sum_{i=1}^{m} P_{0m+1-i}H_i \\ & -(P_{0m+1}H_0 + ... + P_{0m+1+s}H_{-s}) - (H_{-s}P_{0m+1+s} + ... + H_0 P_{0m+1}) \end{aligned} \tag{17}$$

for $m = 0, 1, ...$.

Proof: Let us take $n = 0$, $m > 0$ and then simplify the last two terms in (16), so that

$$\begin{aligned} & \frac{1}{2\pi i}\int\limits_{\Gamma_1} z_1^{-n-1}A^{\#}(z_1)[P_{0m+1} + z_1 P_{0m+2} + ...]dz_1 \\ & = \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{1}{z_1}[\frac{1}{z_1^s}H_{-s} + ... + \frac{1}{z_1}H_{-1} + H_0 + ...][P_{0m+1} + z_1 P_{0m+2} + ...]dz_1 \\ & = \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{1}{z_1}\frac{1}{z_1^{m+1}}\frac{1}{z_1^s}[H_{-s}P_{0m+1+s} + ... + H_0 P_{0m+1}]dz_1 \\ & = H_{-s}P_{0m+1+s} + ... + H_0 P_{0m+1} \end{aligned} \tag{18}$$

The last term can be dealt with in a similar fashion.

$$\frac{1}{2\pi i}\int\limits_{\Gamma_2} z_2^{-m-1}[P_{0n+1}+z_2P_{0n+2}+...]A^{\#}(z_2)dz_2$$
$$=\frac{1}{2\pi i}\int\limits_{\Gamma_2} z_2^{-m-1}[P_{01}+z_2P_{02}+...]A^{\#}(z_2)dz_2$$
$$=\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{1}{z_2}\frac{1}{Z_2^m}[P_{01}+z_2P_{02}+...][\frac{1}{z_1^s}H_{-s}+...+\frac{1}{z_1}H_{-1}+H_0+...]dz_2$$
$$=\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{1}{z_2}[P_{01}H_m+P_{02}H_{m-1}+...+P_{0m}H_1+P_{0m+1}H_0+...+P_{0m+1+s}H_{-s}]dz_2$$
$$=\sum_{i=1}^{m}P_{0m+1-i}H_i+(H_{-s}P_{0m+1+s}+...+H_0P_{0m+1}) \tag{19}$$

Substituting (18) and (19) into (16) with $n=0$ and $m>0$, we obtain

$$\sum_{k=1}^{\infty}\sum_{i=0}^{k-1}H_{n-i}A_kH_{m+i-k+1}=-H_{m+1}-(H_{-s}P_{0m+1+s}+...+H_0P_{0m+1})$$
$$-(P_{0m+1}H_0+...+P_{0m+1+s}H_{-s})-\sum_{i=1}^{m}P_{0m+1-i}H_i.$$

Rearranging terms in the above expression, we get

$$\begin{aligned} H_{m+1} \quad &= \quad -\sum_{i=0}^{m+s}\left(\sum_{j=0}^{s}H_{-j}A_{i+j+1}\right)H_{m-i}-\sum_{i=1}^{m}P_{0m+1-i}H_i \\ &\quad -(P_{0m+1}H_0+...+P_{0m+1+s}H_{-s})-(H_{-s}P_{0m+1+s}+...+H_0P_{0m+1}) \end{aligned}$$

which is the recursive formula (17).

□

If the perturbed operator $A(z)$ is invertible for $0<|z|<z_{max}$, then the inverse $A^{-1}(z)$ can be expanded as a Laurent series

$$A^{-1}(z)=\frac{1}{z^{-s}}H_{-s}+...+\frac{1}{z}H_{-1}+H_0+zH_1+... \tag{20}$$

and the formula (17) becomes

$$H_{m+1}=-\sum_{i=0}^{m+s}\left(\sum_{j=0}^{s}H_{-j}A_{i+j+1}\right)H_{m-i},\quad m=0,1,...$$

Furthermore, if the perturbed operator is invertible and the perturbation is linear $A(z)=A_0+zA_1$ we retrieve the recursive formula from [22]

$$H_{m+1}=(-H_0A_1)H_m,\quad m=0,1,...\ .$$

5 Reduction process

We have seen that the regular terms in the Laurent series expansion (2) of $A^{\#}(z)$ can be computed recursively by (17). However, to apply (17), one first needs to compute the terms $H_{-s}, \dots, H_{-1}, H_0$, that is, the terms of the singular part.

The complex analytic approach allows us to treat the cases regular and singular perturbations in a unified framework. In fact, let us first obtain some results on the regular perturbation which will be useful in the reduction process for the singular perturbation.

Regular case. Let us apply analytic function techniques to express the power series for the Drazin generalized inverse of the perturbed operator in the case of *regular* perturbation, that is, when the dimension of the null space of the matrix does not change if the perturbation parameter deviates from zero. In other words, there is no splitting of the zero eigenvalue of the perturbed matrix at $z = 0$. The latter implies that the expansion (13) is valid in some neighbourhood of $z_0 = 0$, and any contour $\Gamma_0 := \Gamma_0(0)$ chosen so that it does not enclose eigenvalues other than zero. Namely, the expansion (13) takes the form

$$A^{\#}(z) = A_0^{\#} + \sum_{n=1}^{\infty} z^n A_n^{\#}, \tag{21}$$

where

$$A_0^{\#} = A^{\#}(0), \quad A_n^{\#} = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{1}{\zeta} [R^{(n)}(\zeta)]\, d\zeta$$

and

$$R^{(n)}(\zeta) \;=\; \sum_{\nu_1+\ldots+\nu_p=n} (-1)^p R(\zeta) A_{\nu_1} R(\zeta) A_{\nu_2} \cdots R(\zeta) A_{\nu_p} R(\zeta).$$

It turns out that it is possible to express the coefficients $A_n^{\#}$ in terms of

- the unperturbed Drazin generalized inverse $A^{\#}(0)$,
- the eigenprojection P_0 corresponding to the zero eigenvalue of A_0,
- the perturbation matrices $A_n, n = 1, 2, \dots$.

The next theorem gives the precise statement.

Theorem 3 *Suppose that the operator A_0 is perturbed analytically as in* (1) *and assume that the zero eigenvalue of A_0 is semi-simple and the perturbation is regular. Then, the matrices $A_n^{\#}, n = 1, 2, \dots$ in the expansion* (21) *are given by the following formula*

$$A_n^{\#} = \sum_{p=1}^{n} (-1)^p \sum_{\substack{\nu_1 + \ldots + \nu_p = n \\ \mu_1 + \ldots + \mu_{p+1} = p+1 \\ \nu_j \geq 1, \mu_j \geq 0}} S_{\mu_1} A_{\nu_1} S_{\mu_2} \ldots A_{\nu_p} S_{\mu_{p+1}}, \tag{22}$$

where $S_0 := -P_0$ and $S_k := (A_0^{\#})^k, k = 1, 2, \ldots$.

Proof: Since Γ_0 encloses only the zero eigenvalue, we have by (21)

$$A_n^{\#} = \frac{1}{2\pi i} \int_{\Gamma_0} \frac{1}{\zeta} R^{(n)}(\zeta) d\zeta = \sum_{\nu_1+\ldots\nu_p=n} (-1)^p \frac{1}{2\pi i} \int_{\Gamma_0} \frac{1}{\zeta} R(\zeta) A_{\nu_1} R(\zeta) \ldots A_{\nu_p} R(\zeta) d\zeta.$$

$$= \sum_{\nu_1+...\nu_p=n} (-1)^p \mathrm{Res}_{\zeta=0}\{\frac{1}{\zeta}R(\zeta)A_{\nu_1}R(\zeta)...A_{\nu_p}R(\zeta)\}$$

In order to compute the above residue, we replace $R(\zeta)$ by its Laurent series (8) in the expression

$$\frac{1}{\zeta}R(\zeta)A_{\nu_1}R(\zeta)...A_{\nu_p}R(\zeta),$$

and collect the terms with $1/\zeta$, that is, the terms

$$\sum_{\sigma_1+...+\sigma_{p+1}=0} S_{\sigma_1+1}A_{\nu_1}S_{\sigma_2+1}...A_{\nu_p}S_{\sigma_p+1}.$$

Next, we change indicies $\mu_k := \sigma_k + 1, k = 1, ..., p+1$, and rewrite the above sum as:

$$\sum_{\mu_1+...+\mu_{p+1}=p+1} S_{\mu_1}A_{\nu_1}S_{\mu_2}...A_{\nu_p}S_{\mu_p},$$

which yields (22).

□

Remark 1 *Of course, formula* (22) *is computationally demanding due to the combinatorial explosion. However, only few terms will be computed by this formula* (*see the arguments developed below*)*.*

Singular case. We now show that by using a *reduction process*, we can transform the original singular problem into a regular one. Our reduction process can be viewed as complimentary to the existing reduction process based on spectral theory (see, e.g., the book of Kato [19]) which is applied to the eigenvalue problem. To the best of our knowledge, applying the reduction technique to analytical perturbations of generalized inverses is new.

To develop the reduction technique in the context of the generalized inverses, we need to introduce a new notion of *group reduced resolvent.* A definition based on spectral representation is as follows:

Definition 2 *Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be a linear operator with the spectral representation* (3)*. Then, the group reduced resolvent $A^{\#\Lambda}$ relative to the group of eigenvalues $\Lambda := \{\lambda_i\}_{i=0}^k$ is defined as follows:*

$$A^{\#\Lambda} \stackrel{def}{=} \sum_{i=k+1}^{p} \left\{ \frac{1}{\lambda_i}P_i + \sum_{j=1}^{m_i-1}(-1)^j \frac{1}{\lambda_i^{j+1}}D_i^j \right\}.$$

We note that the Drazin generalized inverse (see formula (10)) is a particular case of the group reduced resolvent. In this case, the group of eigenvalues consists only of the zero eigenvalue.

From our definition, the properties of a group reduced resolvent follow easily. In particular, in the next theorem, we will obtain an alternative analytic expression of the group reduced resolvent that will play a crucial role in perturbation analysis.

Theorem 4 *Let A be a linear operator with representation* (3)*. Then, the group reduced resolvent relative to the eigenvalues $\Lambda = \{\lambda_i\}_{i=0}^k$ is given by*

$$A^{\#\Lambda} = \frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta}(A-\zeta I)^{-1}d\zeta, \tag{23}$$

where Γ is a contour in the complex plane which encloses the set of eigenvalues $\{\lambda_i\}_{i=0}^k$, but none of the other eigenvalues $\{\lambda_i\}_{i=k+1}^p$.

Proof: It is a well known fact [19] that the resolvent can be represented by

$$-\sum_{i=0}^{p}\left[\frac{1}{\zeta-\lambda_i}P_i+\sum_{j=1}^{m_i}\frac{1}{(\zeta-\lambda_i)^{j+1}}D_i^j\right].$$

Substituting the above expression into the integral of (23) yields

$$\frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta}(A-\zeta I)^{-1}d\zeta \;=\; -\frac{1}{2\pi i}\int_\Gamma \sum_{i=0}^{p}\left[\frac{1}{\zeta(\zeta-\lambda_i)}P_i+\sum_{j=1}^{m_i}\frac{1}{\zeta(\zeta-\lambda_i)^{j+1}}D_i^j\right]d\zeta.$$

Using the fact that for every positive integer l

$$\mathrm{Res}_{\zeta=0}\frac{1}{\zeta(\zeta-\lambda)^l}=\frac{1}{(-\lambda)^l}\quad\text{and}\quad \mathrm{Res}_{\zeta=\lambda}\frac{1}{\zeta(\zeta-\lambda)^l}=-\frac{1}{(-\lambda)^l},$$

we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma \frac{1}{\zeta}(A-\zeta I)^{-1}d\zeta \;&=\; \sum_{i=0}^{k}\mathrm{Res}_{\zeta=\lambda_i}\left\{-\sum_{i=0}^{p}\left[\frac{1}{\zeta(\zeta-\lambda_i)}P_i+\sum_{j=1}^{m_i}\frac{1}{\zeta(\zeta-\lambda_i)^{j+1}}D_i^j\right]\right\}\\ &=\; \sum_{i=k+1}^{p}\left\{\frac{1}{\lambda_i}P_i+\sum_{j=1}^{m_i-1}(-1)^j\frac{1}{\lambda_i^{j+1}}D_i^j\right\}.\end{aligned}$$

According to Definition 2, the latter expression is equal to the group reduced resolvent, so that the proof is complete.

□

Lemma 2 *Let* $P=\sum_{i=0}^{k}P_i$ *be the projection corresponding to the group of eigenvalues* $\Lambda=\{\lambda_i\}_{i=0}^{k}$*, then*

$$A^{\#\Lambda}=(A[I-P])^{\#}.$$

Proof: Since $P_iP_j=\delta_{ij}P_i$ and $D_iP_j=\delta_{ij}D_i$, we have $P_i[I-P]=0$, $D_i[I-P]=0$ for $i=0,...,k$ and $P_i[I-P]=P_i$, $D_i[I-P]=D_i$ for $i=k+1,...,p$. Then, (3) and above yields

$$(A[I-P])^{\#}=\left(\sum_{i=0}^{p}(\lambda_iP_i+D_i)[I-P]\right)^{\#}=\left(\sum_{i=k+1}^{p}(\lambda_iP_i+D_i)\right)^{\#}.$$

Using formula (10), we obtain

$$(A[I-P])^{\#}=\sum_{i=k+1}^{p}\left\{\frac{1}{\lambda_i}P_i+\sum_{j=1}^{m_i-1}(-1)^j\frac{1}{\lambda_i^{j+1}}D_i^j\right\}.$$

The latter is equal to the group reduced resolvent $A^{\#\Lambda}$ by Definition 2.

□

Now equipped with this new notion of *group reduced resolvent*, we go back to perturbation analysis. The group of the perturbed eigenvalues $\lambda_i(z)$ such that $\lambda_i(z)\to 0$ as $z\to 0$ is called the 0-group. We denote the 0-group of eigenvalues by Ω. The eigenvalues of the 0-group split from zero when the perturbation parameter differs from zero. Since the eigenvalues of the perturbed operator are algebraic functions of the perturbation parameter [19], each eigenvalue of the 0-group (other than 0) can be written as

$$\lambda_i(z)=z^\nu\lambda_{i\nu}+o(z^\nu), \tag{24}$$

with $\lambda_{i\nu} \neq 0$ and ν is a positive rational number. The reduction technique is essentially based on the semi-simplicity assumption of reduced operators [19], which will be introduced below. Under that assumption, the power ν in (24) must be an integer. The latter implies that we can partition the 0-group into subsets that we call z^l-groups. Namely, we say that the eigenvalue $\lambda_i(z)$ belongs to the z^l-group if $\lambda_i(z) = z^l \lambda_{il} + o(z^l)$, with $\lambda_{il} \neq 0$. We denote z^l-group by Λ_l.

Consider now the spectral representation of the perturbed reduced resolvent.

$$A^{\#}(z) = \sum_{i=1}^{p} \left\{ \frac{1}{\lambda_i(z)} P_i(z) + \sum_{j=1}^{m_i-1} (-1)^j \frac{1}{\lambda_i^{j+1}(z)} D_i^j(z) \right\},$$

where $\{\lambda_i(z)\}_{i=1}^k$ is the 0-group. From the above formula one can see that in this case, the Laurent expansion for the reduced resolvent $A^{\#}(z)$ will possess terms with negative powers of z. Moreover, it turns out that under our assumptions, the z^k-group eigenvalues contribute to the terms of the Laurent expansion for $A^{\#}(z)$ with negative powers $-k, -k+1, ..., -1$ as well as to the regular part of the Laurent expansion.

The basic idea is to first treat the part of the perturbed operator corresponding to the eigenvalues which do not tend to zero as $z \to 0$. Then we treat subsequently the parts of the perturbed operator corresponding to the eigenvalues which belong to the z-group, z^2-group and so on.

What is helpful is that to treat the part of $A(z)$ corresponding to z^{k+1}-group we have to perform the same algorithm as for the part of the perturbed operator corresponding to the z^k-group. These steps constitute the (finite) reduction process.

Now we implement the general idea we have just briefly outlined above. Consider a fixed contour Γ_0 that encloses only the zero eigenvalue of the unperturbed operator A_0. Note that by continuity of eigenvalues the 0-group of eigenvalues of the perturbed operator $A(z)$ lies inside Γ_0 for z sufficiently small. Therefore, we may define the group reduced resolvent relative to the 0-group of eigenvalues as follows:

$$A^{\#\Omega}(z) \;=\; \frac{1}{2\pi i} \int_{\Gamma_0} \frac{1}{\zeta} R(\zeta, z) d\zeta \;=\; \frac{1}{2\pi i} \int_{\Gamma_0} \frac{1}{\zeta} (A(z) - \zeta I)^{-1} d\zeta.$$

Since $A^{\#\Omega}(z)$ is an analytic function in some neighbourhood of the origin, it can be expanded as a power series

$$A^{\#\Omega}(z) \;=\; A_0^{\#\Omega} + \sum_{i=1}^{\infty} z^i A_i^{\#\Omega}. \tag{25}$$

Note that $A_0^{\#\Omega} = (A_0)^{\#}$ and from Theorem 3 it follows that the other coefficients $A_i^{\#\Omega}, i = 1, 2, ...$ can be calculated by the formula (22). We would like to emphasize that in general the group reduced resolvent $A^{\#\Omega}(z)$ is different from the reduced resolvent $A^{\#}(z)$. However, we note that $A^{\#\Omega}(z)$ does coincide with $A^{\#}(z)$ in the case of regular perturbations.

Another operator that is used extensively in the reduction process is the *group projection*

$$P(z) \;=\; \frac{1}{2\pi i} \int_{\Gamma_0} R(\zeta, z)\, d\zeta,$$

which describes the subspace corresponding to the eigenvalues which split from zero. The group projection is an analytic function in some small neighbourhood of the origin (see e.g. [19]).

Next, as in the classical reduction process [7, 19], we define the restriction $B(z)$ of the operator $A(z)$ to the subspace determined by the group projection $P(z)$, that is,

$$B(z) := \frac{1}{z} A(z) P(z) \;=\; \frac{1}{2\pi i z} \int_{\Gamma_0} \zeta R(\zeta, z)\, d\zeta,$$

where Γ_0 is some fixed contour enclosing only the zero eigenvalue of the unperturbed operator A_0. For the operator $B(z)$ to be analytic at zero, we need the following assumption.

Assumption S1 *The zero eigenvalue of the operator A_0 is semi-simple, that is the nilpotent operator D_0 corresponding to $\lambda_0 = 0$ is equal to zero.*

Note that this assumption is not too restrictive. For example, in the case of a self-adjoint perturbation operator, the zero eigenvalue of A_0 is semi-simple. This is in particular the case when one studies the Moore-Penrose generalized inverse of an analytically perturbed matrix since it reduces to study a symmetric perturbation of the Drazin inverse (see Section 6). Whenever Assumption S1 is satisfied, the operator $B(z)$ can be expressed as a power series [19]

$$B(z) = B_0 + \sum_{i=1}^{\infty} z^i B_i,$$

with $B_0 = P(0)A_1P(0)$, and

$$B_n \;=\; -\sum_{p=1}^{n+1}(-1)^p \sum_{\substack{\nu_1+\ldots+\nu_p=n+1 \\ \mu_1+\ldots+\mu_{p+1}=p-1 \\ \nu_j\geq 1,\,\mu_j\geq 0}} S_{\mu_1}A_{\nu_1}S_{\mu_2}\ldots A_{\nu_p}S_{\mu_{p+1}}, \tag{26}$$

where $S_0 := -P(0)$ and $S_k := ((A_0)^{\#})^k$.

Since the operator $B(z)$ is analytic in some neighbourhood of the origin, we can again construct the expansion for its group reduced resolvent

$$B^{\#\Omega}(z) \;=\; (B_0)^{\#} + \sum_{i=1}^{\infty} z^i B_i^{\#\Omega}. \tag{27}$$

The coefficients $B_i^{\#\Omega}, i = 1, 2, \ldots$ are calculated by the formula given in Theorem 3. This is the first reduction step. To continue, we must distinguish between two cases:

(i) If the splitting of the zero eigenvalue terminates, and consequently $B(z)$ is a *regular* perturbation of B_0, then $B^{\#\Omega}(z) = B^{\#}(z)$ and the Drazin inverse of the perturbed operator $A(z)$ is given by

$$A^{\#}(z) = A^{\#\Omega}(z) + \frac{1}{z}B^{\#}(z). \tag{28}$$

By substituting the series expansions (25) and (27) for $A^{\#\Omega}(z)$ and $B^{\#}(z)$ into (28), we obtain the Laurent series expansion for $A^{\#}(z)$, which has a simple pole at zero.

(ii) If the zero eigenvalue splits further, the expression

$$A^{\#\Omega\backslash\Lambda_1}(z) = A^{\#\Omega}(z) + \frac{1}{z}B^{\#\Omega}(z)$$

represents *only* the *group reduced resolvent* relative to the eigenvalues constituting the 0-group but not the z-group, and we have to continue the reduction process. In fact, we now consider $B(z)$ as a singular perturbation of B_0, and repeat the procedure with $B(z)$. The 0-group of eigenvalues of B_0 contains all the z^k-groups of $A(0)$ (with $k \geq 2$), but *not* the z-group. Specifically, we construct the next-step reduced operator

$$C(z) = z^{-1}B(z)Q(z),$$

where $Q(z)$ is the eigenprojection corresponding to the 0-group of the eigenvalues of $B(z)$. Again, to ensure that $C(z)$ is an analytic function of z, we assume

Assumption S2 *The zero eigenvalue of B_0 is semi-simple.*

We would like to emphasize that the subsequent reduction steps are totally identical with the first one. At each reduction step, we make the assumption Sk that the analogue of B_0 at step k has a semi-simple 0-eigenvalue. The final result is stated in the next theorem.

Theorem 5 *Let Assumptions Sk hold. Then, the reduction process terminates after a finite number of steps, say s, and the perturbed Drazin inverse $A^{\#}(z)$ has the following expression:*

$$A^{\#}(z) = A^{\#\Omega}(z) + \frac{1}{z}B^{\#\Omega}(z) + \frac{1}{z^2}C^{\#\Omega}(z) + ... + \frac{1}{z^s}Z^{\#}(z). \tag{29}$$

Proof: Consider the first reduction step. Since $R(P(z))$ and $R(I - P(z))$ represent a direct decomposition of $\mathbb{C}^n$ and the subspace $R(P(z))$ is invariant under the operator $A(z)$, we can write

$$\begin{aligned} A^{\#}(z) = (A(z)[I - P(z)] + A(z)P(z))^{\#} &= (A(z)[I - P(z)])^{\#} + (A(z)P(z))^{\#} \\ &= A^{\#\Omega}(z) + z^{-1}(z^{-1}A(z)P(z))^{\#}, \end{aligned}$$

where Lemma 2 was used to get the first term of the right-hand-side. In view of Assumption $S1$, the operator $B(z) = z^{-1}A(z)P(z)$ is analytic in z and hence, one can apply the next reduction step. Similarly, Assumptions $Sk, k = 1, 2, \ldots$ guarantee that the reduction process can be carried out. Since the splitting of the zero eigenvalue has to terminate after a finite number of steps [19], we conclude that the reduction process has to terminate after a finite number of steps as well. Indeed, we successively eliminate the eigenvalues of the z-group, the z^2-group, etc ... Let $\lambda_i(z) = z^s\lambda_{is} + ...$ be the last eigenvalue which splits from zero. Then the corresponding reduced operator $Z(z)$ is regularly perturbed and the associated reduced resolvent $Z^{\#}(z)$ has the power series defined by Theorem 3. This completes the proof.

□

Summarizing, to obtain the Laurent series for $A^{\#}(z)$, there are two cases to distinguish. First, if one needs only few regular terms of $A^{\#}(z)$, then it suffices to replace $A^{\#\Omega}(z), B^{\#\Omega}(z), ...$ in (29) by their respective power series (25) computed during the reduction process. Note that only few terms of the power series $A^{\#\Omega}(z), B^{\#\Omega}(z), ...$ are needed. Otherwise, if one wishes to compute a significant number of regular terms, then compute only $H_{-s}, \ldots, H_{-1}, H_0$ as above (in which case, again, only a few terms of $A^{\#\Omega}(z), B^{\#\Omega}(z), ...$ are needed) and then use the recursive formula (17). Of course, one needs first to compute the power series expansion of the eigenprojection $P_0(z)$, which can be obtained by a number of methods [1, 7, 19].

Remark 2 *If the operator $A(z)$ has an inverse for $z \neq 0$, then the above algorithm can be used to calculate its Laurent expansion. Hence, the inversion problem $A^{-1}(z)$ is a particlular case of the presented above complex analytic approach.*

Example 1 *As was mentioned in the introduction, the perturbation analysis of the reduced resolvent can be applied directly to the theory of singularly perturbed Markov chains [5, 14, 17, 25]. Namely, the reduced resolvent of the generator of a Markov chain is just with minus sign the deviation matrix of this chain. The deviation matrix plays a crucial role in the Markov chain theory. For example, it is used to obtain mean first-passage times. Taking into account the above remark, we consider an example of a perturbed Markov chain. Let us consider the following perturbed operator.*

$$A(z) = A_0 + zA_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix} + z\begin{bmatrix} 2 & -1 & -1 \\ -3 & 1 & 2 \\ -4 & 3 & 1 \end{bmatrix}$$

Note that $-A(z)$ is the generator of a Markov chain. The zero eigenprojection and the reduced resolvent of the unperturbed matrix A_0 is given by

$$P(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \qquad A_0^{\#} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$$

In this instance, the Laurent expansion for $A^{\#}(z)$ has a simple pole. Using the method of [16] for the determination of the singularity order of the perturbed Markov chains, one can check that

$$A^{\#}(z) = \frac{1}{z}H_{-1} + H_0 + zH_1 + ...$$

By applying the reduction process, we compute the singular coefficient H_{-1} and the first regular coefficient H_0. Since the zero eigenvalues of the reduced operators are always semi-simple in the case of perturbed Markov chains [14], we conclude from Theorem 5 that

$$H_{-1} = B_0^{\#}, \quad \text{and} \quad H_0 = A_0^{\#} + B_1^{\#\Omega}.$$

To compute $B_0^{\#}$ and $B_1^{\#\Omega}$, we need to calculate the first two terms of the expansion for the reduced operator $B(z)$. In particular,

$$B_0 = P(0)A_1P(0) = \begin{bmatrix} 2 & -1 & -1 \\ -3.5 & 1.75 & 1.75 \\ -3.5 & 1.75 & 1.75 \end{bmatrix}$$

$$\begin{aligned} B_1 &= -(A_0^{\#}A_1P(0)A_1P(0) + P(0)A_1A_0^{\#}A_1P(0) + P(0)A_1P(0)A_1A_0^{\#}) \\ &= \frac{1}{8}\begin{bmatrix} 0 & 4 & -4 \\ -24 & 5 & 19 \\ 20 & -17 & -3 \end{bmatrix} \end{aligned}$$

calculated with the help of (26). Next, we calculate the eigenprojection corresponding to the zero eigenvalue of the operator B_0, that is,

$$Q(0) = \frac{1}{22}\begin{bmatrix} 14 & 4 & 4 \\ 14 & 15 & -7 \\ 14 & -7 & 15 \end{bmatrix}.$$

Now using formula (22) from Theorem 3, we obtain

$$B_1^{\#\Omega} = Q(0)B_1(B_0^{\#})^2 - B_0^{\#}B_1B_0^{\#} + (B_0^{\#})^2B_1Q(0) = \frac{1}{2662}\begin{bmatrix} -16 & 52 & -36 \\ -236 & 41 & 195 \\ 248 & -201 & -47 \end{bmatrix}.$$

Thus, we finally obtain

$$H_{-1} = B_0^{\#} = \frac{1}{5.5^2}\begin{bmatrix} 1 \\ -1.75 \\ -1.75 \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \end{bmatrix} = \frac{1}{121}\begin{bmatrix} 8 & -4 & -4 \\ -14 & 7 & 7 \\ -14 & 7 & 7 \end{bmatrix}$$

and

$$H_0 = B_1^{\#\Omega} + A_0^{\#} = \frac{1}{1331}\begin{bmatrix} -8 & 26 & -18 \\ -118 & 686 & -568 \\ 124 & -766 & 642 \end{bmatrix}.$$

Note that the above H_{-1} coincides with the one given in [25].

If we have in hand the expansion for the ergodic projection, we can use the recursive formula (17) to compute the regular coefficients. Let us compute by the recursive formula the coefficient H_1 for our example. First, applying the reduction process for the eigenproblem [1, 7, 19], one can compute the coefficients for the expansion of the ergodic projection associated with z and z^2.

$$P_{01} = \frac{1}{121}\begin{bmatrix} 2 & -12 & 10 \\ 2 & -12 & 10 \\ 2 & -12 & 10 \end{bmatrix} \quad P_{02} = \frac{1}{1331}\begin{bmatrix} 32 & -192 & 160 \\ 32 & -192 & 160 \\ 32 & -192 & 160 \end{bmatrix}$$

Then, according to formula (17), we have

$$\begin{aligned} H_1 &= -(H_0 A_1) H_0 - (P_{01} H_0 + P_{02} H_{-1}) - (H_{-1} P_{02} + H_0 P_{01}) \\ &= \frac{1}{14641}\begin{bmatrix} -368 & 1856 & -1488 \\ -2128 & 12416 & -10288 \\ 1744 & -10816 & 9072 \end{bmatrix}. \end{aligned}$$

6 Perturbation of the Moore-Penrose generalized inverse and the group inverse

Note that the Laurent series for the perturbed group inverse does not always exist. Indeed, the existence of the group inverse of the unperturbed operator does not imply the existence of the group inverse of the perturbed operator.

Example 2 *Consider*

$$A(z) = \begin{bmatrix} 0 & z & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The space $\mathbb{C}^n$ can be decomposed in a direct sum of the null space and range of $A(0)$ but no such decomposition exists if $z \neq 0$. Thus, the unperturbed operator $A(0)$ has a group inverse and the perturbed operator does not.

The following is a general sufficient condition for the existence of the Laurent series for the perturbed group inverse.

Theorem 6 *Let the group inverse $A^g(z)$ of the analytically perturbed matrix $A(z)$ exist in some non-empty (possibly punctured) neighbourhood of $z = 0$. Then the group inverse $A^g(z)$ can be expanded as a Laurent series around $z = 0$ with a non-zero radius of convergence.*

Proof: Whenever the group inverse exists, it coincides with the Drazin generalized inverse. Therefore, the existence of a Laurent series follows from Theorem 1.

□

As one can see from the following example, even though the Moore-Penrose generalized inverse always exists, it might be not analytic function of the perturbation parameter.

Example 3 *Let*

$$A(z) = \begin{bmatrix} 0 & z \\ 0 & 1 \end{bmatrix}.$$

Its Moore-Penrose generalized inverse is given by

$$A^{\dagger}(z) = \frac{1}{1 + z\bar{z}}\begin{bmatrix} 0 & 0 \\ \bar{z} & 1 \end{bmatrix},$$

which is not analytic since it depends on $\bar{z}$.

However, if we restrict $A_k, k = 0, 1, ...$ to the matrices with real entries and if z belongs to some interval of the real line, we can state the following existence result.

Theorem 7 *Let $A^\dagger(\varepsilon)$ be the Moore-Penrose generalized inverse of the analytically perturbed matrix*

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + ... \ ,$$

where $A_k \in \mathbb{R}^{n\times m}$, $\varepsilon \in \mathbb{R}$ and the series converges for $0 < |\varepsilon| < \varepsilon_{max}$. Then, $A^\dagger(\varepsilon)$ possesses a Laurent series expansion

$$A^\dagger(\varepsilon) = \frac{1}{\varepsilon^s} B_{-s} + ... + \frac{1}{\varepsilon} B_{-1} + B_0 + \varepsilon B_1 + ... \tag{30}$$

in some non-empty punctured vicinity around $\varepsilon = 0$.

Proof: Applying the formula $A^\dagger = (A^T A)^g A^T$ [1] for the perturbed operator $A(\varepsilon)$ yields

$$A^\dagger(\varepsilon) = (A^T(\varepsilon)A(\varepsilon))^g A^T(\varepsilon). \tag{31}$$

Note that the group inverse of a symmetric matrix always exists. Hence, by Theorem 6, $(A^T(\varepsilon)A(\varepsilon))^g$ has a Laurent series expansion and so the Moore-Penrose generalized inverse $A^\dagger(\varepsilon)$ does.

□

We would like to emphasize that according to (31), computing the Laurent series of the perturbed Moore-Penrose generalized inverse $A^\dagger(\varepsilon)$ reduces to computing the Laurent series of a group inverse. Moreover, $A^T(\varepsilon)A(\varepsilon)$ is a symmetric perturbation, that is each term of its power series has a symmetric matrix coefficient. This guarantees that the reduction process defined in the previous section and restricted to the real line is indeed applicable in this case.

7 Asymptotics for $P_0(z)$

We know that the eigenprojection $P_0(z)$ of the perturbed operator corresponding to the identically zero eigenvalue is analytic in some (punctured) neighbourhood of $z = 0$ (see [1] and [19]), that is,

$$P_0(z) \ = \ P_{00} + \sum_{k=1}^{\infty} z^k P_{0k}, \tag{32}$$

for z sufficiently small but different from zero. For regular perturbations, P_{00} is just $P_0(0)$, and the group projection coincides with the eigenprojection. This is not the case for singular perturbations.

Therefore, an interesting question is how P_{00} in (32) relates to the original matrix $P_0(0)$ in the general case and how the power series (32) can be computed. The answers to these questions are provided below.

Proposition 1 *The coefficients of the power series* (32) *for the perturbed eigenprojection are given by*

$$P_{0k} = -\sum_{i=0}^{s+k} A_i H_{k-i}, \quad k = 1, 2, ... \ .$$

Proof: The above formula is obtained by equating the terms with the same power of z in the identity (6) for the perturbed operators.

□

Corollary 2 *As $z \to 0$, the limit eigenprojection matrix P_{00} satisfies*

$$P_{00} \;=\; P_0(0)\,[I - \sum_{i=1}^{s} A_i H_{-i}]. \tag{33}$$

where $P_0(0)$ is the 0-eigenprojection of the unperturbed matrix A_0.

Proof: When substituting the Laurent series expansion (14) into (6)-(7), and equating the terms of same power, one obtains in particular,

$$I - P_{00} \;=\; A_0 H_0 + A_1 H_{-1} + ... + A_s H_{-s}\ . \tag{34}$$

In addition, from $A(z)P_0(z) = 0$, we immediately obtain:

$$A_0 P_{00} \;=\; 0 \;\text{ so that }\; P_{00} \;=\; P_0(0)V, \tag{35}$$

for some matrix V. Moreover, as $P_0(0)^2 = P_0(0)$, we also have $P_0(0)P_{00} = P_0(0)^2 V = P_0(0)V = P_{00}$. Therefore, pre-multiplying both sides of (34) by $P_0(0)$, and using $P_0(0)A_0 = 0$, one obtains (33), the desired result.

□

Hence, (33) relates in a simple manner, the limit matrix P_{00} to the original 0-group $P_0(0)$, in terms of the perturbation matrices $A_k, k = 1, ..., s$, the original matrix $P_0(0)$, and the coefficients $H_{-k}, k = 1, ..., s$ of the singular part of $A(z)^{\#}$. This shows how the perturbed 0-eigenvectors compare to the unperturbed ones, for small z. Observe that in the case of a linear (or first-order) perturbation, only the singular term H_{-1} is involved. Finally, the regular case is obtained as a particular case, since then, $H_{-k}, k = 1, \dots, s$ vanish so that $P_{00} = P_0(0)$.

Appendix: The proof of Theorem 2

To prove Theorem 2 we use the Cauchy contour integration and the residue technique. First we present some auxiliary results.

Lemma 3 *Let Γ_1 and Γ_2 be two closed positively-oriented contours in the complex plane around zero and let $z_1 \in \Gamma_1, z_2 \in \Gamma_2$. Furthermore, assume that the contour Γ_2 lies inside the contour Γ_1. Then the following formulae hold:*

$$\frac{1}{2\pi i}\int_{\Gamma_2} \frac{z_2^{-m-1}}{z_2 - z_1} dz_2 = -\eta_m z_1^{-m-1}, \tag{36}$$

$$\frac{1}{2\pi i}\int_{\Gamma_1} \frac{z_1^{-m-1}}{z_2 - z_1} dz_1 = -(1-\eta_n) z_2^{-m-1}, \tag{37}$$

with

$$\eta_m := \begin{cases} 0, & m < 0, \\ 1, & m \geq 0. \end{cases}$$

and

$$\frac{1}{2\pi i}\int_{\Gamma_2} \frac{z_2^{-m-1} P_0(z_2)}{z_2 - z_1} dz_2 = \begin{cases} 0, & m < 0, \\ -z_1^{-m-1}[P_{00} + z_1 P_{01} + ... + z_1^m P_{0m}], & m \geq 0. \end{cases} \tag{38}$$

$$\frac{1}{2\pi i}\int_{\Gamma_1} \frac{z_1^{-n-1} P_0(z_1)}{z_2 - z_1} dz_1 \;=\; \begin{cases} -z_2^{-n-1} P_0(z_2), & n < 0, \\ -[P_{0n+1} + z_2 P_{0n+2} + z_2^2 P_{0n+3} + ...], & n \geq 0. \end{cases} \tag{39}$$

PROOF: The proof of formulae (36),(37) is given in the book of Kato [19] and in the book of Korolyuk and Turbin [20].

Let us establish the auxiliary integral (38). If $m < 0$, then the function $\frac{z_2^{-m-1}P_0(z_2)}{z_2-z_1}$ is analytic inside the area enclosed by the contour Γ_2 and hence the auxiliary integral (38) is equal to zero by the Cauchy Integral Theorem. To deal with the case $m \geq 0$, we first expand the function $\frac{z_2^{-m-1}P_0(z_2)}{z_2-z_1}$ as a Laurent series.

$$\frac{z_2^{-m-1}P_0(z_2)}{z_2-z_1} = -\frac{z_2^{-m-1}P_0(z_2)}{z_1(1-z_2/z_1)} = -z_1^{-1}z_2^{-m-1}[P_{00}+z_2P_{01}+z_2^2P_{02}+...][1+\frac{z_2}{z_1}+\frac{z_2^2}{z_1^2}+...] =$$

$$= z_2^{-m-1}(-z_1^{-1})P_{00}+...+z_2^{-1}(-z_1^{-1})[\frac{1}{z_1^m}P_{00}+\frac{1}{z_1^{m-1}}P_{01}+...+P_{0m}]+...$$

Then, according to the Residue Theorem, we have

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{z_2^{-m-1}P_0(z_2)}{z_2-z_1}dz_2 &= (-z_1^{-1})[\frac{1}{z_1^m}P_{00}+\frac{1}{z_1^{m-1}}P_{01}+...+P_{0m}] \\ &= -z_1^{-m-1}[P_{00}+z_1P_{01}+...+z_1^mP_{0m}].\end{aligned}$$

Thus, we have calculated the integral (38). The same method is applied to calculate the auxiliary integral (39).

□

THE PROOF OF THEOREM 2: Each coefficient of the Laurent series (2) can be represented by the contour integral formula

$$H_n = \frac{1}{2\pi i}\int_\Gamma z^{-n-1}A^{\#}(z)dz, \quad \Gamma \in D, \tag{40}$$

where Γ is a closed positively-oriented contour in the complex plane, which encloses zero but no other eigenvalues of A_0. Let us substitute (40) into the following expression

$$\sum_{k=1}^{\infty}\sum_{i=0}^{k-1}H_{n-i}A_kH_{m+i-k+1} = \sum_{k=1}^{\infty}\sum_{i=0}^{k-1}\frac{1}{2\pi i}\int\limits_{\Gamma_1}z_1^{-n+i-1}A^{\#}(z_1)dz_1A_k\frac{1}{2\pi i}\int\limits_{\Gamma_2}z_2^{-m-i+k-2}A^{\#}(z_2)dz_2.$$

As in Lemma 3, we assume without loss of generality that the contour Γ_2 lies inside the contour Γ_1. Then, we can rewrite the above expressions as double integrals

$$\sum_{k=1}^{\infty}\sum_{i=0}^{k-1}H_{n-i}A_kH_{m+i-k+1} = \sum_{k=1}^{\infty}\sum_{i=0}^{k-1}\left(\frac{1}{2\pi i}\right)^2\int\limits_{\Gamma_1}\int\limits_{\Gamma_2}z_1^{-n+i-1}z_2^{-m-i+k-2}A^{\#}(z_1)A_kA^{\#}(z_2)dz_2dz_1$$

$$= \left(\frac{1}{2\pi i}\right)^2\int\limits_{\Gamma_1}\int\limits_{\Gamma_2}z_1^{-n-1}z_2^{-m-1}\sum_{k=1}^{\infty}\sum_{i=0}^{k-1}z_1^iz_2^{k-i-1}A^{\#}(z_1)A_kA^{\#}(z_2)dz_2dz_1$$

$$= \left(\frac{1}{2\pi i}\right)^2\int\limits_{\Gamma_1}\int\limits_{\Gamma_2}\frac{z_1^{-n-1}z_2^{-m-1}}{z_2-z_1}\sum_{k=1}^{\infty}(z_2^k-z_1^k)A^{\#}(z_1)A_kA^{\#}(z_2)dz_2dz_1.$$

Using the resolvent-like identity (15), we obtain

$$\sum_{k=1}^{\infty}\sum_{i=0}^{k-1}H_{n-i}A_kH_{m+i-k+1} =$$

$$= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} z_1^{-n-1} z_2^{-m-1} \left[\frac{A^{\#}(z_1) - A^{\#}(z_2)}{z_2 - z_1} - \frac{A^{\#}(z_1)P_0(z_2)}{z_2 - z_1} + \frac{P_0(z_1)A^{\#}(z_2)}{z_2 - z_1}\right] dz_2 dz_1.$$

Thus, we obtain

$$\sum_{k=1}^{\infty}\sum_{i=0}^{k-1} H_{n-i}A_k H_{m+i-k+1} = I_1 - I_2 + I_3,$$

with:

$$\begin{aligned}
I_1 &:= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} z_1^{-n-1} z_2^{-m-1} \frac{A^{\#}(z_1) - A^{\#}(z_2)}{z_2 - z_1} dz_2 dz_1, \\
I_2 &:= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} z_1^{-n-1} z_2^{-m-1} \frac{A^{\#}(z_1)P_0(z_2)}{z_2 - z_1} dz_2 dz_1, \\
I_3 &:= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} z_1^{-n-1} z_2^{-m-1} \frac{P_0(z_1)A^{\#}(z_2)}{z_2 - z_1} dz_2 dz_1.
\end{aligned}$$

Let us separately calculate the integrals I_1, I_2 and I_3. The integral I_1 can be written as

$$\begin{aligned}
I_1 &= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} \frac{z_1^{-n-1} z_2^{-m-1}}{z_2 - z_1} A^{\#}(z_1) dz_2 dz_1 - \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} \frac{z_1^{-n-1} z_2^{-m-1}}{z_2 - z_1} A^{\#}(z_2) dz_2 dz_1 \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1} \left[\int_{\Gamma_2} \frac{z_2^{-m-1}}{z_2 - z_1} dz_2\right] z_1^{-n-1} A^{\#}(z_1) dz_1 \\
&\quad - \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_2} \left[\int_{\Gamma_1} \frac{z_1^{-n-1}}{z_2 - z_1} dz_1\right] z_2^{-m-1} A^{\#}(z_2) dz_2
\end{aligned}$$

In the last equality we used Fubini Theorem to change the order of integration. Using the auxiliary integrals (36) and (37), we obtain

$$\begin{aligned}
I_1 &= \frac{1}{2\pi i}\int_{\Gamma_1} (-\eta_m z_1^{-m-1}) z_1^{-n-1} A^{\#}(z_1) dz_1 - \frac{1}{2\pi i}\int_{\Gamma_2} (-(1-\eta_n) z_2^{-n-1}) z_2^{-m-1} A^{\#}(z_2) dz_2 \\
&= -\frac{\eta_n + \eta_m - 1}{2\pi i} \int_{\Gamma_1} z_1^{-n-m-2} A^{\#}(z_1) dz_1 = -(\eta_n + \eta_m - 1) H_{n+m+1},
\end{aligned}$$

where the second integral can be taken over Γ_1 by the principle of deformation of contours. We calculate the second integral I_2 as follows:

$$\begin{aligned}
I_2 &= \frac{1}{2\pi i}\int_{\Gamma_1} z_1^{-n-1} A^{\#}(z_1) \frac{1}{2\pi i}\int_{\Gamma_2} \frac{z_2^{-m-1} P_0(z_2)}{z_2 - z_1} dz_2 dz_1 \\
&= \begin{cases} \frac{1}{2\pi i}\int_{\Gamma_1} 0 z_1^{-n-1} dz_1, & m < 0, \\ -\frac{1}{2\pi i}\int_{\Gamma_1} z_1^{-n-1} A^{\#}(z_1) z_1^{-m-1}[P_{00} + z_1 P_{01} + ... + z_1^m P_{0m}] dz_1, & m \geq 0 \end{cases} \\
&= \begin{cases} 0, & m < 0, \\ -\frac{1}{2\pi i}\int_{\Gamma_1} z_1^{-n-m-2} A^{\#}(z_1)[P_{00} + z_1 P_{01} + ... + z_1^m P_{0m}] dz_1, & m \geq 0 \end{cases}
\end{aligned}$$

$$= \begin{cases} 0, & m < 0, \\ -\frac{1}{2\pi i} \int\limits_{\Gamma_1} z_1^{-n-m-2} A^{\#}(z_1)[P_0(z_1) - z_1^{m+1} P_{0m+1} - z_1^{m+2} P_{0m+2} - ...]dz_1, & m \geq 0 \end{cases}$$

$$= \begin{cases} 0, & m < 0, \\ \frac{1}{2\pi i} \int\limits_{\Gamma_1} z_1^{-n-m-2} A^{\#}(z_1)[z_1^{m+1} P_{0m+1} + z_1^{m+2} P_{0m+2} + ...]dz_1, & m \geq 0 \end{cases}$$

$$= \begin{cases} 0, & m < 0, \\ \frac{1}{2\pi i} \int\limits_{\Gamma_1} z_1^{-n-1} A^{\#}(z_1)[P_{0m+1} + z_1 P_{0m+2} + ...]dz_1, & m \geq 0 \end{cases}$$

where, in the above expressions, the auxiliary integral (38) and the property $A^{\#}(z)P_0(z) = 0$ has been used. Now, we calculate the last integral I_3 with the help of the auxiliary integral (39).

$$\begin{aligned} I_3 &= \frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{z_1^{-n-1} P_0(z_1)}{z_2 - z_1} dz_1 z_2^{-m-1} A^{\#}(z_2) dz_2 \\ &= \begin{cases} -\frac{1}{2\pi i} \int\limits_{\Gamma_2} z_2^{-n-m-2} P_0(z_2) A^{\#}(z_2) dz_2, & n < 0, \\ -\frac{1}{2\pi i} \int\limits_{\Gamma_2} z_2^{-m-1} [P_{0n+1} + z_2 P_{0n+2} + z_2^2 P_{0n+3} + ...] A^{\#}(z_2) dz_2, & n \geq 0, \end{cases} \\ &= \begin{cases} 0, & n < 0, \\ -\frac{1}{2\pi i} \int\limits_{\Gamma_2} z_2^{-m-1} [P_{0n+1} + z_2 P_{0n+2} + z_2^2 P_{0n+3} + ...] A^{\#}(z_2) dz_2, & n \geq 0. \end{cases} \end{aligned}$$

Finally, summing up the three integrals I_1, I_2 and I_3, we obtain the relation (16).

□

References

[1] K.E. Avrachenkov, *Analytic perturbation theory and its applications*, PhD Thesis, University of South Australia, 1999.

[2] K.E. Avrachenkov, M. Haviv and P.G. Howlett, *Inversion of analytic matrix functions that are singularly at the origin*, SIAM Journal on Matr. Anal. Appl., **22**, (2001) 11751189.

[3] K.E. Avrachenkov and M. Haviv, *Perturbation of null spaces with applications to the eigenvalue problem and generalized inverses*, Lin. Alg. Appl., **369**, (2003) 1-25.

[4] K.E. Avrachenkov and J.B. Lasserre, *Perturbation analysis of reduced resolvents and generalized inverses*, CNRS-LAAS Research Report no.98520, December 1998.

[5] K.E. Avrachenkov and J.B. Lasserre, *The fundamental matrix of singularly perturbed Markov chains*, Advances in Applied Probability, **31**, (1999) 679-697.

[6] H. Bart, *Meromorphic operator valued functions*, Thesis. Vrije Universiteit, Amsterdam, 1973, (Math. Center Tracts 44, Mathematical Center, Amsterdam, 1973).

[7] H. Baumgartel, *Analytic perturbation theory for matrices and operators,* (Birkhauser, Basel 1985).

[8] E. Bohl and P. Lancaster, *Perturbation of spectral inverses applied to a boundary layer phenomenon arising in chemical networks*, Lin. Alg. Appl., **180**, (1993) 35-59.

[9] S.L. Campbell and C.D. Meyer, *Generalizes Inverses of Linear Transformation*, (Pitman, London 1979).

[10] N. Castro-González, *Additive perturbations results for the Drazin inverse*, Lin. Alg. Appl., **397**, (2005) 279-297.

[11] N. Castro-González, E. Dopazo, M.F. Martínez Serrano, *On the Drazin inverse of the sum of two operators and its application to operator matrices*, J. Math. Anal. Appl., **350**, (2009) 207-215.

[12] M. Coderh, A.S. Willsky, S.S. Sastry and D.A. Castanon, *Hierarchical aggregation of linear systems with multiple time scales*, IEEE Trans. Autom. Contr., **AC-28**, (1983) 1029-1071.

[13] A.S. Deif, *Sensitivity analysis in linear systems*, (Springer-Verlag, Berlin 1986).

[14] F. Delebecque, *A reduction process for perturbed Markov chain*, SIAM J. Appl. Math., **43**, (1983) 325-350.

[15] I. Gohberg, M. A. Kaashoek and F. van Schagen, *On the local theory of regular analytic matrix functions*, Lin. Alg. Appl., **182**, (1993) 9-25.

[16] R. Hassin and M. Haviv, *Mean passage times and nearly uncoupled Markov chains*, SIAM J. Disc. Math., **5**, (1992) 368-397.

[17] M. Haviv and Y. Ritov, *Series expansions for stochastic matrices*, unpublished manuscript, (1989).

[18] P.G. Howlett, *Input retrieval in finite dimentional linear systems*, J. Austral. Math. Soc. (Series B), **23**, (1982) 357–382.

[19] T. Kato , *Perturbation theory for linear operators*, (Springer-Verlag, Berlin 1966).

[20] V.S. Korolyuk and A.F. Turbin, *Mathematical foundations of the state lumping of large systems*, (Naukova Dumka, Kiev 1978).

[21] P. Lancaster, *Inversion of lambda-matrices and application to the theory of linear vibrations*, Arch. Rational Mech. Anal., **6**, (1960) 105-114.

[22] C.E. Langenhop, *The Laurent expansion for a nearly singular matrix*, Linear Algebra Appl., **4**, (1971) 329–340.

[23] A.I. Markushevich, *Theory of functions of a complex variable*, (Chelsea Publishing Company, New York 1977).

[24] A.A. Pervozvanski and V.G. Gaitsgori, *Theory of Suboptimal Decisions*, (Kluwer Academic Publishers, Dordrecht 1988).

[25] P.J. Schweitzer, *Perturbation series expansions for nearly completely-decomposable Markov chains*, in Teletraffic Analysis and Computer Performance Evaluation, eds. O.J. Boxma, J.W. Cohen, H.C. Tijms, (Elsevier, 1986).

[26] P.J. Schweitzer and G.W. Stewart, *The Laurent expansion of pencils that are singular at the origin*, Linear Algebra Appl., **183**, (1993) 237–254.

[27] G.W. Stewart, *On the perturbation of pseudo-inverses, projections and linear least squares problems*, SIAM Review, **19**, (1977) 634-662.

[28] M.M. Vainberg and V.A. Trenogin, *Theory of branching of solutions of non-linear equations*, (Noordhoff International Publishing, 1969).

[29] M.I. Vishik and L.A. Lyusternik, *The solution of some perturbation problems in the case of matrices and self-adjoint and non-self-adjoint differential equations*, Uspechi Mat. Nauk, **15**, (1960) 3-80.

[30] G. Wang, Y. Wei and S. Qiao, *Generalized Inverses: Theory and Computations*, Graduate Series in Mathematics **5**, Science Press, Beijing/New York, 2004.