SINGULARLY PERTURBED FINITE MARKOV CHAINS WITH GENERAL ERGODIC STRUCTURE

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Abstract We analyse singularly perturbed Markov chains. Most previous research has been done under the assumption that the perturbed Markov chain is either ergodic or unichain. In this paper we do not impose any restrictions on the ergodic structure of the perturbed chain. The present approach is based on the inversion of analytic matrix-valued functions.

Introduction

In this paper we consider a perturbed Markov chain with the transition probability matrix $P(\varepsilon) = P(0) + \varepsilon C$, where $P(0) \in \mathbb{R}^{n \times n}$ is the transition matrix of the unperturbed chain and ε is a *small* perturbation parameter. The aim of this paper is to calculate the asymptotic expansions for the stationary distributions and the absorption probabilities of the perturbed Markov chain. We are mostly interested in the case of *singular* perturbations when the number of ergodic classes of the perturbed chain is less than the number of ergodic classes of the original chain. This typically happens when some of the ergodic classes of the original chain become connected with small transition probabilities corresponding to perturbation εC. Often for large state space Markov chains one can naturally introduce the perturbation parameter and then, by using singular perturbation techniques, significantly reduce complexity of the problem. The applications of singularly perturbed Markov chains are discussed in [1, 5, 7, 9]. Most previous research [1, 4, 5, 6, 7, 9, 10, 12, 13] was carried out under the assumption that the perturbed Markov chain is either ergodic or unichain, that is all ergodic classes become connected after the perturbation. Here we remove this assumption and study the perturbed Markov chains with a general structure. The general case was also studied in [8]. However, the approach of [8] based on the reduction technique of [11] leads to combinatorial explosion. Here we avoid combinatorial explosion by providing simple recurrent formulae. Furthermore, the approach of [8] cannot be applied (at least directly) for the computation of the absorption probabilities.

Main Results

Let us write the perturbed transition matrix in the canonical form

$$P(\varepsilon) = \begin{array}{c} \left[\begin{array}{cccc} P_1(\varepsilon) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & P_\ell(\varepsilon) & 0 \\ R_1(\varepsilon) & \cdots & R_\ell(\varepsilon) & S(\varepsilon) \end{array} \right] \end{array} \begin{array}{l} \} \, \Omega_1 \\ \quad \vdots \\ \} \, \Omega_\ell \\ \} \, \Omega_T \end{array} \qquad (1)$$

where $P_i(\varepsilon) = P_i(0)+\varepsilon C_i$, $R_i(\varepsilon) = R_i(0)+\varepsilon C_{Ri}$, $S(\varepsilon) = S(0)+\varepsilon C_S$. Now note that all stationary distributions $\mu_i(\varepsilon)$ of the perturbed Markov chain can be immediately constructed from the stationary distributions of the ergodic classes associated with stochastic matrices $P_i(\varepsilon), i = 1, ..., \ell$. Namely, $\mu_i(\varepsilon) = [0 \cdots 0 \ \pi_i(\varepsilon) \ 0 \cdots 0]$, where $\pi_i(\varepsilon)$ is uniquely determined by the system

$$\begin{cases} \pi_i(\varepsilon)P_i(\varepsilon) = \pi_i(\varepsilon), \\ \quad \pi_i(\varepsilon)\underline{1} = 1, \end{cases}$$

where $\underline{1}$ is a vector of ones. This is exactly the perturbation problem under the assumption that there are several ergodic classes plus possibly a set of transient states coupled by the perturbation into a single ergodic class. As was mentioned above, there exist a number of methods [6, 7, 8, 9, 12, 10, 13] to calculate the asymptotic expansions $\pi_i(\varepsilon) = \pi_i^{(0)} + \varepsilon\pi_i^{(1)} + ..., i = 1, ..., \ell$.

The challenging task that we face in this case of multi-chain perturbed process is the computation of power series for the *right* 0-eigenvectors of the perturbed Markov chains. The perturbed right 0-eigenvectors contain the probabilities of being *absorbed* in each of the ergodic classes under the perturbed processes. These quantities exhibit the multi-chain ergodic properties of the perturbed process. For instance, nothing conceptually new would emerge had the set of transient states in the perturbed process been empty.

The right 0-eigenvectors of the perturbed chain can be written in the following form

$$q_i(\varepsilon) = \begin{bmatrix} 0 \\ \underline{1} \\ 0 \\ \varphi_i(\varepsilon) \end{bmatrix} \begin{matrix} \} \ \Omega_i \\ \\ \} \ \Omega_T \end{matrix} \tag{2}$$

where subvector $\varphi_i(\varepsilon)$ is given by

$$\varphi_i(\varepsilon) = (I - S(\varepsilon))^{-1}(\varepsilon)R_i(\varepsilon)\underline{1}, \tag{3}$$

Note that if some ergodic classes become transient after the perturbation, then the matrix valued function $(I - S(\varepsilon))^{-1}$ has a singularity at $\varepsilon = 0$. To further elaborate on this phenomenon, let us consider the structure of the substochastic matrix

$$S(0) = \begin{bmatrix} \mathrm{diag}\{\tilde{P}_1, ..., \tilde{P}_m\} & 0 \\ \tilde{R}_1 \cdots \tilde{R}_m & \tilde{S} \end{bmatrix}.$$

Blocks $\tilde{P}_1, ..., \tilde{P}_m$ represent the ergodic classes of the original Markov chain which merge with the transient set after the perturbation. Since each of $\tilde{P}_1, ..., \tilde{P}_m$ is a stochastic matrix, we conclude that matrix $I - S(0)$ has zero as an eigenvalue with multiplicity of at least m. Of course, $I - S(0)$ is not invertible. However, the matrix $(I - S(\varepsilon))^{-1}$ exists for small positive (but not zero) values of ε. From the results of [2, 3, 14], it follows that one can expand $(I - S(\varepsilon))^{-1}$ as a Laurent series at $\varepsilon = 0$

$$(I - S(\varepsilon))^{-1} = \frac{1}{\varepsilon^p}U^{(-p)} + ... + U^{(0)} + \varepsilon U^{(1)} + ... \tag{4}$$

One can use the methods of [2, 3, 14] to calculate the coefficients of the above series. These methods are computationally stable even for large matrices. Actually, as shown below, one needs to compute only $U^{(-1)}$ and $U^{(0)}$. Substituting the expression $R_i(\varepsilon) = R_i(0) + \varepsilon C_{Ri}$ and the Laurent series (4) into formula (3), we obtain the asymptotic expansion

$$\varphi_i(\varepsilon) = \varphi_i^{(0)} + \varepsilon\varphi_i^{(1)} + \varepsilon^2\varphi_i^{(2)} + ... , \tag{5}$$

where

$$\varphi_i^{(k)} = U^{(k)}R_i^{(0)} + U^{(k-1)}C_{Ri}, \quad k \geq 0. \tag{6}$$

The coefficients of the regular part of Laurent series (4) satisfy the recursion $U^{(k)} = (U^{(0)}C_S)U^{(k-1)}, k \geq 1$ [14]. This recursion together with (6) leads to the next updating formula

$$\varphi_i(\varepsilon) = U^{(-1)}C_{Ri}\underline{1} + [I - \varepsilon U^{(0)}C_S]^{-1}U^{(0)}R_i(\varepsilon)\underline{1}.$$

The above updating formula and the formula (6) are valid in the most general setting. Some interesting particular cases are discussed in [2].

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