Rate of convergence for the square root law in TCP

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Let $(\chi_n)_{n=0}^{\infty}$ be a sequence of iid random variables with $P\{\chi_n = \text{success}\} = 1 - p$ and $P\{\chi_n = \text{failure}\} = p$, and let the stochastic process $(W_{p,n})_{n=0}^{\infty}$ be defined by

$$W_{p,n+1} = \begin{cases} W_{p,n} + \frac{b}{W_{p,n}^{\alpha}} & \text{if } \chi_n = \text{success}, \\ \beta W_{p,n} & \text{if } \chi_n = \text{failure}. \end{cases} \tag{1}$$

For $b = 1, \alpha = 1, \beta = \frac{1}{2}$ we recognize the behavior of the TCP congestion window. (We make the asssumption of either one halving of the congestion window per lost packet, or of newReno, with modified counting of "good" and "full" acknowledgments only). As generalization to the square root law for TCP we have:

If $\alpha \geq 0, b > 0, 0 < \beta < 1$ are fixed and $p \downarrow 0$ the stationary distribution of the process $(\frac{p(W_{p,n})^{\alpha+1}}{(\alpha+1)b})_{n=0}^{\infty}$ converges weakly to the distribution of the random variable Z defined by

$$E[exp\{-sZ\}] = \prod_{k=0}^{\infty} \frac{1}{1 + \beta^{k(\alpha+1)}s}. \tag{2}$$

(Z is the sum of independent rescaled exponentially distributed random variables.)

It also is possible to give a "rate of convergence" result. Let U_p have the stationary distribution of $\frac{p(W_{p,n})^{\alpha+1}}{(\alpha+1)b}$. Then:

Theorem The joint distribution of Z and U_p can be chosen such, that for every sequence $(p_k)_{k=0}^{\infty}$ $(0 < p_k < 1$ for all $k)$ with $p_k \downarrow 0$ for $k \uparrow \infty$ there is a sub-sequence p_{k_j} such, that the sequence of random variables

$$D_j = \frac{|Z - U_{p_{k_j}}|}{p_{k_j}} \tag{3}$$

converges weakly to a random variable D which has the properties that $E[D^m] < \infty$ for all $m \geq 0$, and that for all $m \geq 0$, $E[\left(\frac{|Z-U_{p_{k_j}}|}{p_{k_j}}\right)^m] \to E[D^m]$ if $j \to \infty$.

Hence, the convergence is not just weak convergence, but also convergence in the mean square (etc), and the distance between U_p and Z is of the order p.

In the proof it is convenient to introduce the random variables X_p of which the distributions are given by

$$E[exp\{-sX_p\}] = \prod_{k=0}^{\infty} \frac{p}{1-(1-p)exp\{-pc^k s\}}, \tag{4}$$

i.e. X_p is the sum of independent rescaled geometrically distributed random variables. $Z - U_p = (Z - X_p) + (X_p - U_p)$, and for both components results similar to the theorem above can be obtained. The joint distributions can be chosen such, that (with probability one) both $Z \geq X_p$ and $U_p \geq X_p$.

More specifically, based on marginal distributions alone we can choose Z and X_p such that (with probability one)

$$0 \leq \frac{Z}{\frac{1}{p}\log\frac{1}{1-p}} - X_p \leq \frac{p}{1-\beta^{\alpha+1}}. \tag{5}$$

To obtain results for $U_p - X_p$ more work based on equation (1) is needed. That work will show that U_p and X_p have a natural joint distribution for which the necessary results will be obtained.

These results still leave open the possibility that $Z - U_p$ is of smaller order than p. However, in special situations that can be proven not to be the case.

The work this talk is based on can be found in the author's web page.