Microscopic behavior of TCP

Vincent Dumas   Fabrice Guillemin   Philippe Robert
Bert Zwart

May, 16 2002

Contents

References

[1] Vincent Dumas, Fabrice Guillemin, and Philippe Robert, A Markovian analysis of additive-increase multiplicative-decrease (AIMD) algorithms, Advances in Applied Probability **34** (2002), no. 1, –.

[2] Fabrice Guillemin, Philippe Robert, and Bert Zwart, AIMD algorithms and exponential functionals, Submitted to the Annals of Applied Probability (2002).

1. Introduction

Setting, Assumptions and Objectives

Setting, Assumptions and Objectives

Setting

- Long TCP connection (Elephant).
- Network is a black box rejecting some packets.
- Analysis of the successive congestion window sizes (W_n) when loss rate α is small.

Setting, Assumptions and Objectives

Setting

- Long TCP connection (Elephant).
- Network is a black box rejecting some packets.
- Analysis of the successive congestion window sizes (W_n) when loss rate α is small.

Assumptions

- Model for packet losses.

Setting, Assumptions and Objectives

Setting

- Long TCP connection (Elephant).
- Network is a black box rejecting some packets.
- Analysis of the successive congestion window sizes (W_n) when loss rate α is small.

Assumptions

- Model for packet losses.

Objectives

- Prove rigourous convergence results.
- Give explicit expressions of the distributions involved.

Congestion avoidance:
The additive increase and multiplicative decrease scheme

Congestion avoidance:
The additive increase and multiplicative decrease scheme

W_n is the size of the nth congestion window.
If $W_n = x \in \mathbb{N}$,

$$W_{n+1} = \begin{cases} \lfloor \delta x \rfloor & \text{if one of the } x \text{ packets is lost,} \\ x + 1 & \text{otherwise} \end{cases}$$

Stochastic models for packet losses: I

An elementary model: the non correlated case

- Each packet has a probability α of being lost.
- Packet losses are independent.

Stochastic models for packet losses: I

An elementary model: the non correlated case

- Each packet has a probability α of being lost.
- Packet losses are independent.

The independence assumption

Stochastic models for packet losses: I

An elementary model: the non correlated case

- Each packet has a probability α of being lost.
- Packet losses are independent.

The independence assumption

Reasonable: the network evolves rapidly.

Stochastic models for packet losses: I

An elementary model: the non correlated case

- Each packet has a probability α of being lost.
- Packet losses are independent.

The independence assumption

Reasonable: the network evolves rapidly.

Not realistic: losses occur by groups.

2. Non correlated case: Convergence results

(W_n^α): sequence of successive window sizes.

Transitions of the Markov chain.

If $W_0^\alpha = x$

— $W_1^\alpha = \lfloor \delta x \rfloor$, with probability $1 - \exp(-\alpha x)$.

— $W_1^\alpha = x + 1$, with probability $\exp(-\alpha x)$.

The invariant probability is $\pi^\alpha = (\pi_n^\alpha)$, W_∞^α is some random variable with distribution π^α.

The congestion window size is $\sim 1/\sqrt{\alpha}$.

The congestion window size is $\sim 1/\sqrt{\alpha}$.

Theorem. *If* $\lim_{\alpha \to 0} \sqrt{\alpha} W_0^\alpha = x$, *and*

$$W^\alpha(t) = \sqrt{\alpha} W^\alpha_{\lfloor t/\sqrt{\alpha} \rfloor},$$

The congestion window size is $\sim 1/\sqrt{\alpha}$.

Theorem. *If* $\lim_{\alpha\to 0}\sqrt{\alpha}W_0^\alpha = x,$ *and*

$$W^\alpha(t) = \sqrt{\alpha}W^\alpha_{\lfloor t/\sqrt{\alpha}\rfloor},$$

then $(W^\alpha(t))$ *converges in distribution to the Markov process* $(\overline{W}(t))$ *whose infinitesimal generator is given by*

$$\Omega(f)(x) = f'(x) + x\Big(f(\delta x) - f(x)\Big)$$

and $\overline{W}(0) = x$.

The embedded Markov chain (V_n^α)

V_n^α is the window size after the nth loss.

The embedded Markov chain (V_n^α)

V_n^α is the window size after the nth loss.

Theorem. *If* $\lim_{\alpha\to 0}\sqrt{\alpha}V_0^\alpha = \overline{v}$, *then* $(\sqrt{\alpha}V_n^\alpha)$ *converges in distribution to the Markov chain* $(\overline{V}_n)$, *with* $\overline{V}_0 = \overline{v}$ *and the transitions*

$$\overline{V}_{n+1} \stackrel{dist.}{=} \delta\left(\overline{V}_n + \overline{G}_{\overline{V}_n}\right),$$

where for $x \geq 0$,

$$\mathbb{P}\left(\overline{G}_x \geq y\right) = e^{-(xy+y^2/2)}.$$

Convergence diagrams

$$
\begin{tikzcd}
\left(\sqrt{\alpha} W^{\alpha}_{\lfloor t/\sqrt{\alpha} \rfloor}\right) \arrow[r, "t \to +\infty"] \arrow[d, "\alpha \to 0"'] & \sqrt{\alpha} W^{\alpha}_{\infty} \\
\left(\overline{W}(t)\right) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\left(\sqrt{\alpha} V^{\alpha}_{n}\right) \arrow[r, "n \to +\infty"] \arrow[d, "\alpha \to 0"'] & \sqrt{\alpha} V^{\alpha}_{\infty} \\
\left(\overline{V}_{n}\right) &
\end{tikzcd}
$$

Convergence diagrams

$$
\begin{tikzcd}
\left(\sqrt{\alpha}W^{\alpha}_{\lfloor t/\sqrt{\alpha}\rfloor}\right) \arrow[r, "t\to+\infty"] \arrow[d, "\alpha\to 0"'] & \sqrt{\alpha}W^{\alpha}_{\infty} \arrow[d] \\
\left(\overline{W}(t)\right) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\left(\sqrt{\alpha}V^{\alpha}_{n}\right) \arrow[r, "n\to+\infty"] \arrow[d, "\alpha\to 0"'] & \sqrt{\alpha}V^{\alpha}_{\infty} \\
\left(\overline{V}_{n}\right) &
\end{tikzcd}
$$

Convergence diagrams

$$\begin{tikzcd}
\left(\sqrt{\alpha}W^{\alpha}_{\lfloor t/\sqrt{\alpha}\rfloor}\right) \arrow[r, "t\to+\infty"] \arrow[d, "\alpha\to 0"'] & \sqrt{\alpha}W^{\alpha}_{\infty} \arrow[d] \\
\left(\overline{W}(t)\right) \arrow[r] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
\left(\sqrt{\alpha}V^{\alpha}_{n}\right) \arrow[r, "n\to+\infty"] \arrow[d, "\alpha\to 0"'] & \sqrt{\alpha}V^{\alpha}_{\infty} \\
\left(\overline{V}_{n}\right) &
\end{tikzcd}$$

Convergence diagrams

$$\begin{tikzcd}
\left(\sqrt{\alpha} W^{\alpha}_{\lfloor t/\sqrt{\alpha} \rfloor}\right) \arrow[r, "t \to +\infty"] \arrow[d, "\alpha \to 0"'] & \sqrt{\alpha} W^{\alpha}_{\infty} \arrow[d] \\
\left(\overline{W}(t)\right) \arrow[r] & \overline{W}(\infty)
\end{tikzcd}$$

$$\begin{tikzcd}
\left(\sqrt{\alpha} V^{\alpha}_{n}\right) \arrow[r, "n \to +\infty"] \arrow[d, "\alpha \to 0"'] & \sqrt{\alpha} V^{\alpha}_{\infty} \\
\left(\overline{V}_{n}\right) &
\end{tikzcd}$$

Convergence diagrams

$$
\begin{tikzcd}
\left(\sqrt{\alpha}W^{\alpha}_{\lfloor t/\sqrt{\alpha}\rfloor}\right) \arrow[r, "t\to+\infty"] \arrow[d, "\alpha\to 0"'] & \sqrt{\alpha}W^{\alpha}_{\infty} \arrow[d] \\
\left(\overline{W}(t)\right) \arrow[r] & \overline{W}(\infty)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\left(\sqrt{\alpha}V^{\alpha}_{n}\right) \arrow[r, "n\to+\infty"] \arrow[d, "\alpha\to 0"'] & \sqrt{\alpha}V^{\alpha}_{\infty} \arrow[d] \\
\left(\overline{V}_n\right) &
\end{tikzcd}
$$

Convergence diagrams

$$\begin{tikzcd}
\left(\sqrt{\alpha}W^{\alpha}_{\lfloor t/\sqrt{\alpha}\rfloor}\right) \arrow[r, "t\to+\infty"] \arrow[d, "\alpha\to 0"'] & \sqrt{\alpha}W^{\alpha}_{\infty} \arrow[d] \\
\left(\overline{W}(t)\right) \arrow[r] & \overline{W}(\infty)
\end{tikzcd}$$

$$\begin{tikzcd}
\left(\sqrt{\alpha}V^{\alpha}_{n}\right) \arrow[r, "n\to+\infty"] \arrow[d, "\alpha\to 0"'] & \sqrt{\alpha}V^{\alpha}_{\infty} \arrow[d] \\
\left(\overline{V}_{n}\right) \arrow[r] & {}
\end{tikzcd}$$

Convergence diagrams

$$\begin{tikzcd}
\left(\sqrt{\alpha}W^{\alpha}_{\lfloor t/\sqrt{\alpha}\rfloor}\right) \arrow[r, "t\to+\infty"] \arrow[d, "\alpha\to 0"'] & \sqrt{\alpha}W^{\alpha}_{\infty} \arrow[d] \\
\left(\overline{W}(t)\right) \arrow[r] & \overline{W}(\infty)
\end{tikzcd}$$

$$\begin{tikzcd}
\left(\sqrt{\alpha}V^{\alpha}_{n}\right) \arrow[r, "n\to+\infty"] \arrow[d, "\alpha\to 0"'] & \sqrt{\alpha}V^{\alpha}_{\infty} \arrow[d] \\
\left(\overline{V}_{n}\right) \arrow[r] & \overline{V}_{\infty}.
\end{tikzcd}$$

Convergence diagrams

$$
\begin{tikzcd}
\left(\sqrt{\alpha}W^{\alpha}_{\lfloor t/\sqrt{\alpha}\rfloor}\right) \arrow[r, "t\to+\infty"] \arrow[d, "\alpha\to 0"'] & \sqrt{\alpha}W^{\alpha}_{\infty} \arrow[d] \\
\left(\overline{W}(t)\right) \arrow[r] & \overline{W}(\infty)
\end{tikzcd}
$$

$$
\begin{tikzcd}
\left(\sqrt{\alpha}V^{\alpha}_{n}\right) \arrow[r, "n\to+\infty"] \arrow[d, "\alpha\to 0"'] & \sqrt{\alpha}V^{\alpha}_{\infty} \arrow[d] \\
\left(\overline{V}_n\right) \arrow[r] & \overline{V}_{\infty}.
\end{tikzcd}
$$

Theorem. *The above diagrams commute.*

Convergence diagrams

$$\begin{tikzcd}
\left(\sqrt{\alpha}W^{\alpha}_{\lfloor t/\sqrt{\alpha}\rfloor}\right) \arrow[r, "t\to+\infty"] \arrow[d, "\alpha\to 0"'] & \sqrt{\alpha}W^{\alpha}_{\infty} \arrow[d] \\
(\overline{W}(t)) \arrow[r] & \overline{W}(\infty)
\end{tikzcd}$$

$$\begin{tikzcd}
(\sqrt{\alpha}V^{\alpha}_{n}) \arrow[r, "n\to+\infty"] \arrow[d, "\alpha\to 0"'] & \sqrt{\alpha}V^{\alpha}_{\infty} \arrow[d] \\
(\overline{V}_{n}) \arrow[r] & \overline{V}_{\infty}.
\end{tikzcd}$$

Theorem. *The above diagrams commute.*

Proof: Exponential estimates of hittings times.

3. Non correlated case: The equilibrium

Proposition. *The following equality in distribution holds*

$$(x + \overline{G}_x)^2 \stackrel{dist.}{=} 2E_1 + x^2,$$

for $x \geq 0$, *where* E_1 *is an exponentially distributed random variable with parameter* 1.

Proposition. *The following equality in distribution holds*

$$(x + \overline{G}_x)^2 \stackrel{dist.}{=} 2E_1 + x^2,$$

for $x \geq 0$, *where* E_1 *is an exponentially distributed random variable with parameter* 1.

Consequence for the embedded chain:

$$\overline{V}_{n+1} = \delta\left(\overline{V}_n + \overline{G}_{\overline{V}_n}\right)$$

Proposition. *The following equality in distribution holds*

$$(x+\overline{G}_x)^2 \stackrel{dist.}{=} 2E_1 + x^2,$$

for $x \geq 0$, *where* E_1 *is an exponentially distributed random variable with parameter* 1.

Consequence for the embedded chain:

$$\overline{V}_{n+1}^{\,2} = \delta\left(\overline{V}_n + \overline{G}_{\overline{V}_n}\right)$$

Proposition. *The following equality in distribution holds*

$$(x + \overline{G}_x)^2 \stackrel{dist.}{=} 2E_1 + x^2,$$

for $x \geq 0$*, where* E_1 *is an exponentially distributed random variable with parameter* 1*.*

Consequence for the embedded chain:

$$\overline{V}_{n+1}^2 = \delta^2 \left(\overline{V}_n + \overline{G}_{\overline{V}_n}\right)^2$$

Proposition. *The following equality in distribution holds*

$$(x+\overline{G}_x)^2 \overset{dist.}{=} 2E_1 + x^2,$$

for $x \geq 0$, *where* E_1 *is an exponentially distributed random variable with parameter* 1.

Consequence for the embedded chain:

$$\begin{aligned}\overline{V}_{n+1}^2 &= \delta^2 \left(\overline{V}_n + \overline{G}_{\overline{V}_n}\right)^2 \\ &\overset{dist.}{=} \delta^2 \left(\overline{V}_n^2 + 2E_n\right)\end{aligned}$$

Proposition. *The following equality in distribution holds*

$$(x + \overline{G}_x)^2 \overset{dist.}{=} 2E_1 + x^2,$$

for $x \geq 0$, where E_1 is an exponentially distributed random variable with parameter 1.

Consequence for the embedded chain:

$$\begin{aligned} \overline{V}_{n+1}^2 &= \delta^2 \left(\overline{V}_n + \overline{G}_{\overline{V}_n}\right)^2 \\ &\overset{dist.}{=} \delta^2 \left(\overline{V}_n^2 + 2E_n\right) \end{aligned}$$

Proposition. *The sequence $\left(\overline{V}_n^2\right)$ is AR (Auto-Regressive):*

$$\overline{V}_{n+1}^2 = \delta^2 \left(\overline{V}_n^2 + 2E_n\right)$$

(E_n) *i.i.d. exponential variables with parameter 1.*

Invariant measure of the AR process

Theorem. *The distribution of the random variable* $\overline{V}_\infty$ *is given by*

$$\overline{V}_\infty^2/2 \stackrel{dist.}{=} \sum_{n=1}^{+\infty} \delta^{2n} E_n \stackrel{dist.}{=} \int_0^{+\infty} \delta^{2N(s)}\, ds.$$

where (E_i) *are i.i.d. exp. random variables with parameter* 1*;* N *Poisson process with parameter* 1*.*

Invariant measure of the AR process

Theorem. *The distribution of the random variable* $\overline{V}_\infty$ *is given by*

$$\overline{V}_\infty^{\,2}/2 \overset{dist.}{=} \sum_{n=1}^{+\infty} \delta^{2n} E_n \overset{dist.}{=} \int_0^{+\infty} \delta^{2N(s)}\, ds.$$

where (E_i) *are i.i.d. exp. random variables with parameter* 1*;*
N *Poisson process with parameter* 1*.*
Its density h_δ *given by*

$$h_\delta(x) = \frac{1}{\prod_{n=1}^{+\infty}(1-\delta^n)} \sum_{n=1}^{+\infty} \frac{1}{\prod_{k=1}^{n-1}(1-\delta^{-2k})} \delta^{-2n} x e^{-\delta^{-2n}x^2/2},$$

for $x \geq 0$*.*

Corollary. *The throughput of the TCP model is given by*

$$\lim_{\alpha\to 0}\sqrt{\alpha}\,\mathbb{E}\left(W_\infty^\alpha\right) \stackrel{def.}{=} \overline{\rho}(\delta) = \frac{1}{\mathbb{E}\left(\overline{V}_\infty\right)} = \sqrt{\frac{2}{\pi}}\prod_{n=1}^{+\infty}\frac{(1-\delta^{2n})}{(1-\delta^{2n-1})}.$$

Corollary. *The throughput of the TCP model is given by*

$$\lim_{\alpha\to 0}\sqrt{\alpha}\,\mathbb{E}\left(W_\infty^\alpha\right) \stackrel{def.}{=} \overline{\rho}(\delta) = \frac{1}{\mathbb{E}\left(\overline{V}_\infty\right)} = \sqrt{\frac{2}{\pi}}\prod_{n=1}^{+\infty}\frac{(1-\delta^{2n})}{(1-\delta^{2n-1})}.$$

$$\overline{\rho}(1/2) \sim 1.3098,$$

Corollary. *The throughput of the TCP model is given by*

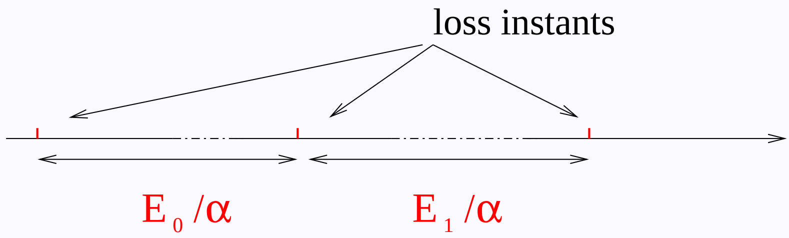
$$\lim_{\alpha \to 0} \sqrt{\alpha}\,\mathbb{E}\left(W_\infty^\alpha\right) \stackrel{def.}{=} \overline{\rho}(\delta) = \frac{1}{\mathbb{E}\left(\overline{V}_\infty\right)} = \sqrt{\frac{2}{\pi}} \prod_{n=1}^{+\infty} \frac{(1-\delta^{2n})}{(1-\delta^{2n-1})}.$$

$$\overline{\rho}(1/2) \sim 1.3098,$$

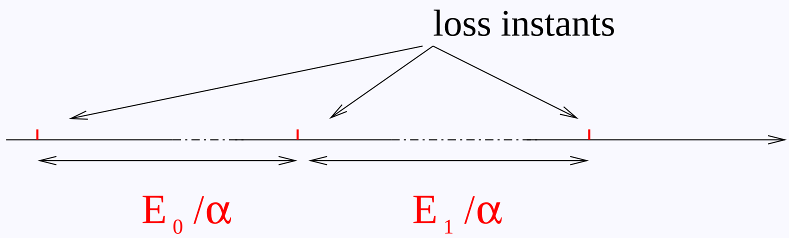
Tail distribution: $\mathbb{P}\left(\overline{W}_\infty \geq x\right) \sim C \exp\left(-x^2/\delta^2\right).$

4. General loss processes

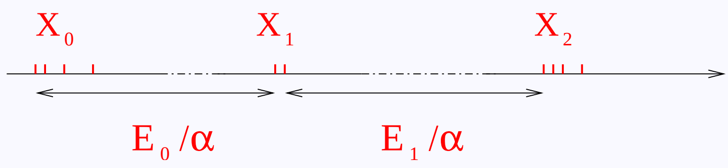
Non correlated model:



Non correlated model:



“Real” loss process (Paxson 1997):



Dynamic of congestion window size

$$\overline{V}_{n+1}^2 = \delta^2 \left(\overline{V}_n^2 + 2E_n \right)$$

Dynamic of congestion window size

$$\overline{V}_{n+1}^2 = \delta^{2X_n}\left(\overline{V}_n^2 + 2E_n\right)$$

Dynamic of congestion window size

$$\overline{V}_{n+1}^{2} = \delta^{2X_n}\left(\overline{V}_{n}^{2} + 2E_n\right)$$

(E_n) i.i.d. exponential variables with parameter 1,

(X_n) i.i.d. integer valued random variables.

Dynamic of congestion window size

$$\overline{V}_{n+1}^2 = \delta^{2X_n}\left(\overline{V}_n^2 + 2E_n\right)$$

(E_n) i.i.d. exponential variables with parameter 1,

(X_n) i.i.d. integer valued random variables.

Equilibrium

$$\overline{V}_\infty^2/2 \overset{dist.}{=} \beta^{X_0}\left(\overline{V}_\infty^2/2 + E_1\right)$$

$$\overline{V}_\infty^2/2 \overset{dist.}{=} I = \int_0^{+\infty} \beta^{X(s)}\,ds,$$

with $\beta = \delta^2$,

$$X(t) = \sum_{k=0}^{N(t)} X_i,$$

N Poisson process with parameter 1.

I is an integral exponential functional of a Lévy process

$$I = \int_0^{+\infty} e^{Y(s)}\, ds.$$

with $Y(t) = (X_1 + \cdots + X_{N(t)}) \log \beta$, N is a Poisson process with parameter 1.

I is an integral exponential functional of a Lévy process

$$I = \int_0^{+\infty} e^{Y(s)}\, ds.$$

with $Y(t) = (X_1 + \cdots + X_{N(t)}) \log \beta$, N is a Poisson process with parameter 1.

When $Y(t) = B(t) - \mu t$, $(B(t))$ Brownian motion and $\mu > 0$, I plays a role in mathematical finance models and statistical physics.

Work by Bertoin, Carmona, Monthus, Petit, Yor (and many others).

The distribution of I

The Laplace transform of I is given by,

$$\mathbb{E}\left(e^{-\lambda I}\right) = \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{\prod_{k=1}^{m}\left(1 - \mathbb{E}(\beta^{kX_1})\right)},$$

The distribution of I

The Laplace transform of I is given by,

$$\mathbb{E}\left(e^{-\lambda I}\right) = \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{\prod_{k=1}^{m}\left(1-\mathbb{E}(\beta^{kX_1})\right)},$$

If $X_1 \in \{1, \ldots, n+1\}$,

$$1 - \mathbb{E}\left(u^{X_1}\right) = (1-u)(1-a_1 u)\ldots)(1-a_n u),$$

$$\mathbb{E}\left(e^{-\lambda I}\right) = \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{(a_1\beta;\beta)_m \ldots (a_n\beta;\beta)_m(\beta;\beta)_m}$$

with the q-factorial $(a;q)_m = (1-a)(1-aq)\ldots(1-aq^{m-1})$ $1 \leq m \leq +\infty$.

The distribution of I

The Laplace transform of I is given by,

$$\mathbb{E}\left(e^{-\lambda I}\right) = \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{\prod_{k=1}^{m}\left(1-\mathbb{E}(\beta^{kX_1})\right)},$$

If $X_1 \in \{1, \ldots, n+1\}$,

$$1 - \mathbb{E}\left(u^{X_1}\right) = (1-u)(1-a_1 u)\ldots)(1-a_n u),$$

$$\mathbb{E}\left(e^{-\lambda I}\right) = \sum_{m=0}^{\infty} \frac{(-\lambda)^m}{(a_1\beta;\beta)_m \ldots (a_n\beta;\beta)_m(\beta;\beta)_m}$$

with the q-factorial $(a;q)_m = (1-a)(1-aq)\ldots(1-aq^{m-1})$ $1 \leq m \leq +\infty$.

The Laplace transform of I is related to a q-hypergeometric function.

Applications

The density of I is given by

$$\sum_{k=0}^{+\infty} r_k \beta^{-k} e^{-\beta^{-k} x},$$

Applications

The density of I is given by

$$\sum_{k=0}^{+\infty} r_k \beta^{-k} e^{-\beta^{-k} x},$$

1. $\mathbb{P}(X_1 = 1) = p = 1 - \mathbb{P}(X_1 = 2)$.

Applications

The density of I is given by

$$\sum_{k=0}^{+\infty} r_k \beta^{-k} e^{-\beta^{-k} x},$$

1. $\mathbb{P}(X_1 = 1) = p = 1 - \mathbb{P}(X_1 = 2)$.

$$r_k = \frac{1}{(-\beta(1-p);\beta)_\infty (\beta;\beta)_\infty (1/\beta;1/\beta)_k} \sum_{m=0}^{k} \frac{(1/\beta;1/\beta)_k}{(1/\beta;1/\beta)_m (1/\beta;1/\beta)_{k-m}} (-(1-p))^m.$$

Applications

The density of I is given by

$$\sum_{k=0}^{+\infty} r_k \beta^{-k} e^{-\beta^{-k} x},$$

1. $\mathbb{P}(X_1 = 1) = p = 1 - \mathbb{P}(X_1 = 2)$.

$$r_k = \frac{1}{(-\beta(1-p);\beta)_\infty (\beta;\beta)_\infty (1/\beta;1/\beta)_k} \sum_{m=0}^{k} \frac{(1/\beta;1/\beta)_k}{(1/\beta;1/\beta)_m (1/\beta;1/\beta)_{k-m}} (-(1-p))^m.$$

2. $\mathbb{P}(X_1 = n) = p^{n-1}(1-p)$, $n \geq 1$.

$$r_k = \frac{1}{(\beta;\beta)_\infty} \frac{(\beta^{k+1} p;\beta)_\infty}{(\beta;\beta)_{k-1}}$$

The fractional moments of I

Proposition. *For any* $s \in \mathbb{R}$, $-s \notin \mathbb{N} - \{0\}$, *if* $\mathbb{E}\left(\beta^{(s+1)X_1}\right) < +\infty$,

$$\mathbb{E}\left(I^s\right) = \Gamma(s+1) \prod_{k=1}^{+\infty} \frac{1 - \mathbb{E}\left(\beta^{(s+k)X_1}\right)}{1 - \mathbb{E}\left(\beta^{kX_1}\right)},$$

for $u \geq \min(s+1, 0)$.

Proposition. *The asymptotic throughput of an AIMD algorithm with multiplicative decrease factor* δ *in a correlated loss model is given by*

$$\overline{\rho}_X = \lim_{\alpha \to 0} \sqrt{\alpha \mathbb{E}(X_1)} \mathbb{E}\left(\overline{W}_\infty^\alpha\right) = \sqrt{\frac{2\mathbb{E}(X_1)}{\pi}} \prod_{n=1}^{+\infty} \frac{1 - \mathbb{E}\left(\delta^{2nX_1}\right)}{1 - \mathbb{E}\left(\delta^{(2n-1)X_1}\right)}.$$

Question: What are the properties of the mapping

$$X \to \overline{\rho}_X(\delta)?$$

Impact of correlation of the loss process

Definition. *[Concave ordering of random variables]*
The inequality $X \leq_{cv} Y$ *holds when*

$$\mathbb{E}(f(X)) \leq \mathbb{E}(f(Y))$$

is true for any non-decreasing concave function on $\mathbb{R}$.

Impact of correlation of the loss process

Proposition. *The asymptotic throughput* $X \to \overline{\rho}_X/\sqrt{\mathbb{E}(X)}$ *is a non-increasing function for the concave order,*

$$X \leq_{cv} Y \qquad \textit{implies} \qquad \frac{\overline{\rho}_X}{\sqrt{\mathbb{E}(X)}} \geq \frac{\overline{\rho}_Y}{\sqrt{\mathbb{E}(Y)}}. \tag{1}$$

In particular, when $\mathbb{E}(X) = \mathbb{E}(Y)$, $X \leq_{cv} Y$ *implies* $\overline{\rho}_X \geq \overline{\rho}_Y$.

Impact of correlation of the loss process

Proposition. *The asymptotic throughput* $X \to \overline{\rho}_X/\sqrt{\mathbb{E}(X)}$ *is a non-increasing function for the concave order,*

$$X \leq_{cv} Y \qquad \textit{implies} \qquad \frac{\overline{\rho}_X}{\sqrt{\mathbb{E}(X)}} \geq \frac{\overline{\rho}_Y}{\sqrt{\mathbb{E}(Y)}}. \tag{1}$$

In particular, when $\mathbb{E}(X) = \mathbb{E}(Y)$, $X \leq_{cv} Y$ *implies* $\overline{\rho}_X \geq \overline{\rho}_Y$.

Corollary. *For a fixed loss rate*

- *The more variable the loss process, the better the throughput.*
- *The non correlated model is worst than any loss model.*

Discussion

- Analogous results for Finite Maximum Window size.
- Slow start period. Duration $-\log_2(\alpha)$: negligible at the time scale $1/\sqrt{\alpha}$.
- Variable RTT's. Under weak assumptions, formulas are still valid even with dependence.
- Occurences of timeouts: $\mathbb{P}(X_1 = +\infty) > 0$.