# Adversarial Control in a Delay Tolerant Network

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## 1. Introduction and the Model

Consider $n$ relay mobile nodes, a source and a destination, assumed to be static. Whenever a relay mobile meets the source, the source may forward a packet to it. A mobile that receives a copy of the packet from the source can forward the same only if it meets the destination (two hop routing). The source meets each relay node according to a Poisson process with a parameter $\lambda$. Each relay node meets the destination according to a Poisson process with parameter $\gamma$. The source maximizes the probability that a packet arrives successfully at a given destination by time ρ. A second transmitter, tries to jam the transmission, and hence minimizes this probability. The jammer is located close to the source. Let $X_t, u_t, w_t$ denote, respectively, the fraction of mobiles with the message, the source's control and the jammer's control, where $u_t$ is the probability to transmit at time *t* if at that time the source meets a relay, $w_t$ is the probability of jamming at time *t*. If jamming and transmission occur simultaneously, then the transmitted packet is lost. Let $x_t = E[X_t]$ be the expected value of $X_t$ and it is generated by

$$\dot{x}_t = u_t(1-w_t)\lambda(n-x_t), \quad x_0 = x \quad (1)$$

During the incremental time interval $[t, t+dt)$, the number of copies of the packet in the network is $X_t$. Then, the number of packets the destination receives during $[t, t+dt)$, is a Poisson random variable with $\gamma X_t dt$ as parameter. So the probability of not receiving any copy of the packet during $[0, \rho]$, conditioned on $\{X_t\}$ is

$$\exp\left(-\int_0^\rho \gamma X_t dt\right).$$

Let *T* denote the first instant a copy reaches the destination. Then the failure probability is

$$P(T>\rho) = E\left[\exp\left(-\int_0^\rho \gamma X_t dt\right)\right].$$

Instead of minimizing $P(T>\rho)$, the failure probability, we will minimize its upper bound, obtained using Jensen's inequality:

$$\exp\left(-E\left[\int_0^\rho \gamma X_t dt\right]\right).$$

Minimizing the latter is equivalent to maximizing:

$$J(u,w) := \int_0^\rho \gamma x_t dt \cdot$$

We assume that the jammer wants to minimize this quantity and the source wants to maximize it. Let $\Pi_c, \Pi_j$ represent the set of policies for the source and the jammer, respectively. We say that $u^* \in \Pi_c$ and $w^* \in \Pi_j$ are saddle-point (SP) policies for the game $(J, \Pi_c, \Pi_j)$ if for every $u \in \Pi_c$ and $u \in \Pi_j$ we have

$$J(u,w^*) \le J(u^*,w^*) \le J(u^*,w).$$

Quantity $J(u^*,w^*)$ is called the value of the game. A policy is said to be *open loop* if it does not depend on the state of the system. It is said to be *Markov* (or a *feedback* policy) if its action at time $t$ depends upon $t$ as well as the state $x_t$. A *pure policy* is one for which the actions at all times are deterministic, i.e., either 0 or 1. Considering soft energy constraints, the source (jammer) maximizes $L_u$ (minimizes $L_w$), where

$$L_u := J(u,w) - \mu\int_0^\rho u_t dt$$

$$\text{and } L_w := J(u,w) + \theta\int_0^\rho w_t \, dt,$$

which thus results in a multi-criteria game. Let

$$L(x,u,w) = J(x,u,w) - \mu\int_0^\rho u_t dt + \theta\int_0^\rho w_t dt.$$

Let $G_{zs}$ be the zero-sum game (ZSG) in which the source maximizes $L(x,u,w)$ while the jammer minimizes it. Note that $L(x,u,w)$ is the result of either adding an extra term to $L_u$ or subtracting a term from $L_w$ and that the addition or the subtraction of these additional terms have not changed the Nash equilibrium (NE) of the multi-criteria game (this argument is not valid if the control policies depend on the state, that is if they are for example feedback policies). Then clearly if $(u^*,w^*)$ is an open-loop NE for $(L_u,-L_w)$ where both players are maximizers, it is also an open-loop NE for $(L,-L)$, and hence an open-loop SP of $L$ (i.e., the game $G_{zs}$ ). Likewise, any open-loop SP solution of the ZSG $G_{zs}$ is also an open-loop NE of $(L_u,-L_w)$. Below, we first consider a game with

$$\mathrm{L(x,u,w)} = \int_0^{\rho} (\gamma \mathrm{x}_t + \mathrm{r}(\mathrm{u}_t, \mathrm{w}_t)) dt,$$

where $r(u,w) = -\mu u + \theta w$ and $x_t$ is the solution of equation (1).

## 2. Static Game

We begin with $u$ and $w$ that are constants in time, in which case (1) has the unique solution:

$$x_t = n + (x_0 - n)\exp(-\lambda\kappa t)$$

where $\kappa := u(1-w)$. Static NE is a SP of $G_{zs}$ and has the following properties:

**Theorem 1** *i) If $x_0 < n$, the game has a SP.*

*ii) If $\gamma(n-x_0)\rho\lambda < \mu$, the game has $(0,0)$ as the unique SP.*

*iii) The game cannot have a SP with $w=1$.*

*iv) If the SP is in the open square $(0,1)\times(0,1)$, then it is unique.* ◊

## 3. NE of Open-Loop and Closed-Loop Dynamic Games

We first consider the open-loop case. Here, every NE is also a SP of $G_{zs}$. Hence, we have a single Hamiltonian:

$$H = -\mu u + \theta w + \gamma x + pu(1-w)\lambda(n-x)$$

which we will be maximizing over $u \in [0,1]$ and minimizing over $w \in [0,1]$. The co-state variable $p$ satisfies the co-state equation:

$$\dot{p} = -\frac{\partial H}{\partial x} = pu(1-w) - \gamma$$

and $x$ satisfies the original state equation (1). The open-loop SP solution is captured below:

**Theorem 2** *i) Let $\theta < \mu$. There exists a $t_s$ such that $u^* = w^* = 0$ for $t > t_s$ and for $t < t_s$, $u^* = \theta / m(t)$ and $w^* = 1 - \mu / m(t)$, where $m(t) = p(t)\lambda(n - \xi(t))$, with $p$ and $\xi$ solving the coupled differential equations:*

$$\dot{\xi} = \frac{\theta\mu}{p^2\lambda(n-\xi)}, \ \xi(0) = x_0, \ \dot{p} = \frac{\theta\mu}{p\lambda^2(n-\xi)} - \gamma,$$

*$p(t_s) = \gamma(\rho - t_s)$ and $t_s$ solves $m(t_s) = \mu$.*

*ii) When $\theta \ge \mu$, there exists an additional threshold $t_{\bar{\bar{s}}}$ with $t_s \le t_{\bar{\bar{s}}}$ such that the source policy changes to $u^* = 1$ when $t \in [t_s, t_{\bar{\bar{s}}}]$. Policy $u^*$ for $t \notin [t_s, t_{\bar{\bar{s}}}]$ and $w^*$ for all $t$ are as in (i).* ◊

We next consider the closed-loop (feedback) case. Here we have to stay with the non-cooperative game framework, and seek for NE. Let $V^u$ and $V^w$ be the value functions. The associated HJB equations are:

$$\frac{\partial \mathrm{V^u}}{\partial \mathrm{t}} + \max_{\mathrm{u}\in[0,1]}\left[\frac{\partial \mathrm{V^u}}{\partial \mathrm{x}}\mathrm{u}(1-\mathrm{w}^*)\lambda(\mathrm{n}-\mathrm{x}) + \gamma\mathrm{x} - \mu\mathrm{u}\right] = 0$$

$$\frac{\partial \mathrm{V^w}}{\partial \mathrm{t}} + \min_{\mathrm{w}\in[0,1]}\left[\frac{\partial \mathrm{V^w}}{\partial \mathrm{x}}\mathrm{u}^*(1-\mathrm{w})\lambda(\mathrm{n}-\mathrm{x}) + \gamma\mathrm{x} + \theta\mathrm{w}\right] = 0.$$

with $\mathrm{V^u}(\rho,\mathrm{x}) \equiv \mathrm{V^w}(\rho,\mathrm{x}) \equiv 0$ (boundary conditions) and where $(\mathrm{u}^*, w^*)$ is a NE. The corresponding NE is the argmax and argmin of these equations and has the following structure:

**Theorem 3** *i) For any NE, $w^* < 1$ for all $t$.*

*ii) If $\gamma\lambda(\mathrm{n}-\mathrm{x}_0)\rho < \mu$, then $u^* \equiv w^* \equiv 0$.*

*iii) Let $t_c(x) := \rho\gamma\lambda(n-x) - \mu / \gamma\lambda(n-x)$. If $\mu - \lambda\theta(\rho - \mathrm{t_c}(\mathrm{x}_0)) > \theta$, the NE exists with the optimal state trajectory given by:*

$$\dot{\mathrm{x}}(\mathrm{t}) = \frac{\mu\,\theta\,(\mathrm{n}-\mathrm{x})1_{\{\mathrm{t}\le \mathrm{t_c}(\mathrm{x})\}}}{(\rho-\mathrm{t})\,(\gamma^2\lambda\,(\rho-\mathrm{t})\,(\mathrm{n}-\mathrm{x})^2 - \mu\,\theta)}$$

*and the optimal controls are given by,*

$$\mathrm{u}^*(\mathrm{t}) = \frac{\theta\ 1_{\{\mathrm{t}\le \mathrm{t_c}(\mathrm{x_t})\}}}{\lambda(\mathrm{n}-\mathrm{x_t})\left(\gamma(\rho-\mathrm{t}) - \dfrac{\mu\theta}{\gamma\lambda(n-x_t)^2}\right)}$$

$$\mathrm{w}^*(\mathrm{t}) = \left(1 - \frac{\mu}{\gamma(\rho-\mathrm{t})\,\lambda\,(\mathrm{n}-\mathrm{x_t})}\right)1_{\{\mathrm{t}\le \mathrm{t_c}(\mathrm{x_t})\}}.$$

*iv) When $\theta$ is larger, the optimal policy has two switch time thresholds as in Theorem 2.iii.* ◊

## 4. Conclusions

We have considered a multi-criteria control problem that arises in DTNs with two adversarial controllers: source and jammer and two types of information structures: closed and open loop, and in the latter case also with restriction to static policies. The structure of the equilibrium is similar under both open-loop and closed-loop structures. When the jammer has a tighter constraint on its energy than the source, the policies have two switch times. After the first switch time, the jammer switches off and the source transmits at maximum probability and after the second switch time, the source also switches off. When the source has a tighter energy constraint, there exists only one switch time after which both are switched off.

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**Reference**

[1] E. Altman, T. Başar and V. Kavitha, "Adversarial Control in a Delay Tolerant Network," GameSec 2010 (Conference on Decision and Game Theory for Security, 22-23 November 2010, Berlin, Germany).

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