Problems of Adamjan–Arov–Krein type on subsets of the circle and minimal norm extensions

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Abstract

We constructively solve a pair of band-limited generalizations of the Adamjan-Arov-Krein problem. The first one consists in extending a function given on a proper subset of the unit circle to the whole circle so as to make it as close as possible to meromorphic with prescribed number of poles, in the *sup* norm, while meeting some gauge constraint. The second consists in directly approximating the given function on the proper subset by the restriction of a meromorphic function, again meeting some gauge constraint.

1 Introduction

1.1 Framework and motivation

The problem under study in this paper may be considered as a means of making the analytic continuation principle computationally effective in the disk for a function which is merely known on a subset of the circle. This issue arises in a variety of inverse problems ranging from deconvolution and the identification of linear control systems or stochastic processes to inverse scattering and Neumann–Dirichlet problems, see *e.g.* [6, 10, 13, 16, 19, 18, 34, 38, 40] and their bibliography. Although the Carleman theory formally provides a solution as the limit of certain singular integrals [2], the question is well-known to be ill-posed [31]: if, for instance, the boundary data do not *exactly* match those of an analytic function, then Carleman's recovery formulas diverge [5]. We shall in fact take up a slightly more general issue by allowing the function to be recovered to have a prescribed number of poles in the disk.

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We will formulate this recovery issue as an extremal problem with *a priori* constraints on the portion of the boundary where no data are available. It appears then as a generalization to arbitrary subsets of the unit circle $\mathbb{T}$ of an Adamjan–Arov–Krein (in short, AAK) problem [1]. Providing as it does a link between meromorphic approximation and Hankel operators, the AAK theory has engaged the interest of many researchers in functional analysis [39, 42] and has already found many applications to approximation, interpolation, control theory and signal processing [22, 26, 34, 40].
The present paper can be seen partly as a sequel to [7, 10], where a version of the Nehari problem, based on data provided on a proper subset of the circle, was applied to the question of band-limited frequency-domain identification of stable linear systems. Here we resolve the problem of finding the best completion of a partial model—i.e., of extending a function to the missing part of the boundary so as to make it as close to meromorphic with N poles as possible in the *sup* norm while meeting some gauge constraint.
In order to keep the solution to this problem constructive, we show that it is generically well behaved in Hölder–Zygmund classes from the numerical viewpoint; this will rest on continuity properties of meromorphic approximation [35, 37] and on the generic non–multipleness of Hankel singular values. An algorithm is described at the end of the paper and used to compute numerical experiments.

1.2 Notations and preliminaries

In the following, $\mathbb{D}$ will denote the unit disk of the complex plane and $\mathbb{T}$ the unit circle. We abbreviate the set of positive real numbers by $\mathbb{R}^+ := \{x \in \mathbb{R},\ x > 0\}$. When $E \subset \mathbb{T}$, we write $C(E)$ for the space of continuous complex-valued functions on E while $L^p(E)$ designates the familiar Lebesgue space for $1 \le p \le \infty$. We let $H^p \subset L^p(\mathbb{T})$ be the Hardy space with exponent p of $\mathbb{D}$ consisting of functions with vanishing Fourier coefficients of negative index. Alternatively, H^p is the set of nontangential limits of functions analytic in $\mathbb{D}$ whose modulus has uniformly bounded L^p-mean over circles centered at 0 included in $\mathbb{D}$; when $p = \infty$, this simply says that the function is bounded in $\mathbb{D}$. When $p = 2$, we also introduce the conjugate Hardy space $\bar{H}_0^2$ which is the orthogonal complement to H^2 in $L^2(\mathbb{T})$, that is to say the subspace of functions with vanishing Fourier coefficients of non-negative index. Equivalently, $f \in \bar{H}_0^2$ if, and only if, $\bar{f} \in H^2$ and $\bar{f}$ has zero mean on $\mathbb{T}$.
The norm on $L^p(E)$ will be the natural one, denoted by $\| \ \|_{L^p(E)}$. In $L^\infty(E)$, we write $d(\phi, S)$ for the distance of the element ϕ to the subset S; we use the same notation regardless of $E \subset \mathbb{T}$, but the context will keep the meaning clear.
The balls in $L^\infty(E)$ are weak-* compact. We use this well-known fact on several occasions in its sequential form: any sequence (f_n) with $\|f_n\|_{L^\infty(E)} \le C$ contains a subsequence (f_{n_k}) such that $\lim_{k\to\infty} \int_E f_{n_k} h \, d\theta = \int_E f h \, d\theta$ for every $h \in L^1(E)$ and some f satisfying $\|f\|_{L^\infty(E)} \le C$. We also recall a theorem of Douglas [20, 25] that the space $H^\infty + C(\mathbb{T})$ is a subalgebra of $L^\infty(\mathbb{T})$.
The subscript $|_E$ applied to a function or to a set of functions indicates restriction to E; for instance, $H^p_{|_E}$ is the space of traces on E of H^p functions. Whenever f is defined on E and h is defined on its complement $\mathbb{T} \setminus E$, then $f \vee h$ stands for the concatenated function which is defined on all of $\mathbb{T}$.

Recall that the modulus of continuity of a function $f \in C(E)$ is defined by

$$\omega_f(\delta) := \sup_{\substack{x,y\in E\\ |x-y|\leq\delta}} |f(x)-f(y)|.$$

For $0<\alpha<1$ fixed, introduce the Hölder–Zygmund[1] class:

$$\Lambda_\alpha(E) := \{f\in C(E)\,,\ \omega_f(\delta)=O(\delta^\alpha)\},$$

with norm

$$\|f\|_\alpha := \|f\|_{L^\infty(E)} + \sup_{\delta>0}\frac{\omega_f(\delta)}{\delta^\alpha}.$$

We define the *separable* Hölder–Zygmund class $\lambda_\alpha(E)$ to be the subspace of those functions f in $\Lambda_\alpha(E)$ such that

$$\frac{\omega_f(\delta)}{\delta^\alpha}\to 0 \qquad \text{as } \delta\to 0.$$

We will use the following subsets:

$$\begin{array}{rcl}
\lambda_\alpha^+(E) & := & \{f\in\lambda_\alpha(E)\,,\ f\geq 0\},\\
\lambda_\alpha^{+,*}(E) & := & \{f\in\lambda_\alpha^+(E)\,,\ \inf_E f>0\},\\
\lambda_\alpha^{+,0}(E) & := & \{f\in\lambda_\alpha^+(E)\,,\ f=0 \text{ at boundary points of } E\}.
\end{array}$$

We also introduce

$$\begin{array}{rcl}
\mathcal{M} & := & \{M\in L^\infty(\mathbb{T}\setminus K)\,,\ M\geq 0 \text{ a.e. on } \mathbb{T}\setminus K\},\\
\mathcal{M}_* & := & \{M\in\mathcal{M}\,,\ \inf_{\mathbb{T}\setminus K} M>0\}.
\end{array}$$

The symbol $\|\ \|_{op}$ will be reserved for the norm of an operator from one Banach space into another.
For $N\geq 0$, we let $P_N[z]$ be the space of complex algebraic polynomials in the variable z of degree at most N, and we further define $R_N\subset L^\infty(\mathbb{T})$ to be the set of rational functions with no poles on $\mathbb{T}$ and at most N poles in $\mathbb{D}$. Finally, $\mathcal{T}$ denotes the space of trigonometric polynomials.

1.3 About AAK approximation

An AAK approximation problem consists by definition in solving for

$$\min_{g\in H^\infty+R_N}\|h-g\|_{L^\infty(\mathbb{T})}\,, \tag{1}$$

where $h\in L^\infty(\mathbb{T})$ is given. The following facts are well-known [33, 39, 42]:
(a) a solution g_N does exist;
(b) the solution need not be unique but it is unique at least if $h\in H^\infty+C(\mathbb{T})$, and the error function $|h-g_N|$ is then constant a.e. on $\mathbb{T}$.

[1] Also commonly called Lipschitz class.

Very much like for conjugate functions, it turns out that g_N need not be continuous even if f is, but g_N will for instance be continuous if f is Dini-continuous[2]. The solution to the AAK problem above is closely related to the *Hankel operator with symbol h*

$$\Gamma_h \,:\, H^2 \to \bar{H}^2_0 \quad \text{given by} \quad \Gamma_h\, v = P_{\bar{H}^2_0}\,(h\, v), \tag{2}$$

where $P_{\bar{H}^2_0}$ is the orthogonal projection from $L^2(\mathbb{T})$ onto $\bar{H}^2_0$. Specifically, a classical theorem of Hartman asserts that Γ_h is compact if and only if $h \in H^\infty + C(\mathbb{T})$, and in this case, if v_N is an eigenvector of $\Gamma_h^*\Gamma_h$ associated with the $N+1$-st largest eigenvalue, we have that

$$g_N = h - \frac{\Gamma_h\, v_N}{v_N};$$

moreover $|h - g_N|$ is equal almost everywhere on $\mathbb{T}$ to the square root of this eigenvalue[3] which is thus the value of problem (1).
A well-known theorem of Kronecker [33, 42] asserts that Γ_h has finite rank if, and only if, h belongs to $H^\infty + R_m$ for some m. This is the only case where AAK approximants can easily be computed, for then the non-zero singular values and their associated vectors can be obtained as those of the finite matrix representing $\Gamma_h^*\Gamma_h$ on the orthogonal space to $\operatorname{Ker}\Gamma_h$ in H^2. This matrix is easily computed from a rational symbol [22, 23, 26]. Due to the lack of continuity of the best approximation operator $C(\mathbb{T}) \to H^\infty + R_N$ [32, 37], situations where no rational symbol is available are much more delicate, but they can be handled generically in smoother classes of functions, see [27, 35] and the last two sections of the present paper.

1.4 Statement of the problems

Let K be a subset of $\mathbb{T}$. We refer to the following as the *Bounded Completion Problem*:

Problem 1 *For $\psi \in L^\infty(\mathbb{T}\setminus K)$ and $M \in \mathcal{M}$, define:*

$$\mathcal{D}_{M,\psi} := \{h \in L^\infty(\mathbb{T}\setminus K)\,,\ |h-\psi| \le M \text{ a.e. on } \mathbb{T}\setminus K\}\,.$$

Given $f \in L^\infty(K)$ and some integer $N \ge 0$ an integer, we seek $h_N \in \mathcal{D}_{M,\psi}$ such that

$$d(f \vee h_N, H^\infty + R_N) = \min_{h\in\mathcal{D}_{M,\psi}} d(f\vee h, H^\infty + R_N) := \gamma_N(f\vee\psi, M)\,. \tag{3}$$

In words, we are given a bounded function on a subset of the circle and a gauge on the complementary subset, and we seek an extended definition to the whole circle that meets the gauge and makes the global function as close as possible to a meromorphic function with N poles. Companion to problem 1 will be the *Bounded Extremal Problem*:

[2]By definition, a function f is Dini-continuous if its modulus of continuity satisfies $\int_0^a \frac{\omega_f(t)}{t}dt < \infty$ for some $a > 0$.

[3]This quantity is the $N+1$-st singular value of Γ_h to be defined in a slightly different but equivalent manner in (14).

Problem 2 *For $\psi \in L^\infty(\mathbb{T} \setminus K)$, $M' \in \mathcal{M}$, and $N \geq 0$ an integer, define:*

$$\mathcal{B}^N_{M',\psi} := \{g \in H^\infty + R_N\,,\ |g - \psi| \leq M' \text{ a.e. on } \mathbb{T} \setminus K\}\,.$$

Given $f \in L^\infty(K)$, we seek $g_N \in \mathcal{B}^N_{M',\psi}$ such that

$$\|f - g_N\|_{L^\infty(K)} = \min_{g \in \mathcal{B}^N_{M',\psi}} \|f - g\|_{L^\infty(K)} := \beta_N(f \vee \psi, M'). \tag{4}$$

The bounded extremal problem is close in spirit to the bounded completion: here again we are given a bounded function on a subset of the circle and a gauge on the complementary subset, but this time we directly seek a meromorphic function with N poles which is closest to the function while meeting the gauge.
When f and ψ are fixed and understood from the context, we occasionally write $\beta_N(M')$ for $\beta_N(f \vee \psi, M')$ and $\gamma_N(M)$ for $\gamma_N(f \vee \psi, M)$.
If $K = \mathbb{T}$, then $\mathcal{B}^N_{M',\psi} = H^\infty + R_N$ and problem 2 reduces to AAK approximation, while problem 1 is devoid of meaning. On the other hand, if $K = \emptyset$, then problem 2 makes no sense and problem 1 amounts to finding a function h_N whose distance γ_N to $H^\infty + R_N$ is minimal in $\mathcal{D}_{M,\psi} \subset L^\infty(\mathbb{T})$. This non-classical question contains *in nuce* all the ingredients of our approach. Specifically, let $w_{M+\varepsilon}$ denote for any $\varepsilon > 0$ the outer factor with modulus $M+\varepsilon$ and σ_{N+1} be the $(N+1)$-st singular value of the Hankel operator with symbol $\psi\, w^{-1}_{M+\varepsilon}$. Then, either $\sigma_{N+1} \leq 1$ for every $\varepsilon > 0$, in which case $(H^\infty + R_N) \cap \mathcal{D}_{M,\psi} \neq \emptyset$ and h_N can be any member of the intersection, or else γ_N is implicitely defined as the unique positive real number ε such that $\sigma_{N+1} = 1$; in this case, the solution is

$$h_N = \frac{M}{M + \gamma_N}\, v_N\, w_{M+\gamma_N} + \left(1 - \frac{M}{M + \gamma_N}\right) \psi\,,$$

where v_N is the best AAK approximant with N poles to $\psi\, w^{-1}_{M+\gamma_N}$. The patient reader will have no problem in verifying this by mimicking the proofs of theorems 3 and 4.
We shall study problem 1 when both K and $\mathbb{T} \setminus K$ have positive Lebesgue measure, and we find that it reduces to problem 2 which in turn reduces, although in an implicit manner, to AAK approximation. When $N = 0$, problem 2 was analysed by the authors in [7] when M' is a constant, and in [10] for positive continuous M'; we shall carry these results over to $N \geq 0$ and to arbitrary constraints $M' \in \mathcal{M}_*$ on our way to solving problem 1. In another connection, we mention that, when $N = 0$, L^2 versions of problems 1 and 2 have been treated in [5, 11, 30] and that 2 was also considered in a Hardy-Sobolev class [4]. But for $N > 0$, where the L^2 analog of AAK approximation is best rational approximation in H^2, the corresponding questions are open; an L^p version of the AAK problem on $\mathbb{T}$ is proposed in [12] for $p \geq 2$ although the occurence of local minima may hinder numerical computations.

The outline of the paper is as follows. In section 2 we prove the existence of a solution to problem 1 and, when the set $\mathcal{B}^N_{M',\psi}$ is not empty, to problem 2. In section 3, we show that problem 1 reduces to problem 2 which is itself implicitly equivalent to AAK approximation. In section 4, assuming that $f \vee \psi \in H^\infty + C(\mathbb{T})$ and K has an interior point, we establish the continuity of γ_N and β_N along with a reciprocity relation that make our reduction to

AAK approximation computationally effective; we then get uniqueness and error circularity for both problems 1 and 2. Section 5 deals with generic non-multipleness of Hankel singular values and contains the technical developments needed to show that problem 1 is well-posed, provided $(f \vee \psi, M)$ lies in some dense and open subset of $\lambda_\alpha(\mathbb{T}) \times \lambda_\alpha^{+,0}(\mathbb{T} \setminus K)$. The α–Lipschitz smoothness required from M and $f \vee \psi$ is reasonable for many applications. That M vanishes at the endpoints of K may seem unnecessarily restrictive at first glance but is really needed if one wants $f \vee h_N$ to be continuous, a mandatory feature in most modelling problems. Section 6 subsequently describes a constructive algorithm in this case which is illustrated in section 7 by numerical experiments.

From now on, we tacitly assume that both K and $\mathbb{T} \setminus K$ have positive Lebesgue measure.

2 Existence of solutions

Recall that a Blaschke product of degree m is a rational function of the form

$$b(z) = e^{i\varphi} \Pi_{j=1}^{m} \frac{z - a_j}{1 - \bar{a}_j\, z}\,,$$

where φ is real and the a_j are complex numbers of modulus less than 1. Such functions may be characterized as being analytic on the closed disk and having modulus 1 on $\mathbb{T}$ [25].

Lemma 1 *For any integer $N \geq 0$,*

$$H^\infty + R_N = \{b^{-1} g \;:\; b,\ g \in H^\infty\,,\ b \text{ a Blaschke product of degree } \leq N\} \tag{5}$$

is weak- closed in $L^\infty(\mathbb{T})$.*

Proof: first, observe that the right–hand side of (5) is indeed equal to $H^\infty + R_N$ since the latter consists of functions meromorphic in $\mathbb{D}$ with at most N poles that remain bounded near $\mathbb{T}$. Let (ϕ_j) be a sequence of functions in $H^\infty + R_N$ which converges weakly-* to $\phi \in L^\infty(\mathbb{T})$. Then, the sequence of Hankel operators (Γ_{ϕ_j}) converges weakly to Γ_ϕ, and it follows from Kronecker's theorem that all of these operators have rank at most N and hence that $\phi \in H^\infty + R_N$. Thus, $H^\infty + R_N$ is weak-* sequentially closed, and the lemma follows because $L^1(\mathbb{T})$ is separable whence the weak-* topology is metrizable on bounded subsets of $L^\infty(\mathbb{T})$ [15, thms III.15, III.25]. ∎

Before producing in the next sections a constructive solution to problem 1, we need the following result.

Theorem 1 *There exists a solution to problem* 1*.*

Proof: let (h_j) in $\mathcal{D}_{M,\psi}$ be a minimizing sequence for (3). By weak-* compactness, (h_j) converges weakly-* to some $h \in \mathcal{D}_{M,\psi}$ up to a subsequence. By property (a) of AAK approximation in section 1.3, there exists for each j a function $k_j \in H^\infty + R_N$ such that

$\|f \vee h_j - k_j\|_{L^\infty(\mathbb{T})} = d(f \vee h_j, H^\infty + R_N)$. Up to a subsequence, (k_j) converges weakly-* to some $k \in H^\infty + R_N$, by lemma 1, and

$$\|f \vee h - k\|_{L^\infty(\mathbb{T})} \leq \lim_{j\to\infty} \|f \vee h_j - k_j\|_{L^\infty(\mathbb{T})} = \gamma_N(f \vee \psi, M)\,,$$

so that h provides us with a solution. ∎

As far as existence is concerned, there is a slight difference between problems 1 and 2, namely the set $\mathcal{B}^N_{M',\psi}$ of candidate approximants may in the latter case be empty. This happens, for example, when $d(\psi, (H^\infty + R_N)_{|\mathbb{T}\setminus K}) > \sup_{\mathbb{T}\setminus K} M'$, which is perfectly possible as there exist functions at positive distance from $(H^\infty + R_N)_{|\mathbb{T}\setminus K}$ in $L^\infty(\mathbb{T} \setminus K)$. Indeed, using lemma 1, the proof of [7, lem. 1] is easily adapted to show that any $\xi \in H^\infty$ verifying

- $\|\xi\|_{L^\infty(\mathbb{T}\setminus K)} \leq 1$,
- $\xi^{-1}_{|\mathbb{T}\setminus K} \in L^\infty(\mathbb{T} \setminus K)$, and
- the zeros of ξ either accumulate at some interior point of $\mathbb{T} \setminus K$ or else accumulate nontangentially at some density point of $\mathbb{T} \setminus K$,

is such that $\xi^{-1}_{|\mathbb{T}\setminus K}$ is at distance at least 1 from $(H^\infty + R_N)_{|\mathbb{T}\setminus K}$. In particular, $\mathcal{B}^N_{M',\xi^{-1}} = \emptyset$ for all $N \geq 0$ whenever $M' \in \mathcal{M}$ verifies $\sup_{\mathbb{T}\setminus K} M' < 1$. Describing those ψ for which $\mathcal{B}^N_{M',\psi} \neq \emptyset$ for all $M' \in \mathcal{M}_*$ (or even for constant $M' > 0$) is equivalent to characterizing the closure of $(H^\infty + R_N)_{|\mathbb{T}\setminus K}$ in $L^\infty(\mathbb{T} \setminus K)$, and this is an open issue even when K is an arc. To check that $\mathcal{B}^N_{M',\psi} \neq \emptyset$, we shall rely on an elementary sufficient condition:

Lemma 2 *If $K \subset \mathbb{T}$ has an interior point and $\psi \in (H^\infty + C(\mathbb{T}))_{|\mathbb{T}\setminus K}$, then $\mathcal{B}^N_{M',\psi} \neq \emptyset$ for every $M' \in \mathcal{M}_*$ and every integer $N \geq 0$.*

Proof: it follows immediately from Runge's theorem that already $\mathcal{B}^0_{M',\psi} \neq \emptyset$ in this case, see the proof of [7, thm 1,(*iii*)]. ∎

Remark 1 Lemma 2 shows that problem 1 would not have any solution in general if we omitted the constraint $h \in \mathcal{D}_{M,\psi}$. Indeed, if $\mathbb{T} \setminus K$ has an interior point and $f \in (H^\infty + C(\mathbb{T}))_{|K}$, we can find a sequence of functions in $H^\infty + R_N$ that will converge to f on K, although the argument in [7, prop. 2] shows this sequence is necessarily unbounded on $\mathbb{T}\setminus K$ if $f \notin (H^\infty + R_N)_{|K}$, see also [28, thm 16]. A similar remark holds for problem 2 if the constraint $g \in \mathcal{B}^N_{M',\psi}$ is omitted.

Theorem 2 *There exists a solution to problem 2 provided $\mathcal{B}^N_{M',\psi} \neq \emptyset$.*

Proof: let (g_j) in $\mathcal{B}^N_{M',\psi}$ be a minimizing sequence for (4). Since it is bounded in $L^\infty(\mathbb{T})$ some subsequence converges weakly-* to a function $g \in H^\infty + R_N$ in view of lemma 1. Up to another subsequence, we may assume that $M' - |\psi - g_j|$ also converges weakly-* in $L^\infty(\mathbb{T}\setminus K)$. Because weak-* limit preserves non-negativity, we now see that $g \in \mathcal{B}_{M',\psi}$ and that:

$$\|f - g\|_{L^\infty(K)} \leq \lim_{j\to\infty} \|f - g_j\|_{L^\infty(K)} = \beta_N(f \vee \psi, M')\,.$$

∎

3 Reduction to AAK approximation

We first show how the bounded completion problem 1 reduces to a bounded extremal one of type 2. It will be convenient formally to define the trace of $\mathcal{B}^N_{M',\psi}$ on K:

$$\mathcal{C}^N_{M',\psi} = \{g_{|K}\,,\ g \in \mathcal{B}^N_{M',\psi}\} \subset L^\infty(K)\,.$$

Theorem 3 *Let $f \in L^\infty(K)$, $\psi \in L^\infty(\mathbb{T} \setminus K)$, and $N \geq 0$ an integer. For $M \in \mathcal{M}$, set*

$$M' = M + \gamma_N(f \vee \psi, M) \text{ on } \mathbb{T} \setminus K\,. \tag{6}$$

Then $\mathcal{B}^N_{M',\psi} \neq \emptyset$ and we have the inequality:

$$\beta_N(M') \leq \gamma_N(M). \tag{7}$$

If $f \notin \mathcal{C}^N_{M,\psi}$, so that $\gamma_N(M) > 0$ whence $M' \in \mathcal{M}_*$, and if g_N is a solution to problem 2, then

$$h_N = \frac{M}{M'}\, g_N + \left(1 - \frac{M}{M'}\right) \psi \tag{8}$$

is a solution to problem 1; in fact,

$$\|f \vee h_N - g_N\|_{L^\infty(\mathbb{T})} = \gamma_N(M).$$

Proof: let $h^1_N \in \mathcal{D}_{M,\psi}$ be a solution to problem 1 and $g^1_N \in H^\infty + R_N$ be such that $\|f \vee h^1_N - g^1_N\|_{L^\infty(\mathbb{T})} = \gamma_N(M)$. By the triangle inequality

$$|\psi - g^1_N| \leq M + \gamma_N(M) = M'\,,\ \text{a.e. on } \mathbb{T} \setminus K\,,$$

so that $\mathcal{B}^N_{M',\psi} \neq \emptyset$ as it contains g^1_N. Moreover, by the very definition of β_N, (7) holds. Consider now the function h_N given by (8). Since

$$|\psi - h_N| = \frac{M}{M'}\,|\psi - g_N| \leq M\,,\ \text{a.e. on } \mathbb{T} \setminus K\,,$$

we see that $h_N \in \mathcal{D}_{M,\psi}$. Furthermore

$$\begin{aligned}\|f \vee h_N - g_N\|_{L^\infty(\mathbb{T})} &= \max\left(\|f - g_N\|_{L^\infty(K)}\,,\ \|h_N - g_N\|_{L^\infty(\mathbb{T}\setminus K)}\right)\\ &\leq \max\left(\beta_N(M')\,,\ \|M' - M\|_{L^\infty(\mathbb{T}\setminus K)}\right) = \gamma_N(M)\end{aligned}$$

where the last equality follows from (6) and (7). ■

Remark 2 If $f \in \mathcal{C}^N_{M,\psi}$, the unique solution to problem 1 is the trace h_N of f on $\mathbb{T} \setminus K$ and $\gamma_N(M) = 0$ in this case; this function is unambiguously defined because members of $H^\infty + R_N$ are uniquely determined by the values they assume on a subset of $\mathbb{T}$ of positive measure.

Next we introduce the AAK approximation to which problem 2 will turn out to be equivalent. For $\rho \in L^\infty(\mathbb{T})$ such that $\rho \geq 0$ and $\log \rho \in L^1(\mathbb{T})$, let w_ρ be the outer factor whose value at the origin is positive and whose modulus on $\mathbb{T}$ is ρ:

$$w_\rho(z) := \exp\left\{ \frac{1}{2\pi} \int_{\mathbb{T}} \frac{e^{it}+z}{e^{it}-z} \log \rho(e^{it})\, dt \right\}, \tag{9}$$

Observe that whenever $\inf_{\mathbb{T}} \rho > 0$, then w_ρ is invertible in H^∞ and $w_\rho^{-1} = w_{1/\rho}$.

Problem 3 *Notation being as in problem 2 with $M' \in \mathcal{M}_*$, and assuming $f \notin \mathcal{C}^N_{M',\psi}$, we seek $v_N \in H^\infty + R_N$ such that*

$$\|(f \vee \psi) w^{-1}_{\beta_N(M') \vee M'} - v_N\|_{L^\infty(\mathbb{T})} = \min_{v \in H^\infty + R_N} \|(f \vee \psi) w^{-1}_{\beta_N(M') \vee M'} - v\|_{L^\infty(\mathbb{T})}. \tag{10}$$

The equivalence we are aiming at is the following:

Theorem 4 *Assume that $M' \in \mathcal{M}_*$, $\mathcal{B}^N_{M',\psi} \neq \emptyset$ and $f \notin \mathcal{C}^N_{M',\psi}$. Any solution v_N to problem 3 is such that $g_N = v_N\, w_{\beta_N(M')\vee M'}$ is a solution to problem 2. Conversely, any solution g_N to problem 2 gives rise to a solution $v_N = g_N\, w^{-1}_{\beta_N(M')\vee M'}$ of problem 3, and the value of that problem is equal to 1.*

Proof: since $\inf_{\mathbb{T}} \beta_N(M') \vee M' > 0$ by assumption, the outer factor $w_{\beta_N(M')\vee M'}$ is invertible in H^∞ and, as a consequence of lemma 1, we have that v_N belongs to $H^\infty + R_N$ if and only if g_N does. Recalling the definition of $\beta_N(M')$, v_N is a solution to problem 3 if and only if

$$\|(f \vee \psi) w^{-1}_{\beta_N(M') \vee M'} - v_N\|_{L^\infty(\mathbb{T})} = \max\Big(\frac{\|f - g_N\|_{L^\infty(K)}}{\beta_N(M')}, \|\frac{\psi - g_N}{M'}\|_{L^\infty(\mathbb{T}\setminus K)} \Big) = 1\,.$$

■

Remark 3 The case where $f \in \mathcal{C}^N_{M',\psi}$ is not covered by theorem 4 since it entails $\beta_N(M') = 0$. However, problem 2 is then trivial, the unique solution being obviously the $H^\infty + R_N$ function g_N such that $f = g_N$ almost everywhere on K.

Theorems 3 and 4 reduce problem 1 to the AAK problem 3. However, it is still unclear whether we made any progress towards a constructive solution for at least two reasons. The first is that our reduction to AAK approximation is implicit for we do not know $\gamma_N(M)$ nor $\beta_N(M + \gamma_N(M))$. The second reason is that, in general, AAK problems can be constructively solved only when the function to be approximated is rational whereas

$$(f \vee \psi) w^{-1}_{\beta_N(M+\gamma_N(M))\vee(M+\gamma_N(M))}$$

has a rather bad singularity at boundary points of K. The remaining of the paper is essentially devoted to filling these gaps when $f \vee \psi$ and M are generic functions in separable Hölder–Zygmund classes and K has non empty interior.

4 Continuity, uniqueness, error circularity in $H^\infty + C(\mathbb{T})$

We assume from now on that:

$$K \textit{ has non–empty interior and } f \vee \psi \in H^\infty + C(\mathbb{T}).$$

In particular, it follows from lemma 2 that $\mathcal{B}^N_{M',\psi} \neq \emptyset$ for all $M' \in \mathcal{M}_*$.
We first study the behavior of β_N and γ_N as functions of f, ψ, M' or M and the links between them. This material will be used to ensure uniqueness of a solution to problem 1 in section 4.2, as well as the generic convergence of our algorithm, in sections 5 and 6.

4.1 Continuity and reciprocity of the problem values

Proposition 1
(i) The map β_N is continuous $(H^\infty + C(\mathbb{T})) \times \mathcal{M}_* \to [0, \infty)$.
(ii) The map γ_N is continuous $(H^\infty + C(\mathbb{T})) \times \mathcal{M} \to [0, \infty)$.
(iii) For every $M \in \mathcal{M}$ such that $f \notin \mathcal{C}^N_{M,\psi}$, we have

$$\beta_N(M + \gamma_N(M)) = \gamma_N(M)\,. \tag{11}$$

Proof: in view of theorem 2, let g_0 be a solution to problem 2 and b a Blaschke product of degree at most N such that $b\,g_0 \in H^\infty$ (see lemma 1). Then, $(b\,f \vee b\,\psi, M')$ belongs to $(H^\infty + C(\mathbb{T})) \times \mathcal{M}_*$ and, since β_0 is continuous on $(H^\infty + C(\mathbb{T})) \times \mathcal{M}_*$ because it is convex and locally bounded (see *e.g.* [17, thm 1.10]), it is readily seen that

$$\beta_N(f \vee \psi, M') = \beta_0(b\,f \vee b\,\psi, M') = \lim_{m\to\infty} \beta_0(b\,f_m \vee b\,\psi_m, M'_m)\,, \tag{12}$$

for all sequences $(f_m \vee \psi_m)$ and (M'_m) converging uniformly to $f \vee \psi$ and M' in $(H^\infty + C(\mathbb{T}))$ and $\mathcal{M}_*$, respectively. Pick $h_m \in \mathcal{B}^0_{M'_m, b\psi_m}$ such that

$$\beta_0(b\,f_m \vee b\,\psi_m, M'_m) = \|b\,f_m - h_m\|_{L^\infty(K)}.$$

By the definition of β_N, we have

$$\beta_N(f_m \vee \psi_m, M'_m) \leq \|f_m - h_m/b\|_{L^\infty(K)} = \beta_0(b\,f_m \vee b\,\psi_m, M'_m)\,,$$

which implies in view of (12) that

$$\limsup_m \beta_N(f_m \vee \psi_m, M'_m) \leq \beta_N(f \vee \psi, M'). \tag{13}$$

This proves the upper semi-continuity of β_N.
Denoting now by g_m a solution to problem 2 for f_m, ψ_m, and M'_m, we get:

$$\|g_m\|_{L^\infty(K)} \leq \beta_N(f_m \vee \psi_m, M'_m) + \|f_m\|_{L^\infty(K)}\,, \qquad \|g_m\|_{L^\infty(\mathbb{T}\setminus K)} \leq M'_m + \|\psi_m\|_{L^\infty(\mathbb{T}\setminus K)}\,,$$

so that $\|g_m\|_{L^\infty(\mathbb{T})}$ is bounded, in view of (13). Some weak-* limit point $g \in \mathcal{B}^N_{M',\psi}$ of $\{g_m\}$ then satisfies

$$\|f - g\|_{L^\infty(K)} \leq \liminf_m \|f - g_m\|_{L^\infty(K)} = \liminf_m \beta_N(f_m \vee \psi_m, M'_m)$$

whence *a fortiori*

$$\beta_N(f \vee \psi, M') \leq \liminf_m \beta_N(f_m \vee \psi_m, M'_m)\,.$$

This proves lower semi-continuity, hence (i) holds. The proof of (ii) is similar and we omit it.
To get (iii), observe that the left hand side of (11) does not exceed the right by theorem 3. To prove equality, pick $M \in \mathcal{M}$, set $M' = M + \gamma_N(M) \in \mathcal{M}_*$ and assume that $\beta_N(M') < \gamma_N(M)$. Because we just proved that β_N is continuous, we can find $\varepsilon > 0$ such that $M'_1 = M' - \varepsilon \in \mathcal{M}_*$ and $\beta_N(M'_1) < \gamma_N(M)$. Let $g_N^{(1)} \in H^\infty + R_N$ be a solution to problem 2 where M'_1 is substituted for M', and set

$$h_N^{(1)} = \frac{M}{M'_1}\, g_N^{(1)} + \left(1 - \frac{M}{M'_1}\right)\psi.$$

It is easily checked that $h_N^{(1)} \in \mathcal{D}_{M,\psi}$ and

$$d(f \vee h_N^{(1)}, H^\infty + R_N) \leq \max(\beta_N(M'_1), \gamma_N(M) - \varepsilon) < \gamma_N(M),$$

a contradiction which proves (11). ∎

Corollary 1 *If $f \notin \mathcal{C}^N_{M,\psi}$, then $f \notin \mathcal{C}^N_{M',\psi}$, with $M' = M + \gamma_N(M)$.*

Proof: whenever $f \in \mathcal{C}^N_{M',\psi} \setminus \mathcal{C}^N_{M,\psi}$, it holds by definition that $\beta_N(M') = 0 < \gamma_N(M)$ thereby contradicting (11). ∎

For $k \geq 1$, we denote by $\sigma_k(\Gamma_\Phi)$ the k-th singular value of the Hankel operator Γ_Φ:

$$\sigma_k(\Gamma_\Phi) := \inf_S \{\|\Gamma_\Phi - S\|_{op}\,,\ \operatorname{rank} S < k\}. \tag{14}$$

When $\Phi \in H^\infty + C(\mathbb{T})$, then Γ_Φ is compact and the singular values of Γ_Φ are the positive square roots of the eigenvalues of $\Gamma_\Phi^*\Gamma_\Phi$. We shall have an occasion to use the following lemma.

Lemma 3 *For $M \in \mathcal{M}$, the map*

$$(G, \gamma) \mapsto \sigma_k(\Gamma_{G\, w^{-1}_{\gamma \vee (M+\gamma)}})$$

is continuous $(H^\infty + C(\mathbb{T})) \times (0, \infty) \to (0, \infty)$ and decreases with γ for fixed G.

Proof: if S_1 and S_2 are any operators, we have the elementary inequality

$$|\sigma_k(S_1) - \sigma_k(S_2)| \leq \|S_1 - S_2\|_{op}. \tag{15}$$

Therefore, if we pick G_1, G_2 in $H^\infty + C(\mathbb{T})$ and γ_1, γ_2 positive, we can write

$$\left|\sigma_k\left(\Gamma_{G_1\, w^{-1}_{\gamma_1 \vee (M+\gamma_1)}}\right) - \sigma_k\left(\Gamma_{G_2\, w^{-1}_{\gamma_2 \vee (M+\gamma_2)}}\right)\right|$$

$$\leq \left\| \Gamma_{(G_2 - G_1)\, w^{-1}_{\gamma_2 \vee (M+\gamma_2)}} \right\|_{op} + \left| \sigma_k \left(\Gamma_{G_1\, w^{-1}_{\gamma_2 \vee (M+\gamma_2)}} \right) - \sigma_k \left(\Gamma_{G_1\, w^{-1}_{\gamma_1 \vee (M+\gamma_1)}} \right) \right| .$$

If we let G_2 and γ_2 tend respectively to G_1 and γ_1 that are kept fixed, the first term above becomes arbitrarily small since $\|w^{-1}_{\gamma_2 \vee (M+\gamma_2)}\|_{L^\infty(\mathbb{T})}$ remains bounded, so it remains to show continuity and monotonicity with respect to γ for fixed G.
For this, note that if $\Phi \in L^\infty(\mathbb{T})$ and $\Psi \in H^\infty$, then $\Gamma_{\Phi\Psi} = \Gamma_\Phi M_\Psi$, where M_Ψ is the operator of multiplication by Ψ, and hence, by [33, cor. 1.5],

$$\sigma_k(\Gamma_{\Phi\Psi}) \leq \sigma_k(\Gamma_\Phi) \|\Psi\|_{L^\infty(\mathbb{T})}. \tag{16}$$

Likewise, if $\Psi^{-1} \in H^\infty$, then

$$\sigma_k(\Gamma_{\Phi\Psi}) \|\Psi^{-1}\|_{L^\infty(\mathbb{T})} \geq \sigma_k(\Gamma_\Phi).$$

Consequently, from

$$\Gamma_{G\, w^{-1}_{\gamma_2 \vee (M+\gamma_2)}} = \Gamma_{G\, w^{-1}_{\gamma_1 \vee (M+\gamma_1)}}\, M_{w_{\frac{\gamma_1 \vee (M+\gamma_1)}{\gamma_2 \vee (M+\gamma_2)}}} ,$$

we deduce for $0 < \gamma_1 < \gamma_2 < \infty$ that

$$A\, \sigma_k(\Gamma_{G\, w^{-1}_{\gamma_1 \vee (M+\gamma_1)}}) < \sigma_k(\Gamma_{G\, w^{-1}_{\gamma_2 \vee (M+\gamma_2)}}) < \sigma_k(\Gamma_{G\, w^{-1}_{\gamma_1 \vee (M+\gamma_1)}}) , \tag{17}$$

where

$$A = \left\| \frac{\gamma_2 \vee (M+\gamma_2)}{\gamma_1 \vee (M+\gamma_1)} \right\|^{-1}_{L^\infty(\mathbb{T})} .$$

As $A \to 1$ when $\gamma_2 - \gamma_1 \to 0$, this shows at the same time the continuity and the monotonicity of the partial map for fixed G. ∎
We turn to an intrinsic characterization of γ_N.

Proposition 2 *If $f \notin \mathcal{C}^N_{M,\psi}$, then $\gamma_N(M)$ is the unique positive real number γ such that*

$$\min_{v \in H^\infty + R_N} \|(f \vee \psi)\, w^{-1}_{\gamma \vee (M+\gamma)} - v\|_{L^\infty(\mathbb{T})} = 1 . \tag{18}$$

Proof: from point (*iii*) of proposition 1, we get $\beta_N(M') = \gamma_N(M)$ for $M' = M + \gamma_N(M)$. Substituting these quantities in the second assertion of theorem 4 proves that (18) holds for $\gamma = \gamma_N(M)$. Since, by AAK theory:

$$\min_{v \in H^\infty + R_N} \|(f \vee \psi)\, w^{-1}_{\gamma \vee (M+\gamma)} - v\|_{L^\infty(\mathbb{T})} = \sigma_{N+1}(\Gamma_{(f\vee\psi)\, w^{-1}_{\gamma \vee (M+\gamma)}}) ,$$

it follows from (17) that the solution to (18) is unique. ∎

Remark 4 Equivalently, if $f \notin \mathcal{C}^N_{M',\psi}$, then $\beta_N(M')$ is the unique positive real number β such that

$$\min_{v \in H^\infty + R_N} \|(f \vee \psi)\, w^{-1}_{\beta \vee M'} - v\|_{L^\infty(\mathbb{T})} = 1 . \tag{19}$$

4.2 Uniqueness and error circularity when $f \vee \psi \in H^\infty + C(\mathbb{T})$

We now translate to problems 1 and 2 property (b) of AAK approximation stated in section 1.3. We begin with the bounded extremal problem:

Theorem 5 *For $M' \in \mathcal{M}_*$, the solution g_N to problem 2 is unique. Whenever $f \notin \mathcal{C}^N_{M',\psi}$, it satisfies*

$$\begin{cases} |f - g_N| & = \beta_N(M') > 0 \quad \text{a.e. on } K\,, \\ |\psi - g_N| & = M' \qquad\qquad\quad \text{a.e. on } \mathbb{T} \setminus K\,. \end{cases} \tag{20}$$

Proof: if $f \in \mathcal{C}^N_{M',\psi}$, then uniqueness is automatic as pointed out in remark 3. Assuming now that $f \notin \mathcal{C}^N_{M',\psi}$, uniqueness follows from theorem 4 and property (b) of AAK approximation. We also know that the error is circular in this case, so that, in view of (19), $|(f \vee \psi)\, w^{-1}_{\beta_N(M') \vee M'} - v_N| = 1$. Substituting $g_N\, w^{-1}_{\beta_N(M') \vee M'}$ for v_N as indicated in theorem 4 and taking into account that $w^{-1}_{\beta_N(M') \vee M'}$ has modulus $1/\beta_N(M')$ on K and $1/M'$ on $\mathbb{T} \setminus K$ leads us to (20). ∎

As for the bounded completion, we get:

Theorem 6 *The solution h_N to problem 1 is unique. Whenever $f \notin \mathcal{C}^N_{M,\psi}$, it satisfies*

$$|\psi - h_N| = M \text{ a.e. on } \mathbb{T} \setminus K\,. \tag{21}$$

Proof: if $f \in \mathcal{C}^N_{M,\psi}$, uniqueness clearly holds, see remark 2. Assume now that $f \notin \mathcal{C}^N_{M,\psi}$. Let h^1_N be a solution to problem 1 and $g_N \in H^\infty + R_N$, be such that

$$\|f \vee h^1_N - g_N\|_{L^\infty(\mathbb{T})} = \gamma_N(M). \tag{22}$$

Clearly $g_N \in \mathcal{B}^N_{M',\psi}$ with $M' = M + \gamma_N(M)$ and $\|f - g_N\|_{L^\infty(K)} \leq \gamma_N(M) = \beta_N(M')$, since (11) holds good from proposition 1. Therefore g_N solves for problem 2 and, according to theorem 5, is uniquely defined by this property. Moreover, we deduce from theorem 3 that h_N given by (8) is also a solution to problem 1. A direct computation yields

$$|\psi - h_N| = \frac{M}{M'} |\psi - g_N|\,,$$

and (20) implies (21). It remains to show that $h^1_N = h_N$. Using (20) again and the definition of h^1_N, we can write

$$M' = |\psi - g_N| \leq |\psi - h^1_N| + |h^1_N - g_N| \leq M + \gamma_N = M' \qquad \text{a.e. on } \mathbb{T} \setminus K$$

so that equality holds throughout, hence we see for almost every $e^{i\theta} \in \mathbb{T} \setminus K$ that $h^1_N(e^{i\theta})$ lies on the segment $[\psi(e^{i\theta}), g_N(e^{i\theta})]$ at a distance $M(e^{i\theta})$ from $\psi(e^{i\theta})$. Since h_N also has this property, we get $h^1_N = h_N$ as desired. ∎

5 Generic well-posedness in separable Hölder–Zygmund classes

Before we can describe a constructive solution to the bounded completion problem, at least for generic λ_α-*data*, we have to address a general difficulty with AAK approximation, namely the map sending a function to its best approximant is discontinuous $C(\mathbb{T}) \to H^\infty + R_N$ at every non-meromorphic argument [32]; in particular, uniformly approximating a continuous function by a rational one does *not* guarantee that the best AAK approximation of the rational function will be uniformly close to that of the original function. The situation improves if one restricts to smoother classes of functions. An attractive class here is the separable Hölder–Zygmund space $\lambda_\alpha(\mathbb{T})$. Indeed, rational approximants can be obtained in $\lambda_\alpha(\mathbb{T})$ upon convolving with approximate identities (for example the Fejér kernel, see lemma 4 below); moreover, AAK approximation maps $\lambda_\alpha(\mathbb{T})$ into itself [37] and is continuous there precisely at those points Φ whose associated Hankel operator Γ_Φ has non-multiple $N+1$-st singular value [35, thm 6]. To warrant this approach to AAK approximation, it only remains to prove the intuitively obvious fact that the multiplicity condition is satisfied most of the time. Therefore, we begin our constructive analysis by establishing that Hankel singular values are generically simple for rather general spaces of continuous functions. In addition to $\lambda_\alpha(\mathbb{T})$, these will comprise all hereditary classes of type 2, in the terminology of [36][4], where the non-multipleness of the $(N+1)$-st singular value is known to be necessary (and in most cases sufficient) for the continuity of best meromorphic approximation with N poles.

Theorem 7 *Let $N \geq 1$ be an integer and let $\mathcal{B}$ be either:*
(i) the set of complex trigonometric polynomials of degree at most $L \geq N$,
or
(ii) a complex Banach space of continuous functions on $\mathbb{T}$ densely containing $\mathcal{T}$, and such that the natural injection $\mathcal{B} \to C(\mathbb{T})$ is continuous.
Define

$$\mathcal{O}_N = \{\Phi \in \mathcal{B};\ \sigma_k(\Gamma_\Phi) \text{ has multiplicity } 1 \text{ for } 1 \leq k \leq N\}.$$

Then $\mathcal{O}_N$ is open and dense in $\mathcal{B}$.

Let us point out the following corollary, which may be of independent interest and is logically weaker than theorem 7 in case (i) and immediately follows in case (ii) from the Baire category theorem (see *e.g.* [41]):

Corollary 2 *The set of functions whose associated Hankel operator has simple singular values is dense in $\mathcal{B}$.*

Proof of theorem **7***:* it follows from the already used inequality (15) that

$$|\sigma_k(\Gamma_{\Phi+\Delta\Phi}) - \sigma_k(\Gamma_\Phi)| \leq \|\Delta\Phi\|_{L^\infty(\mathbb{T})} \leq C\|\Delta\Phi\|_{\mathcal{B}}\,, \tag{23}$$

[4]For example Besov classes $B^s_{p,p}$ for $1 \leq p < \infty$, $s > 1/p$, as well as B^1_1 or the Wiener algebra $\mathcal{W}$.

where C is a positive constant provided by the assumptions and $\| \ \|_{\mathcal{B}}$ denotes the norm on $\mathcal{B}$. If the first N singular values of Γ_Φ are simple and if we set

$$\min_{1\leq j\leq N} \sigma_j(\Gamma_\Phi) - \sigma_{j+1}(\Gamma_\Phi) = \epsilon > 0,$$

it is readily seen from (23) that the first N singular values of $\Gamma_{\Phi+\Delta\Phi}$ will remain simple provided $\|\Delta\Phi\|_{\mathcal{B}} < \epsilon/2C$. This proves the openness of $\mathcal{O}_N$.
It remains to show that $\mathcal{O}_N$ is dense in $\mathcal{B}$. Since in case (ii) the set $\mathcal{T}$ is dense in $\mathcal{B}$, it is enough to treat case (i), so we set $\mathcal{B}$ to be the space of trigonometric polynomials of degree at most L, where $L \geq N$. Since $\mathcal{B}$ is finite dimensional, the choice of the norm is irrelevant here; to fix ideas, we shall identify $\mathcal{B}$ with $\mathbb{C}^{2L+1}$ equipped with the Euclidean topology. We shall prove that trigonometric polynomials whose associated Hankel operator has simple first L singular values form a dense subset of $\mathcal{B}$; this will *a fortiori* imply what we want.
Notice that the Hankel operator associated to the trigonometric polynomial

$$\sum_{k=-L}^{L} q_k \, z^{-k} \ \in \mathcal{B}\,, \qquad z = e^{i\theta}\,,$$

is equal to Γ_q where q is the anti-analytic projection

$$q(z) = q_1 z^{-1} + \cdots + q_L z^{-L} \ \in z^{-1}\, P_{L-1}[1/z]\,.$$

We single out the subset $\mathcal{M}_L$ of those $q \in z^{-1}\, P_{L-1}[1/z]$ for which $q_L \neq 0$. It is clear that $\mathcal{M}_L$ is open and dense in $z^{-1}\, P_{L-1}[1/z]$ when the latter is identified with $\mathbb{C}^L$, and it is enough to show that the first L singular values of Γ_q are simple for q in a dense subset of $\mathcal{M}_L$.
The first L singular values of Γ_q are the singular values of the finite Hankel matrix

$$\mathcal{H}_q = \begin{pmatrix} q_1 & q_2 & \cdots & q_{L-1} & q_L \\ q_2 & q_3 & \cdots & q_L & 0 \\ \vdots & \vdots & . & 0 & \vdots \\ \vdots & q_L & 0 & \vdots & \vdots \\ q_L & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{24}$$

which is obviously nonsingular as $q_L \neq 0$ by assumption; the remaining singular values of Γ_q are of course zero.
For $1 \leq j \leq L$ we write $q_j = r_j + is_j$, where r_j and s_j are real. Then the singular values of $\mathcal{H}_q$ are the positive roots of $\det(\mathcal{H}_q^*\mathcal{H}_q - \sigma^2 I) = 0$, which is a polynomial equation in σ whose coefficients are themselves polynomials in the variables r_j and s_j.
Now a polynomial has no repeated roots if and only if its discriminant Δ is nonzero (see for example [14, 24]). As Δ is a polynomial in the variables $r_1, \ldots, r_L, s_1, \ldots, s_L$, we have to prove that it is not the zero polynomial, because then the set of coefficients $q_1, \ldots, q_L$ for which it does not vanish will be dense in $\mathbb{C}^L$ and we shall be done. Now, if Δ was identically zero, that would mean that all $L \times L$ Hankel matrices $\mathcal{H}_q$ defined by (24) have repeated singular values. However we can easily construct one that does not by induction on L. For $L = 1$ the result is clear; for the induction step, supposing that we have a $L \times L$ Hankel

matrix with top row $q_1, \dots, q_L$ whose singular values are simple and nonzero, we choose q_{L+1} very small but non-zero. It is plain if $|q_{L+1}|$ is sufficiently small, using the continuity of the roots with respect to the coefficients, that the first N singular values of the $(L+1) \times (L+1)$ Hankel matrix with parameters $q_1, \dots, q_{L+1}$ stay close to those of the $L \times L$ matrix with parameters $q_1, \dots, q_L$ and therefore remain distinct, while the $(L+1)$-st is smaller than the previous ones but non-zero. ■

For many applications of AAK approximation, in particular to engineering and system theory, one deals with functions having *real* Fourier coefficients and theorem 7 tells nothing in this case because such functions already form a small subset of $\mathcal{B}$. To warrant such applications, we observe that both theorem 7 and corollary 2 remain true when $\mathcal{B}$ is either the set of trigonometric polynomials with real coefficients of degree at most $L \geq N$ or a real Banach space of continuous functions on $\mathbb{T}$ densely containing the trigonometric polynomials with real coefficients. The proof of this would be virtually identical to the one of theorem 7, the only difference being that coefficients are real throughout.

In order to apply theorem 7 to $\mathcal{B} = \lambda_\alpha(\mathbb{T})$ and to the symbol

$$(f \vee \psi)\, w^{-1}_{\gamma_N(M) \vee (M + \gamma_N(M))},$$

we need a couple of lemmas. Though the first one is apparently well-known, we could not locate it in the literature.

Lemma 4 *If $\overline{\mathcal{T}}^\alpha$ denotes the closure of $\mathcal{T}$ in $\Lambda_\alpha(\mathbb{T})$, then $\overline{\mathcal{T}}^\alpha = \lambda_\alpha(\mathbb{T})$, the separable Hölder–Zygmund class.*

Proof: clearly $\mathcal{T} \subset \lambda_\alpha$, since functions in $\mathcal{T}$ are differentiable. If $f \in \overline{\mathcal{T}}^\alpha$ and $\epsilon > 0$, pick $g \in \mathcal{T}$ such that $\|f - g\|_\alpha < \epsilon/2$. Since $\omega_g(\delta) < \epsilon\delta^\alpha/2$ for δ small enough, we get for such δ and $0 < \eta \leq \delta$:

$$\begin{aligned} |f(\theta+\eta) - f(\theta)| &\leq |g(\theta+\eta) - g(\theta)| + |(f-g)(\theta+\eta) - (f-g)(\theta)| \\ &< \epsilon\delta^\alpha/2 + \epsilon\delta^\alpha/2 = \epsilon\delta^\alpha \end{aligned}$$

for every $\theta \in (-\pi, \pi]$. Hence $\omega_f(\delta) = o(\delta^\alpha)$ so that $f \in \lambda_\alpha(\mathbb{T})$.
We now show that for $f \in \lambda_\alpha(\mathbb{T})$, we have $\|f_n - f\|_\alpha \to 0$, where f_n is a sequence of trigonometric polynomials obtained by convolving f with an approximate identity K_n that may be for instance the Fejér kernel, see *e.g.* [29, 43]. By the well–known properties of such a kernel, f_n converges uniformly to f, so it is enough to show that

$$\sup_{\delta > 0} \frac{\omega_{f - f_n}(\delta)}{\delta^\alpha} \to 0.$$

Let $\epsilon > 0$ be given, and write $\omega_f(\delta) = k_\delta \delta^\alpha$, where k_δ is bounded and $k_\delta \to 0$ as $\delta \to 0$. Thus $k_\delta < \epsilon/4$ for $\delta \leq \delta_1$, say. Put for simplicity $(S_\eta f)(\theta) = f(\theta + \eta)$. We have by definition:

$$\|S_\eta f - f\|_{L^\infty(\mathbb{T})} \leq k_\delta \delta^\alpha\,, \text{ for } 0 < \eta \leq \delta\,. \tag{25}$$

Since convolving with K_n commutes with S_η and does not increase the *sup* norm, we get $\|S_\eta f_n - f_n\|_{L^\infty(\mathbb{T})} \le k_\delta \delta^\alpha$. Hence for $\delta \le \delta_1$:

$$\|S_\eta(f-f_n) - (f-f_n)\|_{L^\infty(\mathbb{T})} < 2k_\delta\delta^\alpha < (\epsilon/2)\delta^\alpha\,.$$

To get a similar estimate for $\delta > \delta_1$, take $0 < \eta \le \delta$ and use $\int_{-\pi}^{\pi} K_n = 1$ to write:

$$S_\eta(f-f_n)(\theta) - (f-f_n)(\theta) = \int_{-\pi}^{\pi} [f(\theta+\eta) - f(\theta+\eta-y) - f(\theta) + f(\theta-y)]K_n(y)\,dy.$$

Split the integral into 2 pieces: I_1 taken over $|y| \le \delta_1$, and I_2 the remainder. By (25) and our choice of δ_1, we get $I_1 < (\epsilon/2)\delta^\alpha$. As to I_2, we get the obvious upper bound:

$$I_2 \le 2k_\delta\delta^\alpha \int_{|y|>\delta_1} K_n(y)\,dy$$

and $\int_{|y|>\delta_1} K_n$ becomes smaller than $\epsilon/4\sup_\delta k_\delta$ for n sufficiently large depending only on δ_1. Adding all this up we get $\sup_{\delta>0}\omega_{f-f_n}(\delta)/\delta^\alpha < \epsilon$ for n large, as desired. ∎

Lemma 5 *The function $\Omega(F,\rho) = Fw_\rho$ is continuous $\lambda_\alpha(\mathbb{T}) \times \lambda_\alpha^{+,*}(\mathbb{T}) \to \lambda_\alpha(\mathbb{T})$. Moreover, for fixed $\rho \in \lambda_\alpha^{+,*}(\mathbb{T})$, the partial map $\Omega_\rho(F) = \Omega(F,\rho)$ is a homeomorphism on $\lambda_\alpha(\mathbb{T})$.*

Proof: since $\lambda_\alpha(\mathbb{T})$ is a Banach algebra, as follows immediately from:

$$\omega_{ab} \le \|b\|_{L^\infty(\mathbb{T})}\,\omega_a + \|a\|_{L^\infty(\mathbb{T})}\,\omega_b\,,$$

the continuity of Ω reduces to show that $\rho \mapsto w_\rho$ is continuous $\lambda_\alpha^{+,*}(\mathbb{T}) \to \lambda_\alpha(\mathbb{T})$. To this effect, observe that on $\mathbb{T}$:

$$w_\rho = \exp(\log\rho + i\,\widetilde{\log\rho})\,.$$

where $\widetilde{f}$ denotes, for any $f \in C(\mathbb{T})$, the boundary value taken by the harmonic conjugate of its Poisson integral. That Hölder–Zygmund classes $\Lambda_\alpha(\mathbb{T})$ are preserved by the conjugation operator is well-known, see e.g. [21, thm 5.8] or (26) below; that the separable Hölder–Zygmund classes $\lambda_\alpha(\mathbb{T})$ inherit this property follows from lemma 4 applied to the inequality [25, thm 1.3,III]:

$$\omega_{\tilde f}(\delta) \le C\left(\int_0^\delta \frac{\omega_f(t)}{t}dt + \delta\int_\delta^\pi \frac{\omega_f(t)}{t^2}dt\right)\,, \tag{26}$$

for a constant C not depending on f. Alternatively, one may use here Privalov's theorem [13, thm 6]. Since taking the logarithm is continuous $\lambda_\alpha^{+,*}(\mathbb{T}) \to \lambda_\alpha(\mathbb{T})$ and taking the exponential is also continuous on $\lambda_\alpha(\mathbb{T})$, gathering things up yields the desired continuity $\rho \mapsto w_\rho$.
Finally, when $\rho \in \lambda_\alpha^{+,*}(\mathbb{T})$, so does $1/\rho$ and the lemma follows from the identity $\Omega_\rho^{-1} = \Omega_{1/\rho}$. ∎

We are now in position to establish the genericity result we are looking for. For both statements and proofs, it is more convenient to handle $G = f \vee \psi$ as a whole.

Proposition 3 *The set $\mathcal{U} \subset \lambda_\alpha(\mathbb{T}) \times \lambda_\alpha^{+,0}(\mathbb{T} \setminus K)$ of pairs (G, M) such that $\gamma_N(G, M) \neq 0$, and also such that the Hankel operator with symbol $Gw^{-1}_{\gamma_N \vee (\gamma_N + M)}$ has simple $N+1$-st singular values, is open and dense.*

Proof: set

$$\mathcal{F} := \{G \in \lambda_\alpha(\mathbb{T}),\ G_{|_K} \in (H^\infty + R_N)_{|_K}\}.$$

We first show that $\mathcal{F}$ is meagre in $\lambda_\alpha(\mathbb{T})$. Indeed, for $n \in \mathbb{N}$, let

$$\mathcal{F}_n := \{G \in \mathcal{F},\ G_{|_K} = h_{|_K} \text{ where } h \in H^\infty + R_N \text{ satisfies } \|h\|_{L^\infty(\mathbb{T})} \leq n\}.$$

Taking into account that the λ_α topology is finer than the L^∞ one, it follows from lemma 1 and weak-* compactness of balls in $L^\infty(\mathbb{T})$ that $\mathcal{F}_n$ is closed in $\lambda_\alpha(\mathbb{T})$ for each n. Moreover $\mathcal{F}_n$ has empty interior, because if $G \in \mathcal{F}_n$ and r_{N+1} is a rational function having $N+1$ poles in $\mathbb{D}$ none of which is a pole of G, then $G + \epsilon r_{N+1}$ cannot belong to $\mathcal{F}_n$ when $\epsilon > 0$ because $(G + \epsilon r_{N+1})_{|_K}$ does not even belong to $(H^\infty + R_N)_{|_K}$. Appealing to the Baire category theorem, $\mathcal{F} = \cup_{n \in \mathbb{N}} \mathcal{F}_n$ is meagre in $\lambda_\alpha(\mathbb{T})$ as announced.
Next, for fixed $G \in \lambda_\alpha(\mathbb{T}) \setminus \mathcal{F}$, we define a map

$$\begin{array}{rccc} \Upsilon_G: & \lambda_\alpha^{+,0}(\mathbb{T} \setminus K) & \rightarrow & \lambda_\alpha^{+,0}(\mathbb{T} \setminus K) \\ & M & \mapsto & \dfrac{M}{\gamma_N(G, M)}. \end{array}$$

We claim that Υ_G is a homeomorphism.
To prove that it is bijective, we construct the inverse using proposition 2, namely $\Upsilon_G^{-1}(Y) = \gamma(Y)\, Y$ where $\gamma(Y)$ is the unique positive real number such that

$$\sigma_{N+1}(\Gamma_{G\, w^{-1}_{\gamma(Y)}\, w^{-1}_{1 \vee (1+Y)}}) = 1.$$

The continuity of Υ_G follows from that of γ_N given by proposition 1. The continuity of Υ_G^{-1} reduces to that of $\gamma(Y)$ defined above, which can be seen as follows. It is readily checked that $\gamma = \gamma(Y)$ is the unique solution to

$$\gamma = \gamma_N(G, \gamma\, Y).$$

Fix $Y_1 \in \lambda_\alpha^{+,0}(\mathbb{T} \setminus K)$ and let $\lambda_1, \lambda_2 > 0$ be such that $\lambda_1 < \gamma(Y_1) < \lambda_2$. Since $\gamma_N(G, \gamma\, Y)$ decreases with γ, we necessarily get that

$$\lambda_1 - \gamma_N(G, \lambda_1 Y_1) < 0,\ \lambda_2 - \gamma_N(G, \lambda_2 Y_1) > 0.$$

By continuity of γ_N, the same holds true for all Y_2 close enough to Y_1 in $\lambda_\alpha^{+,0}(\mathbb{T} \setminus K)$ implying that also $\lambda_1 < \gamma(Y_2) < \lambda_2$.
This establishes the claim.
By theorem 7, (ii), and lemma 4, the subset $\mathcal{O}_{N+1} \subset \lambda_\alpha(\mathbb{T})$ of those F for which Γ_F has simple $N+1$-st singular values is open and dense. Consequently, by lemma 5, for fixed $\rho \in \lambda_\alpha^{+,*}(\mathbb{T})$, the subset $\mathcal{O}_\rho = \Omega_\rho^{-1} \mathcal{O}_{N+1} \subset \lambda_\alpha(\mathbb{T})$ of those G for which Γ_{Gw_ρ} has simple

$N+1$-st singular values is also open and dense. Let Q be a countable dense subset[5] of $\lambda_\alpha^{+,0}(\mathbb{T}\setminus K)$. Define a countable subset of $\lambda_\alpha^{+,*}(\mathbb{T})$ by:

$$\mathcal{Q} := \{(1 \vee (1+\Upsilon_G^{-1}(q)))^{-1}\,,\ q \in Q\}\,.$$

Finally, let

$$\mathcal{G} := \cap_{\rho\in\mathcal{Q}}\mathcal{O}_\rho \cap (\lambda_\alpha(\mathbb{T}) \setminus \mathcal{F})\,.$$

By the Baire category theorem, $\mathcal{G}$ is dense in $\lambda_\alpha(\mathbb{T})$ hence the fibered product:

$$\mathcal{P} := \{(G,M);\ G \in \mathcal{G},\ M \in \Upsilon_G^{-1}(Q)\}$$

is also dense in $\lambda_\alpha(\mathbb{T}) \times \lambda_\alpha^{+,0}(\mathbb{T}\setminus K)$ because Q is dense in $\lambda_\alpha^{+,0}(\mathbb{T}\setminus K)$ by definition and Υ_G is a homeomorphism. Moreover, for each pair $(G,M) \in \mathcal{P}$, the Hankel operator with symbol

$$G\,w^{-1}_{\gamma_N(G,M)\vee(\gamma_N(G,M)+M)} = G\,\gamma_N(G,M)^{-1}\,w_{(1\vee(1+\Upsilon_G(M)))^{-1}}$$

has simple $N+1$-st singular values. Since $\mathcal{P} \subset \mathcal{U}$, it only remains for us to show that $\mathcal{U}$ is open, and this is easily done. Indeed, in view of proposition 1, the map γ_N is *a fortiori* continuous $\lambda_\alpha(\mathbb{T}) \times \lambda_\alpha^{+,0}(\mathbb{T}\setminus K) \to \mathbb{R}$ thus the condition $\gamma_N(G,M) > 0$ defines an open set. The condition of non–multipleness of the Hankel singular values does the same because it behaves continuously in $\lambda_\alpha^{+,*}(\mathbb{T})$ with respect to the symbol:

$$G\,w^{-1}_{\gamma_N(G,M)\vee(\gamma_N(G,M)+M)}\,,$$

which in turn is continuous with respect to $(G,M) \in \lambda_\alpha(\mathbb{T}) \times \lambda_\alpha^{+,0}(\mathbb{T}\setminus K)$. ■

Remark 5 An analogous of proposition 3 holds for $\beta_N = \beta_N(G,M')$ and the Hankel operator with symbol $Gw^{-1}_{\beta_N\vee M'}$, with respect to the pairs (G,M') belonging to

$$\lambda_\alpha(\mathbb{T}) \times \{M' \in \lambda_\alpha^{+,*}(\mathbb{T}\setminus K)\,,\ M' = \beta_N \text{ at boundary points of } K\}\,.$$

6 A generic algorithm in Hölder classes

We now describe a convergent procedure to compute the solution to problem 1 in the case where $(f \vee \psi, M)$ belongs to the open and dense subset $\mathcal{U}$ of $\lambda_\alpha(\mathbb{T}) \times \lambda_\alpha^{+,0}(\mathbb{T}\setminus K)$ defined in corollary 3.

Write $G = f \vee \psi$. *We shall assume that we are provided with a sequence (r_m) of rational functions that converges to G in $\lambda_\alpha(\mathbb{T})$.* Let (ε_m) be a sequence of positive numbers decreasing to 0. An algorithm to solve problem 1 is obtained by performing, for each m, the following steps.

1. Find a positive number $\tilde{\gamma}_m$ such that

$$|1 - \sigma_{N+1}(\Gamma_{r_m w^{-1}_{\tilde{\gamma}_m\vee(\tilde{\gamma}_m+M)}})| \le \varepsilon_m. \tag{27}$$

[5] Such a subset exists because $\lambda_\alpha^{+,0}(\mathbb{T}\setminus K)$ is separable by lemma 4 and every $M \in \lambda_\alpha^{+,0}(\mathbb{T}\setminus K)$ is the trace on $\mathbb{T}\setminus K$ of some member of $\lambda_\alpha(\mathbb{T})$ obtained for instance by concatenating with the zero function on K.

2. Compute the best AAK approximant v_m to $r_m\, w^{-1}_{\tilde\gamma_m \vee(\tilde\gamma_m+M)}$ in $H^\infty + R_N$.

3. Set
$$h_m = \frac{M}{\tilde\gamma_m + M}\, v_m\, w_{\tilde\gamma_m\vee(\tilde\gamma_m+M)} + \frac{\tilde\gamma_m}{\tilde\gamma_m+M}\psi.$$

Then h_m converges in $\lambda_\alpha(\mathbb{T}\setminus K)$ to the solution of problem 1 as m goes to infinity.

Because $(G,M)\in\mathcal{U}$, it holds that $f\notin \mathcal{C}^N_{M,\psi}$. We show first that $(\tilde\gamma_m)$ is bounded away from 0 and ∞. It follows from (16) with $\Phi = r_m$, $\Psi = w^{-1}_{\tilde\gamma_m\vee(\tilde\gamma_m+M)}$, and $k=N+1$ that if $\tilde\gamma_m\to\infty$ then $\sigma_{N+1}(\Gamma_{r_m w^{-1}_{\tilde\gamma_m\vee(\tilde\gamma_m+M)}})\to 0$; the same applies to a subsequence, thereby contradicting (27).
Further $\tilde\gamma_m$ cannot go to 0. Indeed, as a consequence of (27), there exists $g_m\in H^\infty+R_N$ such that
$$\|r_m\, w^{-1}_{\tilde\gamma_m\vee(\tilde\gamma_m+M)} - g_m\|_{L^\infty(\mathbb{T})} \le 1+\varepsilon_m\,.$$
Let
$$f_m = g_m\, w_{\tilde\gamma_m\vee(\tilde\gamma_m+M)} \in H^\infty + R_N\,.$$
It then holds that
$$\begin{array}{lll} |r_m - f_m| \le (1+\varepsilon_m)\,\tilde\gamma_m & \text{a.e. on} & K\,,\\ |r_m - f_m| \le (1+\varepsilon_m)\,(\tilde\gamma_m+M) & \text{a.e. on} & \mathbb{T}\setminus K\,. \end{array}$$

Hence (f_m) is bounded and admits a weak-* convergent subsequence with limit in $H^\infty+R_N$, in view of lemma 1. This limit function is equal to f a.e. on K and belongs also to $\mathcal{B}^N_{M,\psi}$, a contradiction.
Extracting any subsequence of $(\tilde\gamma_m)$ with limit γ, say, it follows from (27) and lemma 3 that
$$\sigma_{N+1}(\Gamma_{G w^{-1}_{\gamma\vee(\gamma+M)}}) = 1\,,$$
hence $\gamma = \gamma_N(G,M)$ by proposition 2. This shows that
$$\lim_{m\to\infty} |\tilde\gamma_m - \gamma_N(G,M)| = 0\,.$$

That v_m introduced in step 2 goes to the solution v_N of the AAK problem (18) with $\gamma = \gamma_N(G,M)$ is ensured by [35, thm 6] whose hypotheses are verified on $\mathcal{U}$ by proposition 3 and lemma 5.
Hence, h_m defined in step 3 converges in $\lambda_\alpha(\mathbb{T}\setminus K)$ to the solution
$$h_N = \frac{M}{\gamma_N+M}\, v_m\, w_{\gamma_N\vee(\gamma_N+M)} + \frac{\gamma_N}{\gamma_N+M}\psi\,,$$
to problem 1, in view of theorems 3, 4, and proposition 1, point (iii).

Last but not least, each of these steps is constructive. Though we do not discuss here how to obtain the sequence (r_m), as this depends much on the way G is given, we simply note that, if the Fourier coefficients of G are available, one can use Fejér approximants, as in the proof of lemma 4, but also Jackson or de La Vallée Poussin trigonometric polynomials. Computing the singular value involved in step 1 and the associated Schmidt pair needed in step 2 (see section 1.3) is straightforward since r_m is rational. Indeed, if $r_m = p_\kappa/q_\ell$ with $p_\kappa \in P_\kappa[z]$, $q_\ell \in P_\ell[z]$, and for arbitrary $\rho \in \lambda_\alpha^{+,*}(\mathbb{T})$, we can write:

$$r_m\, w_\rho = \frac{t_{\ell-1}}{q_\ell} + h$$

where h belongs to H^∞ and $t_{\ell-1} \in P_{\ell-1}[z]$ is easily computed from the values of p_m, w_ρ, and possibly a certain number of their derivatives at the roots of q_ℓ. Now, the outer function $w_{\gamma\vee(\gamma+M)}$ being numerically available through (9), a solution $\tilde{\gamma}_m$ to (27) can be computed using a dichotomy procedure.

From the numerical viewpoint, a natural concern with generic algorithms is whether their conditioning is uniformly poor, in other words whether all admissible data are close to the singular set. At the cost of a slightly lengthier proof, it can be shown in the present case that the complement of $\mathcal{U}$ in $\lambda_\alpha(\mathbb{T}) \times \lambda_\alpha^{+,0}(\mathbb{T} \setminus K)$ has in fact the structure of a Banach manifold. This ensures at least an abundance of data for which the algorithm is well-behaved, although it is of course quite difficult to characterize them.

An analogous algorithm works to solve problem 2.

7 Numerical experiments

The above algorithm has been implemented using Matlab (release 5.3) and applied to experimental data measured on a hyperfrequencies filter by the French National Space Agency (CNES Toulouse) for harmonic identification purposes.
More precisely, after performing some conformal mapping to the circle, we are given 801 noisy pointwise complex values of the so-called transfer function $\mathcal{H}$ of the filter in a bandwidth $K = [e^{i\pi/2}, e^{i3\pi/2}]$ and we want to robustly recover $\mathcal{H}$ in the whole unit disk (these pointwise data are plotted as dots "." on the figures).
For physical reasons (due to the form of the solutions to Maxwell equations near resonant frequencies), $\mathcal{H}$ is expected to be close to meromorphic with 8 poles in $\mathbb{D}$, see [3, 10] where this application has also been handled.
As a preliminary step, we interpolate these pointwise data in K by cubic splines to obtain f (plotted as a straight line on the figures), and we likewise interpolate between $e^{i\pi/2}$ and $e^{i3\pi/2}$] on $\mathbb{T}\setminus K$ in order to get a reference behaviour ψ (plotted as a dotted-dashed line on the figures). The anti-analytic part of $f \vee \psi$ is then approximated by its truncation r_m of order m of its Fourier series. The gauge constraint on $\mathbb{T} \setminus K$ is taken to be $M(e^{i\theta}) = M_{max}\cos^2\theta$. The results of our algorithm for $m = 100$, $\varepsilon_{100} = 10^{-4}$ (see section 6), and $M_{max} = .02$ are presented on figures 1, 2, 3 for $N = 2, 4, 8$ both through their Nyquist (image of the unit circle parametrized by θ) real and imaginary parts of the function values) and Bode (modulus as a function of θ) diagrams.

In the notation of section 6, this enables us to check the behaviour of the approximants h_{100} and g_{100} to the solutions h_N and g_N of problems 1 and 2, respectively plotted as a straight and dotted line, for $N = 2, 4, 8$.

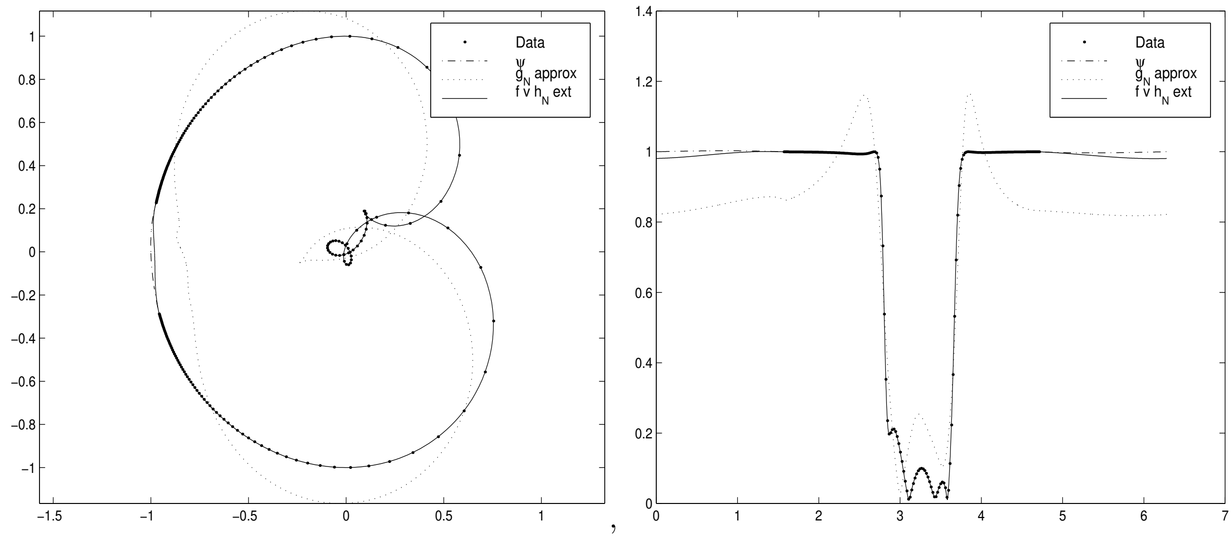

Figure 1: N=2, $\gamma_N = 0.167$

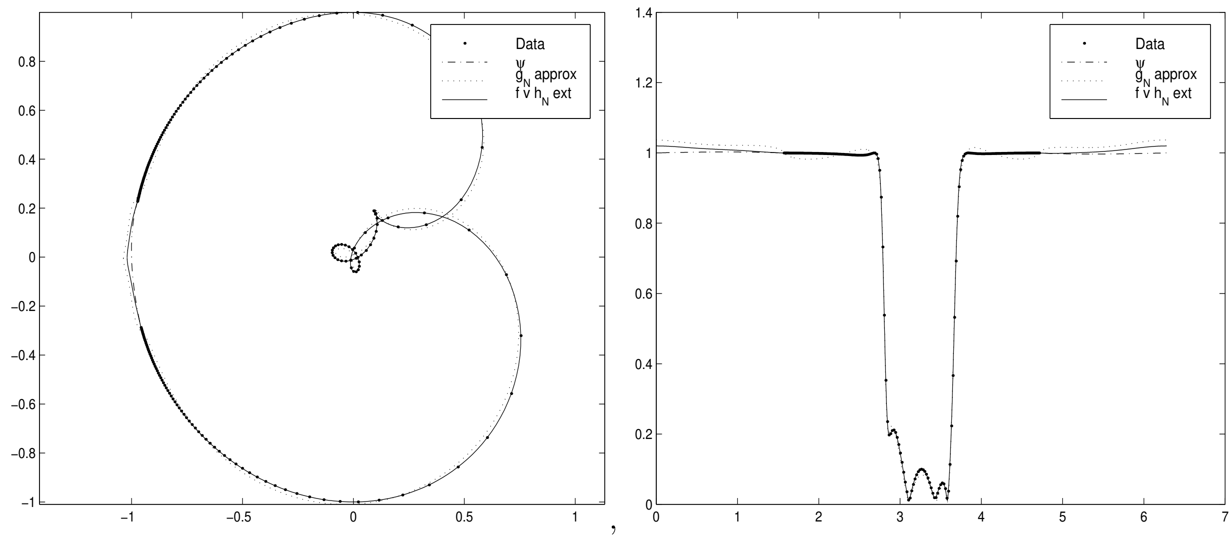

Figure 2: N=4, $\gamma_N = 0.017$

8 Conclusion

We have given constructive solutions to a pair of extremal problems in L^∞ that can be regarded as band-limited versions of the Adamjan–Arov–Krein approximation problems. Such approximation problems can be used to design robust algorithms for modelling experimental

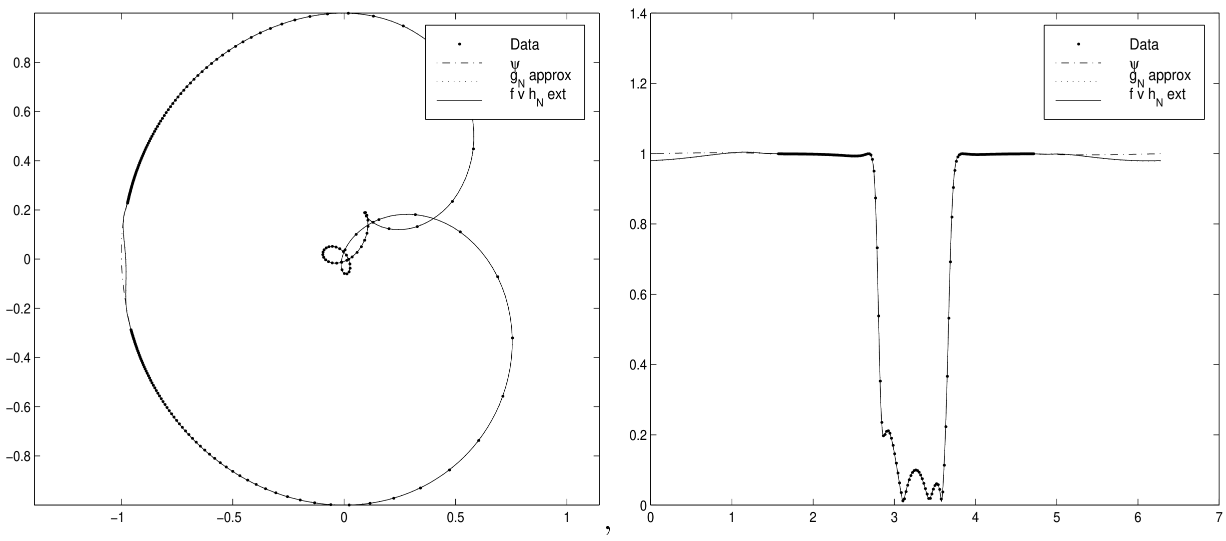

Figure 3: N=8, $\gamma_N = 0.0009$

data. They also have applications to related interpolation questions [9, 34], and can be used, for example, to test whether experimental data (either frequency response measurements or input/output measurements) are consistent with certain a priori assumptions on the system model.

Finally, we mention as an open question that of finding the closest to analytic extension with fixed real part: this would be useful for the numerical solution of Dirichlet–Neumann problems from incomplete data, which might typically be specified on the arc of the unit circle. Note the possible applications to physical problems such as fault detection, [6]. Another pending issue is the rate of convergence of our algorithms. In the H^2 case, the decay of the error when the norm of M increases and N is fixed has recently been taken up in [8].

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