Hardy approximation to L^p functions on subsets of the circle with $1 \leq p < \infty$.

Laurent Baratchart and Juliette Leblond

Abstract

We consider approximation of L^p functions by Hardy functions on subsets of the circle, for $1 \leq p < \infty$. After some preliminaries on the possibility of such an approximation which are connected to recovery problems of the Carleman type, we prove existence and uniqueness of the solution to a generalized extremal problem involving norm constraints on the complementary subset.

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1 Introduction

We investigate hereafter a generalization to arbitrary subsets of the unit circle $\mathbb{T}$ of classical extremal problems in Hardy spaces H^p of the disk for $1 \leq p < \infty$. More precisely, if K is a subset of $\mathbb{T}$ and if $\mathbb{T} \setminus K$ denotes its complement, the question we address can be stated as follows:

For $1 \leq p < \infty$, let us be given $f \in L^p(K)$, $h \in L^p(\mathbb{T} \setminus K)$, and $M > 0$. We seek $g \in H^p$ whose distance to h in $L^p(\mathbb{T} \setminus K)$ does not exceed M and which is as close as possible to f in the $L^p(K)$ metric under this constraint.

This article is a sequel to [8], which deals with the corresponding problem for $p = \infty$ and gives on the way some properties of traces of H^∞ functions on subsets of $\mathbb{T}$. The present paper, however, is self–contained.

When $K = \mathbb{T}$, then h and M play no role and we recognize a standard (dual) extremal problem (see e.g. [11, chap.8], [12, chap.IV], [14, chap.VII]). We shall refer to the above generalization as a bounded extremal problem (in short: BEP). If K is not of full measure, then traces of Hardy functions are dense in $L^p(K)$ (see *e.g.* [8, prop.1]). This makes it clear why the constraint $\|g - h\|_{L^p(\mathbb{T}\setminus K)} \leq M$ is needed for BEP to have a solution. Yet it is

natural to ask how this density property can be made effective, in other words how does one construct an H^p approximant to a function in $L^p(K)$?
When f is the trace on K of a Hardy function, we face a recovery problem that was originally approached by Carleman and then by Goluzin, Krylov, later by Lavrentyev, and more recently by Patil (see [1] and the bibliography therein, see also [16]). We will show in section 3 that certain recovery procedures of the Carleman type can be construed into approximation schemes to $f \in L^p(K)$, and that the boundedness of such schemes provides *criteria* to decide whether f is the trace of some H^p function or not, a so–called Loewner type question that was taken up in [21], [16], [17], [18], [15], [19], [1], [3], for instance.
We then turn to BEP and establish existence and uniqueness of a solution in section 4. Subsequently, in section 5, we solve BEP when $p = 2$ – the only case in which we were able to do so beyond $p = \infty$ which is handled in [8]. For $p = 2$, the problem was treated in [15] when $f = 0$ while $h = 0$ is solved in [2]; it may come as a surprise that extremal functions in the latter case are provided by the family of approximants that we shall derive from Aizenberg's account of Patil's recovery scheme [1].

The interested reader will find in [2] and [8] a description of the links between bounded extremal problems and identification of linear time–invariant dynamical control systems. An analoguous question was also studied in [5] for a Hardy–Sobolev class in H^2.

We shall be working entirely on the disk, but it is clear how to derive half–plane versions of our results, either by direct inspection or using conformal mapping (see e.g. [13, chap.8]).

2 Notation and statement of the problem

For $F \subset E$, we write $E \setminus F$ for the set–theoretic complement of F in E. Call $\mathbb{T}$ the unit circle and $\mathbb{D}$ the unit disk. We let μ be the Lebesgue measure on $\mathbb{T}$ and, for $1 \leq p \leq \infty$, we designate by H^p the Hardy space with exponent p of the unit disk. For $E \subset \mathbb{T}$, we denote by $L^p(E)$ the familiar Lebesgue space and by $\|.\|_{L^p(E)}$ the corresponding norm. The symbol $< \, , \, >_{L^2(E)}$ indicates the scalar product in $L^2(E)$. Further, $C(E)$ is the space of continuous complex–valued functions and $\mathcal{M}_E$ is the space of complex Borel measures on E. If f is a function defined on some set containing E, the symbol $f_{|_E}$ indicates the restriction of f to E, and we write without ambiguity $\|f\|_{L^p(E)}$ for the norm of this restriction. Accordingly, we set $H^p_{|_E}$ to mean the space of traces on E of H^p functions. *Whenever f is defined on E and h is defined on $\mathbb{T} \setminus E$, the notation $f \vee h$ is used for the concatenated function which is defined on all of $\mathbb{T}$.*

Let $K \subset \mathbb{T}$ and $1 \leq p < \infty$. For $f \in L^p(K)$, $h \in L^p(\mathbb{T} \setminus K)$, and M a positive real number, we define a subset $\mathcal{B}^p_{M,h}$ of H^p by:

$$\mathcal{B}^p_{M,h} := \{g \in H^p \, , \; \|h - g\|_{L^p(\mathbb{T}\setminus K)} \leq M\} \, .$$

With these notations, the bounded extremal problem we consider can be stated as :

BEP For K of positive Lebesgue measure and f, h, M as above, find $g_0 \in \mathcal{B}^p_{M,h}$ such that

$$\|f - g_0\|_{L^p(K)} = \min_{g \in \mathcal{B}^p_{M,h}} \|f - g\|_{L^p(K)} \,. \tag{1}$$

When $\mu(\mathbb{T} \setminus K) = 0$, we face a classical (dual) extremal problem for which a solution does exist and is unique for $1 \leq p < \infty$. In this paper, we shall carry these existence and uniqueness properties over to **BEP**.
Standard proofs when $K = \mathbb{T}$ rely on duality in Hardy spaces, interpreting the *infimum* in (1) as an operator norm (see e.g. [11], [12], [14]). Although such an interpretation is not available here, it can be supplemented by uniform convexity properties if $1 < p < \infty$ and, if $p = 1$, by combining differentiation with more refined results on Douglas algebras due to Axler.

When $\mu(K) > 0$, it follows from the density of $H^p_{|_{\mathbb{T}\setminus K}}$ in $L^p(\mathbb{T} \setminus K)$ already mentioned that $\mathcal{B}^p_{M,h} \neq \emptyset$. On the contrary, $\mathcal{B}^p_{M,h}$ may well be empty if $\mu(K) = 0$, a case that was formally ruled out in **BEP**.

3 H^p approximants in $L^p(K)$

Let us first recall a simple result which is proved for example in [8, prop.1].

Proposition 1 *Let K be a subset of $\mathbb{T}$ such that $\mu(\mathbb{T} \setminus K) > 0$, and let $1 \leq p < \infty$. Then $H^p_{|_K}$ is dense in $L^p(K)$.*

Remark 1 If in addition $\mu(K) > 0$, then $H^p_{|_K} \neq L^p(K)$. Indeed, it is easy to construct a non–zero $f \in C(K)$ that vanishes on a set of positive measure, see e.g. [8, rmk.1].

When $p > 1$, a constructive proof of proposition 1 can also be obtained in connexion with Carleman type integral formulae whose aim is to *reconstruct* analytic functions from their boundary values on a subset of $\mathbb{T}$ [1]. This theory requires auxiliary functions as follows:
Definition A function $\varphi \in H^\infty$ is said to be a quenching function if and only if $|\varphi| > 1$ in $\mathbb{D}$ and $|\varphi| = 1$ a.e. on $\mathbb{T} \setminus K$.

To every function $f \in H^p$ and every quenching function φ, one can then associate the family:

$$f_m(z) = \frac{1}{2\,i\,\pi} \int_K \left(\frac{\varphi(\xi)}{\varphi(z)}\right)^m \frac{f(\xi)\,d\xi}{\xi - z}\,,\ \forall z \in \mathbb{D}\,,\ m \geq 0. \tag{2}$$

If $1 < p < \infty$, let $P_{H^p} \,:\, L^p(\mathbb{T}) \to H^p$ denote the natural projection. For such p's, (2) may be rewritten as:

$$f_m = \frac{1}{\varphi^m}\, P_{H^p}\left(\varphi^m\, f \vee 0\right). \tag{3}$$

We then have the following result.

Theorem 1 ([1, thms1.1,1.11]) *Let K be a subset of $\mathbb{T}$ such that $\mu(K) > 0$. For $f \in H^p$ and φ a quenching function, define a sequence $(f_m)_{m \geq 0}$ of H^p functions by (2). It holds that:*
(i) for $1 < p < \infty$, f_m converges to f in H^p as m goes to infinity,
(ii) for $p = 1$, $f_m(z)$ tends to $f(z)$ uniformly over compact subsets of $\mathbb{D}$.

Theorem 1 remains valid for more general Hardy spaces. Point (ii) holds in $H^1(D)$ for any simply connected $D \subset \mathbb{C}$ with rectifiable boundary [1, thm.1.1]; this result is due to Goluzin and Krylov. Point (i) generalizes a recovery result of Patil [16, thm.1] and is stated in [1, thm.1.11] for any simply connected D whose boundary δD is regular enough to ensure that the projection $L^p(\delta D) \to H^p(D)$ is continuous.

Now, we observe that formula (2) makes perfectly good sense for any f in $L^p(K)$ – the values of f outside K being irrelevant – and we shall elaborate on theorem 1 to obtain an *approximation* procedure for $p > 1$.

Proposition 2 *Let $K \subset \mathbb{T}$ be such that $\mu(K) > 0$ and $\mu(\mathbb{T} \setminus K) > 0$. For $1 < p < \infty$, let $f \in L^p(K)$ and φ a quenching function of constant modulus a.e. on K. If $(f_m)_{m \geq 0}$ is the sequence of H^p functions defined by (2), then $f_{m|_K}$ converges to f in $L^p(K)$ as m goes to infinity.*

Proof: denote by ϱ the constant value of $|\varphi|$ a.e. on K. Using (3), we obtain:

$$\|f_m\|_{L^p(K)} = \frac{1}{\varrho^m} \|P_{H^p}(\varphi^m f \vee 0)\|_{L^p(K)} \leq \frac{1}{\varrho^m} \|P_{H^p}(\varphi^m f \vee 0)\|_{L^p(\mathbb{T})}$$
$$\leq \frac{C_p}{\varrho^m} \|\varphi^m f \vee 0\|_{L^p(\mathbb{T})} = C_p \|f\|_{L^p(K)}, \tag{4}$$

where the constant C_p is given by the M. Riesz theorem (see e.g. [13, chap. 9]). The family of operators $f \to f_m$ is thus uniformly bounded on $L^p(K)$ and converges pointwise to the identity on $H^p_{|K}$ by (i) of theorem 1. Since $H^p_{|K}$ is dense in $L^p(K)$, the conclusion follows. ■

Since it is outer, the quenching function with constant modulus ϱ on K is uniquely determined up to a multiplicative constant of modulus 1 by:

$$\varphi(z) = \exp\left\{\frac{\log \varrho}{2\pi} \int_K \frac{e^{it} + z}{e^{it} - z}\, dt\right\}, \; z \in \mathbb{D}. \tag{5}$$

Point (i) of theorem 1 was originally stated by Patil [16, thm.1] for this particular quenching function. The corresponding sequence (f_m) will turn out to have the *remarkable* property of being extremal for **BEP** when $p = 2$ and $h = 0$.

We show now that the approximants provided by propositions 1 and 2 are bound to have wild behaviour outside K. Recall that H^p is reflexive for $1 < p < \infty$ whence closed balls are weakly compact. Moreover, closed balls are also weak–* compact in H^1 when imbedded in the space $\mathcal{M}_{\mathbb{T}}$ of complex measures, as this is essentially the content of the F. and M. Riesz theorem (see e.g.[14, VII.A]).

Proposition 3 *Let $K \subset \mathbb{T}$ be such that $\mu(K) > 0$ and let $f \in L^p(K)$, with $1 \leq p < \infty$. Suppose (g_n) is a sequence of H^p functions such that ${g_n}_{|K}$ converges to f in $L^p(K)$. If f is not the trace of an H^p function, then $\lim_{n\to\infty} \|g_n\|_{L^p(\mathbb{T}\setminus K)} = \infty$.*

Remark 2 Proposition 3 remains true for $p = \infty$ and proof is the same [8, prop.2].

Proof: suppose the conclusion does not hold. Extracting a subsequence if necessary, we may assume that $(\|g_n\|_{L^p(\mathbb{T}\setminus K)})$ is bounded, hence that (g_n) is bounded in H^p. Then there exists a subsequence (g_{n_k}) converging weakly (weakly–$*$ if $p = 1$) toward some $g \in H^p$. Consider first the case where $p \neq 1$. Then it is clear that $({g_{n_k}}_{|K})$ converges weakly to $g_{|K}$ in $L^p(K)$, and since it also converges strongly to f by hypothesis, we get $f = g$ a.e. on K. Suppose now that $p = 1$. The weak–$*$ convergence of (g_{n_k}) means that:

$$\lim_{k\to\infty} \int_{\mathbb{T}} g_{n_k} w \, d\mu = \int_{\mathbb{T}} g w \, d\mu \quad \text{for all } w \in C(\mathbb{T}). \tag{6}$$

Upon multiplying f, g, and g_{n_k} by $e^{i\theta}$, we may as well assume that $g(0) = g_{n_k}(0) = 0$. Then, g and g_{n_k} are orthogonal to H^∞ because of the Cauchy formula. Therefore, (6) remains valid for $w \in H^\infty + C(\mathbb{T})$. Since every measurable subset of $\mathbb{T}$ is almost everywhere the zero set of some function in $H^\infty + C(\mathbb{T})$ [4], there exists $w_0 \in H^\infty + C(\mathbb{T})$ with the property that $w_0(e^{i\theta}) = 0$ a.e. on $\mathbb{T} \setminus K$ and $w_0(e^{i\theta}) \neq 0$ a.e. on K. As $H^\infty + C(\mathbb{T})$ is an algebra [12, IX,thm 2.2] and since the support of $w_0 w$ lies in K, we obtain

$$\int_{\mathbb{T}} g w_0 w \, d\mu = \lim_{k\to\infty} \int_{\mathbb{T}} g_{n_k} w_0 w \, d\mu = \lim_{k\to\infty} \int_K g_{n_k} w_0 w \, d\mu = \int_K f w_0 w \, d\mu \tag{7}$$

whenever $w \in H^\infty + C(\mathbb{T})$. In particular:

$$\int_{\mathbb{T}} (g - f \vee 0) \, w_0 \, w \, d\mu = 0 \quad \text{for all } w \in C(\mathbb{T}) \,,$$

implying that $(g - f \vee 0) \, w_0 = 0$ a.e. on $\mathbb{T}$. Hence $f = g_{|K}$, for $w_0 \neq 0$ a.e. on K. ■

Proposition 3 contains a clue to characterize traces on K of H^p functions:

construct for each $f \in L^p(K)$ a family (g_ρ) of H^p functions such that
(i) ${g_\rho}_{|K}$ converges to f in $L^p(K)$ when ρ tends to some value ρ_0,
(ii) g_ρ converges in H^p as $\rho \to \rho_0$ whenever $f \in H^p_{|K}$;
then, $f \in H^p_{|K}$ if and only if $\|g_\rho\|_{L^p(\mathbb{T}\setminus K)}$ remains bounded.

This principle can be illustrated at several places in the literature. It is implicit in [16, thm.2] where the boundedness of f_m defined by (2) and (5) is a necessary and sufficient condition for f to belong to $H^p_{|K}$; that (i) is satisfied in this case is just proposition 2.
Another example when $p = 2$ lies with the work in [18] where a process based on Mellin transforms initiated in [21] is generalized into a recovery procedure of Hardy functions on the half–plane from their values on a Borel subset of the real line; this procedure may again be construed into an approximation scheme satisfying (i), (ii) [7, prop.2].

Yet another instance of this kind of characterization is [15, thm.4.1]; the problem considered by these authors on the half–plane translates to the disk into **BEP** with $p = 2$ and $f = 0$. The cited theorem – to be found also in [1, thm.27.23] – is equivalent to asserting that $h \in H^2_{|_{\mathbb{T}\setminus K}}$ if and only if the solution to **BEP** remains bounded as M goes to zero; dually, [17, thm.1] amounts to the fact that $f \in H^2_{|_K}$ if and only if the solution to **BEP** remains bounded as M goes to infinity.

4 Bounded extremal problems

Let us denote by $\mathcal{C}^p_{M,h}$ the trace of $\mathcal{B}^p_{M,h}$ on K:

$$\mathcal{C}^p_{M,h} := \{g_{|_K}\,,\ g \in \mathcal{B}^p_{M,h}\}.$$

Theorem 2 *Assume that both K and $\mathbb{T} \setminus K$ are of positive measure. There exists a unique solution g_0 to* **BEP***; we have that:*

$$\|h - g_0\|_{L^p(\mathbb{T}\setminus K)} = M \tag{8}$$

whenever $f \notin \mathcal{C}^p_{M,h}$.

The proof of theorem 2 will occupy the remainder of this section. We will have to distinguish between the cases $1 < p < \infty$ and $p = 1$, the latter being more difficult.

4.1 Some preliminary results

Denote by q the conjugate exponent of p and let $H^q_0 := e^{i\theta}H^q$ be the subspace of H^q consisting of functions vanishing at the origin. The following closedness property will be used to establish theorem 2 for $p > 1$.

Theorem 3 *For K an arbitrary subset of $\mathbb{T}$ and $1 < p < \infty$, $\mathcal{C}^p_{M,h}$ is closed in $L^p(K)$. Specifically, we have:*

$$\mathcal{C}^p_{M,h} = \bigcap_{\substack{v \in H^q_0 \\ \|v\|_{L^q(\mathbb{T}\setminus K)\le 1}}} \{w \in L^p(K)\,;\ \left|\int_K wv\,d\mu + \int_{\mathbb{T}\setminus K} hv\,d\mu\right| \le M\}\,. \tag{9}$$

Remark 3 Equation (9) remains true for $p = \infty$ and proof is the same [8, thm.3].

Proof: if $\mu(K) = 0$, then $\mathcal{C}^p_{M,h} = L^p(K) = 0$ and the conclusion trivially holds. Assume that $\mu(K) > 0$. Since $q < \infty$, it follows from proposition 1 that $H^q_{|_{\mathbb{T}\setminus K}}$ is dense in $L^q(\mathbb{T} \setminus K)$ and it is easily seen that the same is true of $H^q_{0\,|_{\mathbb{T}\setminus K}}$.
Now, take $w \in \mathcal{C}^p_{M,h}$ and write $w = g_{|_K}$ with $g \in \mathcal{B}^p_{M,h}$. By duality and using the Cauchy formula:

$$M \ge \|h - g\|_{L^p(\mathbb{T}\setminus K)} = \sup_{\substack{v \in H^q_0 \\ \|v\|_{L^q(\mathbb{T}\setminus K)\le 1}}} \left|\int_{\mathbb{T}\setminus K}(h-g)v\,d\mu\right| = \sup_{\substack{v \in H^q_0 \\ \|v\|_{L^q(\mathbb{T}\setminus K)\le 1}}} \left|\int_{\mathbb{T}}(w \vee h)v\,d\mu\right|.$$

Conversely, let $w \in L^p(K)$ be such that:

$$\sup_{\substack{v \in H_0^q \\ \|v\|_{L^q(\mathbb{T}\setminus K)\leq 1}}} \left|\int_{\mathbb{T}} (w \vee h) v \, d\mu\right| \leq M\,.$$

Then the map $L_w : H^q_{0|_{\mathbb{T}\setminus K}} \to \mathbb{C}$ defined by

$$L_w(v_{|_{\mathbb{T}\setminus K}}) := \int_{\mathbb{T}} (w \vee h)\, v \, d\mu \tag{10}$$

extends to a continuous linear form on $L^q(\mathbb{T} \setminus K)$ whose norm is at most M. Thus, there exists $w' \in L^p(\mathbb{T} \setminus K)$ such that $\|w'\|_{L^p(\mathbb{T}\setminus K)} \leq M$ and

$$L_w(v_{|_{\mathbb{T}\setminus K}}) = \int_{\mathbb{T}\setminus K} w' v \, d\mu \quad \text{ for } v \in H_0^q. \tag{11}$$

Substracting (11) from (10), we see that the function $w \vee (h - w')$ belongs to H^p hence to $\mathcal{B}^p_{M,h}$. Consequently, w is in $\mathcal{C}^p_{M,h}$ and we obtain (9). ■

To handle the case $p = 1$ in theorem 2, we shall need two lemmas.

Lemma 1 *Let (g_n) be a sequence of functions converging weakly–* to g in H^1. For E a compact subset of $\mathbb{T}$ and $f \in L^1(E)$, we have that*

$$\|f - g\|_{L^1(E)} \leq \liminf_n \|f - g_n\|_{L^1(E)}. \tag{12}$$

Proof: upon multiplying all functions by $e^{i\theta}$, we may assume that $g_n(0) = g(0) = 0$. As in the proof of proposition 3, weak–* convergence sharpens to

$$\lim_{n\to\infty} \int_{\mathbb{T}} g_n w \, d\mu = \int_{\mathbb{T}} g w \, d\mu \quad \text{ for all } w \in H^\infty + C(\mathbb{T}). \tag{13}$$

Let $\Lambda(E)$ denote the closed unit ball of $C(E)$ endowed with the sup norm. By the Tietze–Urysohn theorem, and since E is closed, we have $\Lambda(E) = \Lambda(\mathbb{T})_{|_E}$. Defining

$$\widetilde{\Lambda(E)} := \{v \in L^\infty(\mathbb{T}) \; ; \; v_{|_E} \in \Lambda(E) \; \text{ and } \; v_{|_{\mathbb{T}\setminus E}} = 0\},$$

this can be rephrased as

$$\widetilde{\Lambda(E)} = \chi_E \Lambda(\mathbb{T}). \tag{14}$$

Also, since any function in $L^\infty(\mathbb{T})$ is the quotient of some member of $H^\infty + C(\mathbb{T})$ by a Blaschke product [4], we can write

$$\chi_E = \bar{b} w_E, \quad w_E \in H^\infty + C(\mathbb{T}), \quad b \text{ a Blaschke product.} \tag{15}$$

Now by duality

$$\|f - g\|_{L^1(E)} = \|b(f - g)\|_{L^1(E)} = \sup_{v \in \widetilde{\Lambda(E)}} \left|\int_{\mathbb{T}} b(f - g) v \, d\mu\right| = \sup_{w \in \Lambda(\mathbb{T})} \left|\int_{\mathbb{T}} (f - g) w_E w \, d\mu\right|,$$

where the last equality uses (14) and (15). Using that $H^\infty + C(\mathbb{T})$ is an algebra, we see from (13) that the argument of the last supremum is just

$$\lim_{n\to\infty} \left| \int_{\mathbb{T}} (f - g_n) w_E w \, d\mu \right| . \tag{16}$$

To establish (12), it remains to prove that (16) is majorized by $\liminf \|f - g_n\|_{L^1(E)}$, but this follows at once from the inequality

$$\left| \int_{\mathbb{T}} (f - g_n) w_E w \, d\mu \right| \le \|f - g_n\|_{L^1(E)} .$$

■

Lemma 2 *For any subset E of $\mathbb{T}$, the map:*

$$\begin{array}{rccl} \mathcal{N}_E : & L^1(E) & \to & \mathbb{R} \\ & v & \mapsto & \|v\|_{L^1(E)} \end{array}$$

is Gâteaux–differentiable at every $v \in L^1(E)$ such that $v \neq 0$ a.e. on E; the derivative in the direction $u \in L^1(E)$ is then:

$$D\mathcal{N}_E(v).[u] = \mathrm{Re} \int_E \frac{\overline{v}}{|v|} u \, d\mu . \tag{17}$$

Proof: since v is a.e. nonzero on E, it has a unique norming functional which is given by (17). The result now follows from [10, 3,I,2,rmk.6]. ■

4.2 Proof of theorem 2: existence

The set $\mathcal{C}^p_{M,h} \subset L^p(K)$ is clearly convex. When $1 < p < \infty$, it is closed by theorem 3. This is enough to obtain at *the same time* existence and uniqueness of g_0 from the fact that there is a best approximation projection on any closed convex subset of a uniformly convex Banach space (see e.g. [10, 3,II,1,prop. 5 and 8]).

When $p = 1$, denote by β_1 the infimum in the right hand–side of (1) and let (g_n) be a minimizing sequence in $\mathcal{B}^1_{M,h}$. Then $\|g_n\|_{L^1(\mathbb{T})}$ is bounded and we may extract a subsequence (g_{n_k}) converging weakly–* in $\mathcal{M}_{\mathbb{T}}$ to $g \in H^1$. There is a slight difficulty here, due to the fact that K is not necessarily locally compact so that duality between $C(K)$ and $\mathcal{M}_K$ may fail. However, there exists a sequence (K_m) of compact subsets of K such that $\mu(K \setminus K_m) \to 0$ (see e.g. [20, thm.2.18]). Since, by definition:

$$\lim_{n\to\infty} \|f - g_n\|_{L^1(K)} = \beta_1 , \tag{18}$$

then *a fortiori*

$$\liminf_n \|f - g_n\|_{L^1(K_m)} \le \beta_1 \quad \text{for each} \quad m ,$$

and by lemma 1

$$\|f-g\|_{L^1(K_m)} \leq \beta_1 \quad \text{for each} \quad m\,.$$

By monotone convergence, it follows now that

$$\|f-g\|_{L^1(K)} \leq \beta_1\,. \tag{19}$$

Because $\|h-g_n\|_{L^1(\mathbb{T}\setminus K)} \leq M$, a similar argument when changing K into $\mathbb{T}\setminus K$ and f into h shows that $\|h-g\|_{L^1(\mathbb{T}\setminus K)} \leq M$, in other words $g \in \mathcal{B}^1_{M,h}$. Thus (19) is an equality and we can take $g_0 = g$. This proves the existence part of theorem 2.

4.3 Proof of theorem 2: saturation of the constraint

Assume that $\|h-g_0\|_{L^p(\mathbb{T}\setminus K)} < M$ and define the map

$$\begin{array}{rccl} \psi_p: & L^p(K) & \longrightarrow & \mathbb{R} \\ & \xi & \longmapsto & \|f-\xi\|_{L^p(K)}\,, \end{array}$$

for $1 \leq p < \infty$. There is a neighborhood of g_0 in H^p which is contained in $\mathcal{B}^p_{M,h}$. Denoting by $R_K : H^p \to L^p(K)$ the natural restriction, this means that the map $\psi_p \circ R_K$ attains a local minimum at g_0 hence a global minimum, for this map is convex. Thanks to proposition 1, we know that f belongs to the closure of $H^p_{|_K}$ in $L^p(K)$ so this global minimum can only be zero. This entails $f = {g_0}_{|_K}$ hence $f \in \mathcal{C}^p_{M,h}$. Thus, (8) holds under the stated assumption.

4.4 Proof of theorem 2: uniqueness

The case $p > 1$ was settled in section 4.2 so we need only consider $p = 1$.
Let g_0 be a solution to **BEP**. Our first task will be to derive a critical point equation for g_0 in this case, under the extra–assumption that

$$(f \vee h) - g_0 \neq 0 \text{ a.e. on } \mathbb{T}. \tag{20}$$

We may thus define a subspace of H^1 by putting

$$T = \{u \in H^1; \qquad \mathrm{Re} \int_{\mathbb{T}\setminus K} \frac{\overline{h-g_0}}{|h-g_0|} u \, d\mu = 0\}\,.$$

Next, since $H^1_{|_{\mathbb{T}\setminus K}}$ is dense in $L^1(\mathbb{T}\setminus K)$ by proposition 1, we can find $v \in H^1$ such that

$$c_v := \mathrm{Re} \int_{\mathbb{T}\setminus K} \frac{\overline{h-g_0}}{|h-g_0|} (v-g_0)\, d\mu > 0\,, \tag{21}$$

for if $v_{|_{\mathbb{T}\setminus K}}$ tends to h in $L^1(\mathbb{T}\setminus K)$ then (21) tends to $\|h-g_0\|_{L^1(\mathbb{T}\setminus K)}$. Fix such a v, choose $u \in T$ and some $t > 0$, and consider the map

$$\begin{array}{rccl} G: & \mathbb{R} & \to & \mathbb{R} \\ & \varepsilon & \mapsto & \|h-g_0+\varepsilon\,(u+t(g_0-v))\|_{L^1(\mathbb{T}\setminus K)}. \end{array}$$

It follows from (20) and lemma 2 that G is differentiable at $\varepsilon = 0$:

$$G(\varepsilon) = G(0) + \varepsilon\, D\mathcal{N}_{\mathbb{T}\setminus K}(h - g_0).[u + t(g_0 - v)] + \varepsilon\, \eta(\varepsilon)\,, \tag{22}$$

where η depends on u, t, and v but in any case satisfies $\lim_{\varepsilon\to 0} \eta(\varepsilon) = 0$. Since $f \notin \mathcal{C}^1_{M,h}$ by (20), we have $G(0) = M$ by section 4.3; using (17), we can rewrite (22) as

$$G(\varepsilon) = M + \varepsilon\, \mathrm{Re} \int_{\mathbb{T}\setminus K} \frac{\overline{h - g_0}}{|h - g_0|}(u + t(g_0 - v))\, d\mu\, + \varepsilon\, \eta(\varepsilon)\,.$$

Because $u \in T$, this reduces to

$$G(\varepsilon) = M - \varepsilon\, t\, c_v + \varepsilon\, \eta(\varepsilon)\,,$$

and since $u\,,\ v \in H^1$ it follows from the definition of G that:

$$g_0 - \varepsilon\, (u + t(g_0 - v)) \quad \in \mathcal{B}^1_{M,h}$$

as soon as $\eta(\varepsilon) < t\, c_v$. Hence, for ε small enough:

$$\psi_1(g_{0|_K}) \le \psi_1 \left(g_{0|_K} - \varepsilon\, (u + t(g_0 - v))_{|_K} \right)\,.$$

If we now consider the map

$$\begin{array}{rccl} P: & \mathbb{R} & \to & \mathbb{R} \\ & \varepsilon & \mapsto & \|f - g_0 + \varepsilon\, (u + t(g_0 - v))\|_{L^1(K)}, \end{array}$$

this means that $P(\varepsilon)$ attains a local minimum on $\mathbb{R}^+$ at $\varepsilon = 0$ and since it is differentiable at this point, thanks again to lemma 2 and (20), we obtain:

$$\frac{dP}{d\varepsilon}(0) = D\mathcal{N}_K(f - g_0).[u + t(g_0 - v)] = \mathrm{Re} \int_K \frac{\overline{f - g_0}}{|f - g_0|}(u + t(g_0 - v))\, d\mu \ge 0.$$

Letting t go to zero, we get

$$\mathrm{Re} \int_K \frac{\overline{f - g_0}}{|f - g_0|} u\, d\mu \ge 0,$$

whence

$$\mathrm{Re} \int_K \frac{\overline{f - g_0}}{|f - g_0|} u\, d\mu = 0 \qquad \forall\ u \in T \tag{23}$$

for T is a linear space. This is the critical point equation we were looking for.

Next, thanks to hypothesis (20) again, we may define two linear forms on $L^1(\mathbb{T})$ by the formulas:

$$\phi_h(\xi) := \int_{\mathbb{T}\setminus K} \frac{\overline{h - g_0}}{|h - g_0|}\, \xi\, d\mu \quad \text{and} \quad \phi_f(\xi) := \int_K \frac{\overline{f - g_0}}{|f - g_0|}\, \xi\, d\mu\,. \tag{24}$$

We have Re $\phi_h(T) = 0$ by the definition of T and also Re $\phi_f(T) = 0$ by (23). Choose $w \in H^1$ such that $w \notin T$. This is possible for otherwise $\mathrm{Re}\, \phi_h$ would vanish on H^1 and, since $\phi_h(\xi)$

depends on $\xi_{|_{\mathbb{T}\setminus K}}$ only, it would vanish identically by the density of $H^1_{|_{\mathbb{T}\setminus K}}$ in $L^1(\mathbb{T}\setminus K)$. Since a linear form is completely determined by its real part, ϕ_h itself would vanish identically in $L^1(\mathbb{T}\setminus K)$ and this cannot happen. Set

$$m := \frac{\text{Re } \phi_f(w)}{\text{Re } \phi_h(w)}, \tag{25}$$

and consider the composite linear form on $L^1(\mathbb{T})$:

$$L_{f,h} := \phi_f - m\,\phi_h. \tag{26}$$

By restriction, $L_{f,h}$ induces a continuous linear form on H^1 whose real part vanishes on T and at w. Because any function $\varphi \in H^1$ can be written as

$$\varphi = \frac{\phi_h(\varphi)}{\phi_h(w)} w + u$$

where u belongs to T, we conclude that Re $L_{h,f}$ vanishes on H^1, hence $L_{h,f}$ itself vanishes on H^1. By duality, there exists a function $\rho \in H_0^\infty$ such that

$$L_{h,f}(\xi) = \int_{\mathbb{T}} \rho\,\xi\,d\mu\,,\ \forall \xi \in L^1(\mathbb{T})\,. \tag{27}$$

Note that $\rho \neq 0$ for $L_{h,f}$ is not the zero form:

$$L_{h,f}((f - g_{0|_K}) \vee 0) = \|f - g_0\|_{L^1(K)} > 0\,.$$

If χ_K and $\chi_{\mathbb{T}\setminus K}$ denote the characteristic functions of K and $\mathbb{T}\setminus K$ respectively, $L_{h,f}$ can be rewritten using (24) and (26) as:

$$L_{h,f}(\xi) = \int_{\mathbb{T}} (\chi_K - m\,\chi_{\mathbb{T}\setminus K})\,\frac{\overline{f \vee h - g_0}}{|f \vee h - g_0|}\,\xi\,d\mu\,,$$

so that necessarily

$$(\chi_K - m\,\chi_{\mathbb{T}\setminus K})\,\frac{\overline{f \vee h - g_0}}{|f \vee h - g_0|} = \rho \in H_0^\infty\,,\ \text{a.e. on } \mathbb{T}\,. \tag{28}$$

Here, it is essential to recognize that T, ϕ_h, ϕ_f, and thus also m, *depend only on the function* $\overline{f \vee h - g_0}/|f \vee h - g_0|$.

We are now in position to establish uniqueness when $p = 1$. If $f \in \mathcal{C}^1_{M,h}$, it is obvious. Otherwise, let g_0 and g_0' be two solution to **BEP**, *where we no longer assume* (20) *to hold.* Let $Z_0, Z_0' \subset \mathbb{T}$ be the zero sets of $f \vee h - g_0$ and $f \vee h - g_0'$ respectively. Since $\mathcal{C}^1_{M,h}$ and ψ_1 are convex, $l\,g_0 + (1-l)\,g_0'$ is again a solution of **BEP** for each $l \in [0,1]$:

$$\int_K |l\,(f - g_0) + (1-l)\,(f - g_0')|\,d\mu = \int_K (l\,|f - g_0| + (1-l)\,|f - g_0'|)\,d\mu = \beta_1\,,$$

and also:

$$\int_{\mathbb{T}\setminus K} |l\,(h-g_0)+(1-l)\,(h-g_0')|\,d\mu = \int_{\mathbb{T}\setminus K} (l\,|h-g_0|+(1-l)\,|h-g_0'|)\,d\mu = M$$

since the constraint gets saturated by section 4.3. In each of the above equations, the integrands must clearly be equal a.e. and this means that the arguments of $f\vee h-g_0$ and $f\vee h-g_0'$ agree whenever both are defined:

$$\frac{\overline{f\vee h-g_0}}{|f\vee h-g_0|} = \frac{\overline{f\vee h-g_0'}}{|f\vee h-g_0'|} \quad \text{a.e. on } \mathbb{T}\setminus(Z_0\cup Z_0')\,. \tag{29}$$

Assume first that $\mu(Z_0)=\mu(Z_0')=0$. Then (20) is satisfied both for g_0 and g_0'. By (29) and a previous remark, equation (28) holds for g_0 and g_0' *with the same* m and thus the same function ρ:

$$(\chi_K - m\,\chi_{\mathbb{T}\setminus K})\,\frac{\overline{f\vee h-g_0}}{|f\vee h-g_0|} = (\chi_K - m\,\chi_{\mathbb{T}\setminus K})\,\frac{\overline{f\vee h-g_0'}}{|f\vee h-g_0'|} = \rho\in H_0^\infty\,,\ \text{a.e. on } \mathbb{T}.$$

Upon multiplying by $f\vee h-g_0$ and $f\vee h-g_0'$ respectively, we get

$$\begin{aligned}\rho\,(f\vee h-g_0) &= (\chi_K - m\,\chi_{\mathbb{T}\setminus K})\,|f\vee h-g_0|\,,\\ \rho\,(f\vee h-g_0') &= (\chi_K - m\,\chi_{\mathbb{T}\setminus K})\,|f\vee h-g_0'|\,,\end{aligned}$$

so that $\rho\,(g_0-g_0')\in H^1$ takes real values a.e. on $\mathbb{T}$ hence is constant. Since $\rho(0)=0$, this constant has to be zero, and $g_0=g_0'$ because $\rho\neq 0$.

Assume now that $\mu(Z_0)>0$.
If $\mu(Z_0\cap Z_0')>0$ then g_0 and g_0' are both equal to $f\vee h$ a.e. over some set of positive measure, so they are equal.
Next, consider the case where $\mu(Z_0\cap Z_0')=0$. For each $l\in(0,1)$,

$$g_l = l\,g_0+(1-l)\,g_0'$$

is also a solution of to **BEP**. Let Z_l be the zero set of $f\vee h-g_l$ and assume that $\mu(Z_l)>0$. If $\mu(Z_l\cap Z_0)>0$, we get $g_0=g_l$ by the previous argument whence $g_0=g_0'$ also. We proceed similarly if $\mu(Z_l\cap Z_0')>0$. Finally, assume that $\mu(Z_l\cap Z_0)=0$ and $\mu(Z_l\cap Z_0')=0$ for each $l\in(0,1)$. Then $f\vee h-g_0$ and $f\vee h-g_0'$ have the same argument a.e. on Z_l by (29). By definition however,

$$f\vee h-g_l = l\,(f\vee h-g_0)+(1-l)\,(f\vee h-g_0') = 0 \quad \text{a.e. on } Z_l\,,$$

implying

$$\frac{f\vee h-g_0'}{f\vee h-g_0} = -\frac{l}{1-l} < 0\,,\ \text{a.e. on } Z_l\,,$$

which contradicts the equality of the arguments.
Hence, either $g_0=g_0'$ or else our initial assumption that $\mu(Z_l)>0$ is false. But if we suppose that $\mu(Z_l)=0$ for each l we get $g_l=g_{l'}$ for any $l,l'\in(0,1)$ by the first case of the proof. Choosing $l\neq l'$ and solving for g_0 and g_0' again leads to $g_0=g_0'$. ∎

5 Solution to BEP for $p = 2$

When $p = 2$, **BEP** was solved when $f = 0$ in [15] and when $h = 0$ in [2]. In this section, we use the results of the last reference to derive a solution in the general case. Let

$$\begin{array}{rccc} \Phi : & H^2 & \to & H^2 \\ & g & \mapsto & P_{H^2}\left(\chi_{\mathbb{T}\setminus K}\, g\right) \end{array}$$

be the Toeplitz operator with symbol $\chi_{\mathbb{T}\setminus K}$. If $h = 0$ and $f \notin \mathcal{C}^2_{M,0}$, it is shown in [2] that the solution to **BEP** is given by the implicit equation:

$$g_0 = (1 + \lambda\,\Phi)^{-1}\, P_{H^2}\,(f \vee 0)\,, \tag{30}$$

where $\lambda \in (-1, +\infty)$ is the unique real number such that $\|g_0\|_{L^2(\mathbb{T}\setminus K)} = M$. The parameter λ is a Lagrange multiplier expressing the dependence between the derivative of the *criterium* $\|f - g\|_{L^2(K)}$ and the normal to the manifold $\mathcal{B}^2_{M,h}$ at g_0. Note that the *spectrum* of Φ is $[0, 1]$ [19, chap.3,p.63] so that $(1 + \lambda\,\Phi)$ is indeed invertible.

Theorem 4 *For $p = 2$ and $f \notin \mathcal{C}^2_{M,h}$, the solution to* **BEP** *is given by the implicit equation:*

$$g_0 = (1 + \lambda\,\Phi)^{-1}\, P_{H^2}\,\left(f \vee (\lambda + 1)\, h\right)\,, \tag{31}$$

where $\lambda \in (-1, +\infty)$ is the unique real number such that $\|g_0 - h\|_{L^2(\mathbb{T}\setminus K)} = M$.

Proof: assume first that $h \in H^2_{|\mathbb{T}\setminus K}$ so that $h = k_{|\mathbb{T}\setminus K}$ for $k \in H^2$, and let $\kappa = f - k_{|K} \in L^2(K)$. Consider the following instance of **BEP**:

$$\min_{g \in \mathcal{B}^2_{M,0}} \|\kappa - g\|_{L^2(K)}\,. \tag{32}$$

Since $\kappa \notin \mathcal{C}^2_{M,0}$, the solution g_κ to (32) is given according to (30) by

$$g_\kappa = (1 + \lambda\,\Phi)^{-1}\, P_{H^2}\,(\kappa \vee 0)\,, \tag{33}$$

for λ such that $\|g_\kappa\|_{L^2(\mathbb{T}\setminus K)} = M$. Set $g_0 = g_\kappa + k \in H^2$; since g_κ is a solution to problem (32), the function g_0 is a solution to **BEP**:

$$\|f - g_0\|_{L^2(K)} = \min_{g \in \mathcal{B}^2_{M,h}} \|f - g\|_{L^2(K)}\,. \tag{34}$$

Using (33) in the definition of g_0 yields (31) when $h \in H^2_{|\mathbb{T}\setminus K}$.
Next, consider any $h \in L^2(\mathbb{T}\setminus K)$. Since $H^2_{|\mathbb{T}\setminus K}$ is dense in $L^2(\mathbb{T}\setminus K)$, there exists a sequence (h_n) in $H^2_{|\mathbb{T}\setminus K}$ which converges to h in $L^2(\mathbb{T}\setminus K)$. To each h_n is associated a **BEP** with a unique solution g_n:

$$\|f - g_n\|_{L^2(K)} = \min_{g \in \mathcal{B}^2_{M,h_n}} \|f - g\|_{L^2(K)}\,,$$

and by what we just said:

$$(1 + \lambda\,\Phi)\, g_n = P_{H^2}\,\left(f \vee (\lambda + 1)\, h_n\right)\,. \tag{35}$$

The sequence (g_n) is bounded in H^2; thus, extracting a subsequence if necessary, (g_n) converges weakly in $L^2(\mathbb{T})$ to some $\gamma \in H^2$. As Φ is self–adjoint, we get that, for every $\xi \in H^2$,

$$((1+\lambda\,\Phi)\,g_n, \xi)_{L^2(\mathbb{T})} = (g_n, (1+\lambda\,\Phi)\,\xi)_{L^2(\mathbb{T})}$$

converges to

$$(\gamma, (1+\lambda\,\Phi)\,\xi)_{L^2(\mathbb{T})} = ((1+\lambda\,\Phi)\,\gamma, \xi)_{L^2(\mathbb{T})}\,.$$

The same argument applies to the right hand side of (35) and we obtain in the limit:

$$(1+\lambda\,\Phi)\,\gamma = P_{H^2}\,\left(f \vee (\lambda+1)\,h\right)\,.$$

Since closed balls are weakly compact in $L^2(\mathbb{T}\backslash K)$, we have that $\gamma \in \mathcal{B}^2_{M,h}$. By the definition of g_0, this implies:

$$\|f-g_0\|_{L^2(K)} \le \|f-\gamma\|_{L^2(K)}\,. \tag{36}$$

If equality holds in (36), we are done by the uniqueness asserted in theorem 2. Assume on the contrary that:

$$\|f-g_0\|_{L^2(K)} < \|f-\gamma\|_{L^2(K)}\,.$$

Since $\|h-g_0\|_{L^2(\mathbb{T}\backslash K)} = M$, we can construct $g_0' \in H^2$ such that $\|h-g_0'\|_{L^2(\mathbb{T}\backslash K)} < M$ and $\|f-g_0'\|_{L^2(K)} < \|f-\gamma\|_{L^2(K)} - \eta$, for some $\eta > 0$ (choose $v \in H^2$ of unit norm such that $< h-g_0, v >_{L^2(\mathbb{T}\backslash K)} > 0$ and $\varepsilon > 0$ so small that $g_0' = g_0 + \varepsilon\, v$ does the job). Then, for n large enough, $\|h_n - g_0'\|_{L^2(\mathbb{T}\backslash K)} < M$ and $\|f-g_n\|_{L^2(K)} \ge \|f-\gamma\|_{L^2(K)} - \eta$ since $\underline{\lim}_{n\to\infty}\|f-g_n\|_{L^2(K)} \ge \|f-\gamma\|_{L^2(K)}$. To recap, $g_0' \in \mathcal{B}^2_{M,h_n}$ and still $\|f-g_0'\|_{L^2(K)} < \|f-g_n\|_{L^2(K)}$: a contradiction. ∎

Corollary 1 *For $p=2$ and $f \notin \mathcal{C}^2_{M,h}$, the solution to* **BEP** *is given by the implicit formula*

$$g_0(z) = \frac{1}{2\,i\,\pi}\int_{\mathbb{T}} \left(\frac{\varphi(\xi)}{\varphi(z)}\right)^{\alpha} (f\vee h)(\xi)\,\frac{d\xi}{\xi - z}\,,\ \forall z \in \mathbb{D}\,, \tag{37}$$

where φ is a quenching function of modulus ϱ on K defined by (5) *and*

$$\alpha = -\frac{\log(\lambda+1)}{2\,\log\varrho}\,,$$

$\lambda \in (-1,\infty)$ *being the unique real number such that* $\|g_0\|_{L^2(\mathbb{T}\backslash K)} = M$.

Remark 4 It is clear from the corollary that the functions defined by (2) are extremal for **BEP** when $h=0$. This is all the more remarkable that this family was introduced in [16] without reference to optimization.

Proof: equation (37) is equivalent to

$$\varphi^{\alpha}\,g_0 = P_{H^2}\,\left(\varphi^{\alpha}\,(f\vee h)\right)\,.$$

Let $\bar{H}^\infty$ denote the Hardy space of the complement of the disk; multiplying by $\bar{\varphi}^\alpha \in \bar{H}^\infty$ and projecting again onto H^2 gives us:

$$P_{H^2}\left(|\varphi|^{2\alpha}\, g_0\right) = P_{H^2}\left(|\varphi|^{2\alpha}(f \vee h)\right) .$$

As

$$\varrho^{2\alpha} = \frac{1}{\lambda + 1}$$

and since

$$|\varphi|^{2\alpha} = \varrho^{2\alpha}\, \chi_K + \chi_{\mathbb{T}\setminus K} ,$$

we obtain

$$P_{H^2}\left(\chi_K\, g_0 + (\lambda+1)\, \chi_{\mathbb{T}\setminus K}\, g_0\right) = P_{H^2}\left(f \vee (\lambda+1)\, h\right)$$

which is nothing but theorem 4. ■

When computing g_0, the implicit character in M of equation (31) may be circumvented by dichotomy using the following result.

Proposition 4 *If $f \notin H^2_{|K}$, then M is a decreasing function of $\lambda \in (-1, \infty)$ onto $(0, \infty)$ whose derivative does not vanish.*

Proof: by (31), the mapping $\lambda \mapsto g_0(\lambda)$ is smooth $(-1, \infty) \to H^2$:

$$\frac{dg_0}{d\lambda} = -(1 + \lambda\, \Phi)^{-2}\, \Phi\, P_{H^2}\left(f \vee (\lambda+1)\, h\right) + (1 + \lambda\, \Phi)^{-1}\, P_{H^2}\left(0 \vee h\right)$$

$$= -(1 + \lambda\, \Phi)^{-1}\left(\Phi\, g_0 - P_{H^2}\left(0 \vee h\right)\right) .$$

Since $M = \|g_0 - h\|_{L^2(\mathbb{T}\setminus K)}$ by theorem 4, we compute:

$$\frac{dM^2}{d\lambda} = -2\,\mathrm{Re} < (1 + \lambda\, \Phi)^{-1}\left(\Phi\, g_0 - P_{H^2}\left(0 \vee h\right)\right), g_0 - h >_{L^2(\mathbb{T}\setminus K)}$$

$$= -2\,\mathrm{Re} < (1 + \lambda\, \Phi)^{-1}\left(\Phi\, g_0 - P_{H^2}\left(0 \vee h\right)\right), \Phi\, g_0 - P_{H^2}\left(0 \vee h\right) >_{L^2(\mathbb{T})} . \tag{38}$$

Because

$$< (1 + \lambda\, \Phi)\, \gamma, \gamma >_{L^2(\mathbb{T})} \geq \min(1, 1+\lambda)\, \|\gamma\|^2_{L^2(\mathbb{T})} \quad \forall \gamma \in H^2 ,$$

we see that (38) is negative unless $\Phi\, g_0 = P_{H^2}\left(0 \vee h\right)$, in other words unless

$$P_{H^2}\left(0 \vee \left(g_{0|_{\mathbb{T}\setminus K}} - h\right)\right) = 0 .$$

This is possible only if $g_{0|_{\mathbb{T}\setminus K}} = h$, contradicting the hypothesis $f \notin H^2_{|K}$ in view of theorem 4. ■

6 Concluding remarks

The present work teams up with [8] to establish existence and uniqueness properties of solutions to bounded extremal problems on the disk and to carry out their computation when $p = 2$, and also when $p = \infty$ provided $f \vee h \in H^\infty + C(\mathbb{T})$. The situation is remarkably similar to that of classical (dual) extremal problems, both regarding existence and uniqueness results as well as constructive aspects. In fact, **BEP** reduces to an implicit classical extremal problem which is solved by projecting onto H^2 when $p = 2$ and by finding a maximizing vector of some Hankel operator when $p = \infty$. For other values of p, a constructive solution to extremal problems is known in special cases only [11].

Related to **BEP** is the following completion problem. *For $1 \leq p \leq \infty$, let us be given $f \in L^p(K)$ and $C > 0$. We seek $h \in L^p(\mathbb{T} \setminus K)$ whose norm does not exceed C and such that $f \vee h$ is as close as possible to H^p under this constraint.*
This has been solved in [9] for $p = 2$ and can be shown to reduce to **BEP** for $p = \infty$ [6].

Let us finally mention a question which arises naturally in this context, namely what is the rate of approximation to $f \in L^p(K)$ by $g \in \mathcal{C}^p_{M,h}$ as a function of M? The answer, which is likely to depend on the smoothness of f, would be particularly interesting from an algorithmic point of view.

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Laurent Baratchart, Juliette Leblond,
INRIA, BP 93, 06902 Sophia–Antipolis Cedex, FRANCE.
{baratcha}{leblond}@sophia.inria.fr