Hardy approximation to L^∞ functions on subsets of the circle.

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MSC: 30D55, 30E99, 93B30, 93C80.

Keywords: uniform analytic approximation, extremal problems in Hardy spaces, Nehari extension.

Abstract

We consider approximation of L^∞ functions by H^∞ functions on proper subsets of the circle. We derive some properties of traces of Hardy classes on such subsets, and then turn to a generalization of classical extremal problems involving norm constraints on the complementary subset.

1 Introduction

1.1 General background

The issue under investigation here may be considered as a generalization to arbitrary subsets of the unit circle $\mathbb{T}$ of a classical extremal problem in the Hardy space H^∞ of the unit disk, known as the Nehari extension. More precisely, if K is a subset of $\mathbb{T}$ and if we denote its complement by $\mathbb{T} \setminus K$, the question we address is as follows.

Let us be given $f \in L^\infty(K)$, $h \in L^\infty(\mathbb{T} \setminus K)$, and $M > 0$. We want to find $g \in H^\infty$ whose distance to h in $L^\infty(\mathbb{T} \setminus K)$ does not exceed M and which is as close as possible to f in the $L^\infty(K)$ metric under this constraint.

When $K = \mathbb{T}$, then h and M play no role and we recognize a standard Nehari problem (see e.g. [3, chap.8], [5, chap.IV], [8, chap.VII]). We shall refer to the above generalization as a bounded extremal problem.

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This article is the first of a pair, the second of which will deal with the analogous problem in H^p and L^p for $1 \le p < \infty$.

Part of the authors motivations for studying such problems originates with some questions in identification of linear dynamical control systems that we would now like to describe. A linear time–invariant dynamical control system, or simply a system, is a convolution operator associating to an input function $u : [0, +\infty) \to \mathbb{R}$ an output function $y : [0, +\infty) \to \mathbb{R}$ given by

$$y(t) = \int_0^{+\infty} r(t-\tau)u(\tau)\, d\tau,$$

where $r : [0, \infty) \to \mathbb{R}$ is called the impulse response of the system. As a general rule, restrictions on the growth of the impulse response express various boundedness properties of the input–output map $u \mapsto y$, and a connection with complex analysis can be made when r is of exponential type, so that we may consider its Laplace transform :

$$\mathcal{F}(s) = \int_0^{\infty} e^{-st} r(t)\, dt,$$

which is analytic for $\operatorname{Re} s$ large enough and is called the transfer function of the system. This is where Hardy classes arise in connection with stability : the input–output map $L^2[0, \infty) \to L^2[0, \infty)$ is bounded if and only if $\mathcal{F}$ belongs to $\mathcal{H}^\infty$ of the right half–plane and the corresponding operator norm, that is the maximum rate of energy transmission, is then precisely $\|\mathcal{F}\|_\infty$.

Now, if $r \in L^1(\mathbb{R}^+)$, a common way of identifying a system from experiments is to plug a periodic signal with frequency ω into it as this produces asymptotically an output which is also periodic and whose phase and modulus are those of the complex number $\mathcal{F}(i\omega)$. However, due to the bandwidth limitation of the system and also to the restricted domain of validity of the linear model, the points $i\omega$ at which the transfer function can be evaluated are confined within a certain range of frequencies Ω, say, and only some rough information on the roll–off rate at other frequencies is available. To recap, we have a description of the $\mathcal{H}^\infty$ function $\mathcal{F}$ on Ω which is accurate up to experimental errors, and an expected behaviour of the same function outside Ω which need not be exceedingly precise. When trying to recover a plausible model from these data, we are naturally led to consider the bounded extremal problem stated before except that the setting is now the half–plane. Note that the Nehari extension already plays an important role in robust identification and has been extensively used in control for several years (see e.g. [4]). In addition, the above identification scheme might be helpful in a design procedure for deciding how feasible a given frequency behaviour may be [2].

There is of course much more to approximation in this identification scheme because $\mathcal{F}$ has to be reconstructed on Ω from pointwise values. For details on how this merges into the framework of robust identification discussed in [6] and [11], we refer the reader to [1] and again to [2].

Observe that if the subset K is invariant under conjugation and f and h are real–symmetric, namely if $\overline{f(z)} = f(\bar{z})$ and similarly for h, a best approximant g in our problem can always be chosen so as to be real–symmetric since $\frac{1}{2}\left(g(z) + \overline{g(\bar{z})}\right)$ does the job as well ; if moreover we know that such a g is unique, we conclude that the best approximant is in fact real–symmetric. Because transfer–functions are real–symmetric, such a property is mandatory for system–theoretic applications.

1.2 Statement of the main problem

We begin with some notation. If E is a set and $F \subset E$, we write $E \setminus F$ for the set–theoretic complement of F in E. When E is a topological space, we let as usual $\overline{F}$ and $\overset{\circ}{F}$ be respectively the closure and the interior of F in E ; further, $C(E)$ will denote the space of continuous complex–valued functions on E. Call $\mathbb{T}$ the unit circle and $\mathbb{D}$ the unit disk. We let λ be the Lebesgue measure on $\mathbb{T}$ and, for $1 \leq p \leq \infty$, we designate by H^p the Hardy space with exponent p of the unit disk while $\mathcal{A}$ stands for the disk algebra. For $E \subset \mathbb{T}$, we denote by $L^p(E)$ the familiar Lebesgue space and by $\|.\|_{L^p(E)}$ the corresponding norm. If f is a function defined on some set containing E, the symbol $f_{|_E}$ indicates the restriction of f to E, and we write without ambiguity $\|f\|_{L^p(E)}$ for the norm of this restriction. Accordingly, we set $H^p_{|_E}$ to mean the space of traces on E of H^p functions and we define $\mathcal{A}_{|_E}$ similarly. For $f \in L^\infty(E)$, the symbol $d(f, H^\infty_{|_E})$ denotes the distance from f to $H^\infty_{|_E}$ in $L^\infty(E)$.
Whenever f is defined on E and h is defined on $\mathbb{T} \setminus E$, the notation $f \vee h$ is used for the concatenated function which is defined on all of $\mathbb{T}$.

The question we address is the following :

Problem 1 *Let K be a subset of $\mathbb{T}$ of positive measure. For $f \in L^\infty(K)$ and $h \in L^\infty(\mathbb{T}\setminus K)$, define a subset $\mathcal{B}_{M,h}$ of H^∞ by :*

$$\mathcal{B}_{M,h} := \{g \in H^\infty\,,\ \|h - g\|_{L^\infty(\mathbb{T}\setminus K)} \leq M\}\,,$$

where M is a positive real number. We seek $g_0 \in \mathcal{B}_{M,h}$ such that

$$\|f - g_0\|_{L^\infty(K)} = \min_{g \in \mathcal{B}_{M,h}} \|f - g\|_{L^\infty(K)}\,. \tag{1}$$

When $K = \mathbb{T}$, we see that h and M play no role and that $\mathcal{B}_{M,h} = H^\infty$, so the bounded extremal problem above reduces to a Nehari problem for which the following facts (see e.g. [5, IV]) are well–known :
(a) a solution does exist ;
(b) the solution need not be unique but is unique at least if $f \in H^\infty + C(\mathbb{T})$, and the error function $|f - g_0|$ is then constant a.e. on $\mathbb{T}$;
(c) g_0 is continuous as soon as f is Dini–continuous (the Carleson–Jacobs theorem).

Our objective in this paper (theorem 2) is to carry properties (a) and (b) over to bounded extremal problems, and also to reduce such problems to implicit Nehari problems which can be solved constructively. As we shall see (remark 3), property (c) has no straightforward generalization.

Note that problem 1 is trivial if there exists $g \in \mathcal{B}_{M,h}$ such that $f = g_{|_K}$, since g is then the unique solution ; also, when $\lambda(\mathbb{T} \setminus K) = 0$, this is just the Nehari problem. We shall rule out these cases in the rest of the paper. It may happen that $\mathcal{B}_{M,h} = \emptyset$, in which case problem 1 obviously has no solution. This will not occur if h belongs to the closure of $H^\infty_{|_{\mathbb{T}\setminus K}}$ in $L^\infty(\mathbb{T} \setminus K)$, and this is why we shall start studying this closure in section 2. In section 3, we show that well–known existence and uniqueness properties of solutions to standard extremal problems do generalize to bounded extremal problems and that these problems reduce to standard Nehari ones in an implicit manner. In section 4, we give an example of non–uniqueness of solution and non saturation of the constraint.

We shall be working entirely on the disk, but it is clear how to derive half–plane versions of our results, either by direct inspection or using conformal mapping (see e.g. [7, chap.8]).

2 Some density results

The main result of this section is the following theorem.

Theorem 1 *Let K be a subset of $\mathbb{T}$ such that $\lambda(K) > 0$. Then :*
(i) $H^\infty_{|_K}$ is not dense in $L^\infty(K)$.
(ii) If K is open, the closure of $H^\infty_{|_K}$ in $L^\infty(K)$ is contained in $H^\infty_{|_K} + C(K)$.
(iii) If K is a proper closed subset of $\mathbb{T}$, then $\mathcal{A}_{|_K}$ is dense in $C(K)$ and the closure of $H^\infty_{|_K}$ in $L^\infty(K)$ contains $(H^\infty + C(\mathbb{T}))_{|_K}$.

Remark 1 Whenever $\lambda(K) > 0$, we have that $\mathcal{A}_{|_K} \neq C(K)$. Indeed, it is easy to construct $f \in C(K)$ which is nonzero but vanishes on some subset of K of positive measure for we can construct two disjoint closed subsets K_1, K_2 of K of positive measure and there exists $f \in C(K)$ which is identically 1 on K_1 and zero on K_2 by the Tietze–Urysohn extension theorem [13, thm.20.4]. In contrast, if K is closed and $\lambda(K) = 0$, a theorem of Rudin asserts that $C(K) = \mathcal{A}_{|_K}$. A proof of Rudin's result can be found e.g. in [7, ch.6].
Let K be an open subset of $\mathbb{T}$ which is not dense in $\mathbb{T}$ and denote by $\overline{H^\infty}_{|_K}$ the closure of $H^\infty_{|_K}$ in $L^\infty(K)$. We deduce from (ii) and (iii) that the restrictions of $\overline{H^\infty}_{|_K}$ and of $H^\infty + C(\mathbb{T})$ to any compact subset of K coincide. However, we do not know when this property holds for K itself.

For the proof of theorem 1, we need a lemma. Recall that a density point of K is a point at which the measure $\chi_K \, d\lambda$ is differentiable with derivative equal to 1, where χ_K denotes the characteristic function of K.

Lemma 1 *If $f \in H^\infty$ is such that $\|f\|_{L^\infty(K)} \leq 1$ and $f_{|_K}^{-1} \in L^\infty(K)$, and either*
(i) the zeroes of f accumulate nontangentially at some density point of K, or
(ii) the zeroes of f accumulate at some interior point of K,
then $d(f^{-1}, H^\infty_{|_K}) \geq 1$.

Proof : let $P_r(\theta)$ be the familiar Poisson kernel :

$$P_r(\theta) := \frac{1-r^2}{1-2\,r\,\cos\theta + r^2}\ , \text{ for } 0 \leq r < 1 \text{ and } 0 \leq \theta \leq 2\pi\,.$$

Let us also define two positive measures ν_1 and ν_2 on $\mathbb{T}$ by

$$d\nu_1 := \chi_{_K}\, d\lambda \qquad \text{and} \qquad d\nu_2 := (1-\chi_{_K})\, d\lambda\,.$$

Assume $d(f^{-1}, H^\infty_{|_K}) < 1$ and denote by (α_n) a sequence of zeroes of f in $\mathbb{D}$ with $\alpha_n = r_n\, e^{i\theta_n}$. Then, there exists $h \in H^\infty$ and $\eta < 1$ such that $|f^{-1} - h| < \eta$ almost everywhere on K, implying that

$$|1 - f\,h| < \eta \ \text{ a.e. on } K\,. \tag{2}$$

Since $f\,h$ lies in H^∞, we get from the Poisson representation formula

$$0 = f\,h(\alpha_n) = \frac{1}{2\,\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\, fh(e^{it})\, dt,$$

so that

$$\left|\frac{1}{2\,\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\, fh(e^{it})\, d\nu_1(t)\right| = \left|\frac{1}{2\,\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\, fh(e^{it})\, d\nu_2(t)\right|. \tag{3}$$

Now, we have on the one hand

$$\left|\frac{1}{2\,\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\, fh(e^{it})\, d\nu_2(t)\right| \leq \|f\,h\|_{L^\infty(\mathbb{T}\setminus K)}\, \frac{1}{2\,\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\, d\nu_2(t), \tag{4}$$

while on the other hand (2) implies

$$\begin{aligned} \left|\frac{1}{2\,\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\, fh(e^{it})\, d\nu_1(t)\right| \ &\geq\ \frac{1}{2\,\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\, \mathrm{Re}[fh(e^{it})]\, d\nu_1(t) \\ &\geq\ (1-\eta)\,\frac{1}{2\,\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\, d\nu_1(t). \end{aligned} \tag{5}$$

Because of (3), (4), and (5), we get

$$(1-\eta)\,\frac{1}{2\,\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\, d\nu_1(t) \leq \|f\,h\|_{L^\infty(\mathbb{T}\setminus K)}\, \frac{1}{2\,\pi}\int_{-\pi}^{\pi} P_{r_n}(\theta_n - t)\, d\nu_2(t). \tag{6}$$

In case (i), we assume that (α_n) converges nontangentially to a point $e^{i\theta_0}$ of K where ν_1 has derivative equal to 1. At such a point, ν_2 also has a derivative which is equal to 0. Invoking a well–known theorem of Fatou (see e.g. [7, chap.3]) shows that the integral in the left hand–side of (6) goes to 2π if n goes to infinity whereas the integral in the right hand–side goes to 0, yielding $1-\eta \leq 0$, a contradiction.

In case (ii), we show that (6) again becomes absurd as soon as the α_n's accumulate at a point $e^{i\theta_0}$ which is *interior* to K (it is not a consequence of what precedes because we do not assume this time that the convergence is nontangential, and this is the reason why we need the stronger hypothesis that $e^{i\theta_0}$ is an interior point and not merely a density point). This actually follows from a classical result, namely that every H^p function with non–negative real part on an arc has an inner factor which is analytic across this arc [5, II, ex.14]. However, we can also do it directly as follows. Because $e^{i\theta_0}$ belongs to $\overset{\circ}{K}$, there exists $\sigma > 0$ such that

$$\{e^{it}\,,\ \theta_0-\sigma < t < \theta_0+\sigma\} \subset K,$$

and we have that $\theta_n \in [\theta_0 - \frac{\sigma}{2}, \theta_0 + \frac{\sigma}{2}]$ when n is large enough. For such an n, we get from the monotonicity of the Poisson kernel with respect to θ that

$$\frac{1}{2\pi}\int_{-\pi}^{\theta_0-\sigma} P_{r_n}(\theta_n - t)\,dt + \frac{1}{2\pi}\int_{\theta_0+\sigma}^{\pi} P_{r_n}(\theta_n - t)\,dt \leq P_{r_n}(\sigma/2) \tag{7}$$

hence also

$$\frac{1}{2\pi}\int_{\theta_0-\sigma}^{\theta_0+\sigma} P_{r_n}(\theta_n - t)\,dt \geq 1 - P_{r_n}(\sigma/2). \tag{8}$$

Using (8) and (7) to minorize and majorize the left and right hand–side of (6), we get

$$(1-\eta)\,(1 - P_{r_n}(\sigma/2)) \ \leq\ \|f\,h\|_{L^\infty(\mathbb{T}\setminus K)}\, P_{r_n}(\sigma/2),$$

which again contradicts the definition of η, for $P_{r_n}(\sigma/2)$ tends to 0 when n goes to infinity. ■

We are now in position to prove the result we are after.

Proof of theorem 1 : to prove (i), observe since $\lambda(K) > 0$ that there exists a density point $e^{i\theta_0} \in K$ by Lebesgue's theorem. Consider a Blaschke product :

$$b(z) := z^k \prod_{n=1}^{\infty} \frac{\bar{\alpha}_n}{|\alpha_n|}\,\frac{\alpha_n - z}{1-\bar{\alpha}_n z}\,, \qquad \text{with} \quad \sum_n (1-|\alpha_n|) < \infty\,, \tag{9}$$

whose zeroes α_n converges nontangentially to $e^{i\theta_0}$. Applying lemma 1 with $f = b$ shows that $d(\bar{b}, H^\infty_{|K}) = 1$.
We turn to the proof of (ii), assuming now that K is open and thus consists entirely of interior points. Recall $j \in H^\infty$ is said to be inner if $|j(e^{it})| = 1$ almost everywhere on $\mathbb{T}$. Let us show that if j is inner and $d(\bar{j}, H^\infty_{|K}) < 1$, then j is continuous on K. When j is a Blaschke product, this follows from the fact that the zeroes of j cannot accumulate on K by (ii) of lemma 1 and from the well–known analytic continuation properties of Blaschke products (see e.g. [7, chap.5]). For the general case, we use Frostman's theorem (see e.g. [5, II, cor. 6.5]) that Blaschke products are dense in the set of inner functions : if b is sufficiently close to j, then $d(\bar{b}, H^\infty_{|K}) < 1$ and b is continuous at $e^{i\theta_0}$. Since j is the uniform limit of such b's, we are done.

Let $\mathcal{C}$ be the closure of $H^\infty_{|K}$ in $L^\infty(K)$ which is a closed subalgebra of $L^\infty(K)$. Since the restriction map $\Pi_K : L^\infty(\mathbb{T}) \to L^\infty(K)$ is continuous, $\mathcal{D} := \Pi_K^{-1}(\mathcal{C})$ is a closed subalgebra of $L^\infty(\mathbb{T})$ containing H^∞ and, by a theorem of Chang on Douglas algebras [5, IX, thm 4.2], we have that

$$\mathcal{D} = H^\infty + [\bar{\mathcal{J}}_\mathcal{D}, \mathcal{J}_\mathcal{D}]\,. \tag{10}$$

where $[\bar{\mathcal{J}}_\mathcal{D}, \mathcal{J}_\mathcal{D}]$ is the self-adjoint algebra generated by $\mathcal{J}_\mathcal{D}$, those inner functions invertible in $\mathcal{D}$.
Clearly, any element j of $\mathcal{J}_\mathcal{D}$ satisfies $d(\bar{j}, H^\infty_{|K}) = 0$. Hence j and $\bar{j}$ are continuous on K by what we saw above. This proves in particular that $\mathcal{C} \subset H^\infty_{|K} + C(K)$, which is (ii).

We finally establish (iii) in which K is now a proper closed subset of $\mathbb{T}$. Since K does not separate $\mathbb{C}$, the density of $\mathcal{A}_{|K}$ in $C(K)$ follows for instance from Mergelyan's theorem [13, thm.20.5]. This shows that $H^\infty_{|K} + C(K)$ is contained in the closure of $H^\infty_{|K}$ in $L^\infty(K)$. But since K is closed, $C(K) = C(\mathbb{T})_{|K}$ by the Tietze–Urysohn extension theorem already quoted. ∎

When $1 \leq p < \infty$, we shall require the following result which is in contrast with part (i) of theorem 1.

Proposition 1 *Let K be a subset of $\mathbb{T}$ such that $\lambda(\mathbb{T} \setminus K) > 0$ and let $1 \leq p < \infty$. Then, $H^p_{|K}$ is dense in $L^p(K)$.*

Proof : let q be conjugate to p, and assume that $f \in L^q(K)$ is orthogonal to $H^p_{|K}$. This may be expressed as

$$\int_\mathbb{T} (f \vee 0)\, g\, d\lambda = 0, \quad \text{for all } g \in H^p.$$

In particular, the Fourier coefficients of $f \vee 0$ of negative index are all zero so $f \vee 0 \in H^q$. Since it vanishes on a set of positive measure, namely $\mathbb{T} \setminus K$, it must vanish identically. ∎

We end up this section by showing that the approximants guaranteed by theorem 1 are bound to have wild behaviour outside K. Recall that H^∞ is weak–* closed in $L^\infty(\mathbb{T})$ whence balls are weak–* compact.

Proposition 2 *Let K be a subset of $\mathbb{T}$ such that $\lambda(K) > 0$, and f be in $L^\infty(K)$. Suppose (g_n) is a sequence of H^∞ functions such that $g_{n|K}$ converges to f in $L^\infty(K)$. If f is not the trace of an H^∞ function, then* $\lim_{n\to\infty} \|g_n\|_{L^\infty(\mathbb{T}\setminus K)} = \infty$.

Proof : suppose the conclusion does not hold. Extracting a subsequence if necessary, we may assume that $(\|g_n\|_{L^\infty(\mathbb{T}\setminus K)})$ is bounded, hence that (g_n) is bounded in H^∞. Then there exists a subsequence (g_{n_k}) converging weakly–* towards some $g \in H^\infty$. In particular, $(g_{n_k|K})$ converges weakly–* to $g_{|K}$ in $L^\infty(K)$, and since it also converges strongly to f by hypothesis, we get $f = g$ a.e. on K so that $f \in H^\infty_{|K}$. ∎

3 Bounded extremal problems

To state our main theorem, let us denote by $\mathcal{C}_{M,h}$ the trace of $\mathcal{B}_{M,h}$ (introduced in problem 1) on K :

$$\mathcal{C}_{M,h} := \{g_{|_K}\,,\ g \in \mathcal{B}_{M,h}\},$$

and define β_∞ to be the value of problem 1, which of course depends on f, h, and M.

Theorem 2 *Concerning the bounded extremal problem stated in section 1.2 as problem 1 where it is assumed that $f \notin \mathcal{C}_{M,h}$ (so that $\beta_\infty > 0$) and $\lambda(\mathbb{T}\setminus K) > 0$, the following assertions hold true :*
(i) A solution g_0 does exist, provided $\mathcal{B}_{M,h} \neq \emptyset$.
(ii) Unless $\beta_\infty = d(f, H^\infty_{|_K})$, then any solution g_0 satisfies

$$\|h - g_0\|_{L^\infty(\mathbb{T}\setminus K)} = M\,. \tag{11}$$

(iii) The solution g_0 is unique at least when $f \vee h$ lies in $H^\infty + C(\mathbb{T})$ and, in this case, the functions $f - g_0$ and $h - g_0$ have constant modulus β_∞ and M a.e. on K and $\mathbb{T}\setminus K$ respectively.

Remark 2 In general, the solution need not be unique nor satisfy $|h - g_0| = M$ a.e. on $\mathbb{T}\setminus K$. More generally, it need not even satisfy (11). This is illustrated by an example in section 4.
Note also that (iii) provides an hypothesis under which the constraint is saturated in a stronger sense than (11). It is perhaps worthwhile noticing that whenever $\mathbb{T}\setminus K$ has an interior point, then $f \vee h \in H^\infty + C(\mathbb{T})$ implies that $d(f, H^\infty_{|_K}) = 0$ in view of part (iii) of theorem 1.

The proof of theorem 2 will occupy the remaining of this section and requires a few steps.

3.1 A closedness property

The following result will be needed to establish part (i) of theorem 2 but we state it separately as it may be of independent interest. To this effect, let H^1_0 be the subspace of H^1 consisting of functions vanishing at zero.

Theorem 3 *$\mathcal{C}_{M,h}$ is weak–* closed in $L^\infty(K)$. More precisely :*

$$\mathcal{C}_{M,h} = \bigcap_{\substack{v \in H^1_0 \\ \|v\|_{L^1(\mathbb{T}\setminus K)\le 1}}} \{w \in L^\infty(K)\,;\ \left|\int_K wv\,d\lambda + \int_{\mathbb{T}\setminus K} hv\,d\lambda\right| \le M\}\,. \tag{12}$$

Proof : it follows from theorem 1 that $H^1_{|_{\mathbb{T}\setminus K}}$ is dense in $L^1(\mathbb{T}\setminus K)$ and it is then easily seen that the same is true of $H^1_{0\,|_{\mathbb{T}\setminus K}}$.

Now, take $w \in \mathcal{C}_{M,h}$ and write $w = g_{|K}$ with $g \in \mathcal{B}_{M,h}$. By duality and using the Cauchy formula :

$$M \geq \|h-g\|_{L^\infty(\mathbb{T}\setminus K)} = \sup_{\substack{v \in H_0^1 \\ \|v\|_{L^1(\mathbb{T}\setminus K) \leq 1}}} \left| \int_{\mathbb{T}\setminus K} (h-g) v \, d\lambda \right| = \sup_{\substack{v \in H_0^1 \\ \|v\|_{L^1(\mathbb{T}\setminus K) \leq 1}}} \left| \int_{\mathbb{T}} (w \vee h) v \, d\lambda \right| .$$

Conversely, let $w \in L^\infty(K)$ be such that :

$$\sup_{\substack{v \in H_0^1 \\ \|v\|_{L^1(\mathbb{T}\setminus K) \leq 1}}} \left| \int_{\mathbb{T}} (w \vee h) v \, d\lambda \right| \leq M .$$

Then the map $L_w : {H_0^1}_{|\mathbb{T}\setminus K} \to \mathbb{C}$ defined by

$$L_w(v_{|\mathbb{T}\setminus K}) := \int_{\mathbb{T}} (w \vee h) \, v \, d\lambda \tag{13}$$

extends to a continuous linear form on $L^1(\mathbb{T}\setminus K)$ whose norm is at most M. Thus, there exists $w' \in L^\infty(\mathbb{T}\setminus K)$ such that $\|w'\|_{L^\infty(\mathbb{T}\setminus K)} \leq M$ and

$$L_w(v_{|\mathbb{T}\setminus K}) = \int_{\mathbb{T}\setminus K} w' v \, d\lambda \quad \text{for } v \in H_0^1. \tag{14}$$

Substracting (14) from (13), we see that the function $w \vee (h - w')$ belongs to H^∞ hence to $\mathcal{B}_{M,h}$. Consequently, w is in $\mathcal{C}_{M,h}$ and we obtain (12).
When $h = 0$, this characterization is in the spirit of the one obtained on the line in [12, thm.4]. It shows that $\mathcal{C}_{M,h}$ is an intersection of weak–* closed sets hence is weak–* closed.

■

3.2 Proof of theorem 2 : existence and saturation of the constraint

As another piece of notation, we introduce the map

$$\begin{array}{rccc} \psi : & L^\infty(K) & \longrightarrow & \mathbb{R} \\ & \xi & \longmapsto & \|f-\xi\|_{L^\infty(K)}. \end{array}$$

The set $\mathcal{C}_{M,h}$ being clearly a convex subset of $L^\infty(K)$, the desired existence property follows from the fact that $\mathcal{C}_{M,h}$ is weak–* closed (theorem 3). Indeed, $\psi(\xi)$ goes to infinity with $\|\xi\|_{L^\infty(K)}$ so that we may restrict our attention to $\mathcal{C}_{M,h} \cap B(0,\varrho)$ where $B(0,\varrho)$ is the closed ball of radius ϱ centered at 0 in $L^\infty(K)$ and where ϱ is large enough. Since $\mathcal{C}_{M,h}$ is weak–* closed, $\mathcal{C}_{M,h} \cap B(0,\varrho)$ is weak–* compact because $B(0,\varrho)$ is. Now, ψ is lower semi–continuous on $L^\infty(K)$ equipped with the weak–* topology because it is the supremum of the family of continuous functions $\xi \to |\int_K (f-\xi) w \, d\lambda|$ when w ranges over the unit ball of $L^1(K)$. Therefore, ψ reaches its minimum on $\mathcal{C}_{M,h} \cap B(0,\varrho)$ hence on $\mathcal{C}_{M,h}$ by the choice of ϱ. This proves (i).

To prove (ii), let us argue by contradiction, assuming that $\|h-g_0\|_{L^\infty(\mathbb{T}\setminus K)} < M$. In this case, there is a neighborhood of g_0 in H^∞ which is contained in $\mathcal{B}_{M,h}$; denoting by $R_K : H^\infty \to L^\infty(K)$ the natural restriction, this means that the map $\psi \circ R_K$ attains a local minimum at g_0 hence a global minimum since this map is convex. This means precisely that $\beta_\infty = d(f, H^\infty_{|K})$ as desired.

3.3 An equivalent Nehari problem

Problem 1 reduces, though in an implicit manner, to a classical extremal problem that we introduce now. This will be the basis for proving part (iii) of theorem 2 and also for constructively solving problem 1. Let us introduce one more piece of notation by setting w_ρ to be the outer factor whose modulus is 1 on $\mathbb{T}\setminus K$ and ρ on K :

$$w_\rho(z) := \exp\left\{\frac{1}{2\,\pi}\,\log\rho\int_K \frac{e^{it}+z}{e^{it}-z}\,dt\right\}, \tag{15}$$

where ρ is a positive constant. Observe that w_ρ is invertible in H^∞ and that $w_\rho^{-1} = w_{1/\rho}$.

Problem 2 *Notations being as in problem* 1 *and assuming that* $f \notin \mathcal{C}_{M,h}$ *(so that* $\beta_\infty > 0$*), we seek* $v_0 \in H^\infty$ *such that*

$$\|(f\vee h)w_{\frac{M}{\beta_\infty}} - v_0\|_{L^\infty(\mathbb{T})} = \min_{v\in H^\infty} \|(f\vee h)w_{\frac{M}{\beta_\infty}} - v\|_{L^\infty(\mathbb{T})}. \tag{16}$$

Now, problems 1 and 2 are related each to the other in view of the following theorem.

Theorem 4 *Any solution* v_0 *to problem* 2 *is such that* $g_0 = v_0\, w^{-1}_{\frac{M}{\beta_\infty}}$ *is a solution to problem* 1 *and, conversely, any solution* g_0 *to problem* 1 *gives rise to a solution* $v_0 = g_0 w_{\frac{M}{\beta_\infty}}$ *of problem* 2.

Proof : suppose that $v_0 \in H^\infty$ is a solution to problem 2. Define

$$\delta := \|(f\vee h)\, w_{\frac{M}{\beta_\infty}} - v_0\|_{L^\infty(\mathbb{T})}\,.$$

Then with $g_0 = v_0 w^{-1}_{\frac{M}{\beta_\infty}} = v_0 w_{\beta_\infty/M}$, we have

$$\|(f\vee h - g_0)\, w_{\frac{M}{\beta_\infty}}\|_{L^\infty(\mathbb{T})} = \delta\,,$$

which, from the definition of $w_{\frac{M}{\beta_\infty}}$, shows that

$$\max\left(\frac{M}{\beta_\infty}\|f-g_0\|_{L^\infty(K)}, \|h-g_0\|_{L^\infty(\mathbb{T}\setminus K)}\right) = \delta.$$

We assert now that $\delta = M$, which implies that g_0 is a solution to problem 1. For it is clearly impossible for δ to be less than M, otherwise this makes $\|f-g_0\|_{L^\infty(K)} < \beta_\infty$ and

$\|h-g_0\|_{L^\infty(\mathbb{T}\setminus K)} < M$, which contradicts the definition of β_∞. Moreover δ cannot exceed M, because there is a function g_1 (i.e. a solution to problem 1 which has been shown to exist in section 3.2) with

$$\max\left(\frac{M}{\beta_\infty}\|f-g_1\|_{L^\infty(K)}, \|h-g_1\|_{L^\infty(\mathbb{T}\setminus K)}\right) = M\,.$$

But then replacing v_0 by $v_1 = g_1 w_{\frac{M}{\beta_\infty}}$ in problem 2 would contradict the optimality of v_0. Hence $\delta = M$ and g_0 is a solution to problem 1.

Conversely, suppose that g_0 is a solution to problem 1. Thus

$$\|f-g_0\|_{L^\infty(K)} = \beta_\infty \text{ and } \|h-g_0\|_{L^\infty(\mathbb{T}\setminus K)} \leq M.$$

With $v_0 = g_0 w_{\frac{M}{\beta_\infty}}$ we then have

$$\|(f\vee h)w_{\frac{M}{\beta_\infty}} - v_0\|_{L^\infty(\mathbb{T})} = M$$

and we assert that v_0 is a solution to problem 2. For if not, letting v_1 be such that

$$\|(f\vee h)w_{\frac{M}{\beta_\infty}} - v_1\|_{L^\infty(\mathbb{T})} < M,$$

and writing $g_1 = v_1 w^{-1}_{\frac{M}{\beta_\infty}}$, we see that $\|f-g_1\|_{L^\infty(K)} < \beta_\infty$ and $\|h-g_1\|_{L^\infty(\mathbb{T}\setminus K)} < M$, which contradicts the definition of g_0 as a solution to problem 1. ∎

This completes the proof of theorem 4. Note that we have been using the existence of a solution to problem 1 to establish the equivalence between this problem and problem 2. Alternatively, we could have deduce this existence from the known existence of a solution to the Nehari problem 2 by using a slight modification of the above reasoning. We leave it to the interested reader.

3.4 End of the proof of theorem 2

We finally prove part (iii) of theorem 2. Having at our disposal the equivalence of problem 1 with the classical Nehari problem 2 (which is theorem 4), all the assertions follow immediately from the known properties (b) and (c) listed just after problem 1, once it is recognized that $f\vee h$ belongs to $H^\infty + C(\mathbb{T})$ if and only if $(f\vee h)w_{\frac{M}{\beta_\infty}}$ does. This, in turn, follows at once from the fact that $H^\infty + C(\mathbb{T})$ is an algebra and $w_{\frac{M}{\beta_\infty}}$ is invertible in H^∞. The proof of theorem 2 is now complete. ∎

4 An example of non–uniqueness

Let $K = [-\frac{\pi}{2}, \frac{\pi}{2}]$, $M = 1$, and consider the $L^\infty(K)$ function

$$f = \begin{cases} -1 & \text{on } [0, \frac{\pi}{2}]\,, \\ 1 & \text{on } [-\frac{\pi}{2}, 0)\,. \end{cases}$$

Take $h \equiv 0$ on $\mathbb{T} \setminus K$. We now show that these data generate at least two solutions of (1). This may not be surprising since $f \vee 0$ is a classical example of a function having at least two best approximants from H^∞ in $L^\infty(\mathbb{T})$ (see e.g. [3, 8.3(iii)], [5, IV.1.5], [8, VII.A.2]), and we merely show that these are solutions of a bounded extremal problem as well. Any of the references we just mentioned establishes that

$$\inf_{g \in H^\infty} \|f \vee 0 - g\|_{L^\infty(\mathbb{T})} = 1 \,. \tag{17}$$

From (17), it follows that :

$$1 \leq \inf_{\substack{g \in H^\infty \\ \|g\|_{L^\infty(\mathbb{T}\setminus K)} \leq 1}} \|f \vee 0 - g\|_{L^\infty(\mathbb{T})} \leq \|f \vee 0\|_{L^\infty(\mathbb{T})} = \|f\|_{L^\infty(K)} = 1 \,,$$

so that equality holds througout and

$$\min_{g \in \mathcal{B}_{1,0}} \psi(g_{|_K}) = \min_{\substack{g \in H^\infty \\ \|g\|_{L^\infty(\mathbb{T}\setminus K)} \leq 1}} \|f - g\|_{L^\infty(K)} \leq \inf_{\substack{g \in H^\infty \\ \|g\|_{L^\infty(\mathbb{T}\setminus K)} \leq 1}} \|f \vee 0 - g\|_{L^\infty(\mathbb{T})} = 1 \,.$$

Let us prove that in fact :

$$\min_{g \in \mathcal{B}_{1,0}} \psi(g_{|_K}) = 1 \,. \tag{18}$$

Indeed, by part (i) of theorem 2 which we already proved, there is a function $g_0 \in \mathcal{B}_{1,0}$ such that

$$\psi({g_0}_{|_K}) = \min_{g \in \mathcal{B}_{1,0}} \psi(g_{|_K}) \,.$$

Assume that $\psi({g_0}_{|_K}) < 1$; then there exists $\nu \in (0,1)$ such that $\psi({g_0}_{|_K}) < 1 - \nu$ so that in particular ${g_0}_{|_K} \neq 0$. For every $l \in (-1, +1)$,

$$\|l \, g_0\|_{L^\infty(\mathbb{T}\setminus K)} < 1 \tag{19}$$

hence $l \, g_0 \in \mathcal{B}_{1,0}$. Choose $l = 1 - \nu \,/\, \|g_0\|_{L^\infty(K)}$ to obtain :

$$\psi(l \, {g_0}_{|_K}) \leq \psi({g_0}_{|_K}) + (1 - l) \, \|g_0\|_{L^\infty(K)} < 1 \,.$$

This, together with (19), implies that $\|f \vee 0 - l \, g_0\|_{L^\infty(\mathbb{T})} < 1$, which contradicts (17), thereby establishing (18).

As a consequence of (18), the zero function is a best approximant to f from $\mathcal{B}_{1,0}$ for which (11) does not hold.
Yet another solution is given here by the conformal mapping g_0 from $\mathbb{D}$ onto the half disk $\{w \in \mathbb{D} \,, \; \mathrm{Im}\, w < 0\}$ such that (see also [3, 8.3(iii)]) :

$$\begin{cases} g_0(-i) & = & 1 \,, \\ g_0(1) & = & 0 \,, \\ g_0(i) & = & -1 \,. \end{cases}$$

Indeed, it maps $\mathbb{T} \setminus K$ onto the arc $\mathbb{T} \cap \{\mathrm{Im}\, w < 0\}$ so that $\|g_0\|_{L^\infty(\mathbb{T}\setminus K)} = 1$ hence $g_0 \in \mathcal{B}_{1,0}$, and one can easily verify that $\|f - g_0\|_{L^\infty(K)} = 1$. This shows that a solution to problem 1 need not be unique.

Remark 3 Unlike (a) and (b), property (c) of the Nehari extension listed in section 1.2 does not generalize to the solution of problem 1 : there is no chance that g_0 be continuous in general, even if $f \vee h$ is very smooth. To see this, assume for instance that $f \vee h$ is Lipschitz of order α on $\mathbb{T}$, $0 < \alpha \leq 1$, and that $K = (-\theta_0, \theta_0)$ is an arc. There exists a polynomial P such that $(f \vee h - P)(\pm\theta_0) = 0$, and it is not hard to see that $(f \vee h - P)\, w_{\frac{M}{\beta_\infty}}$ is again Lipschitz of order α. In particular, it is Dini–continuous and, by the Carleson–Jacobs theorem, its best H^∞ approximant in $L^\infty(\mathbb{T})$, say u_0, is continuous. But we have, by theorem 4, that $g_0 = P + u_0\, w^{-1}_{\frac{M}{\beta_\infty}}$ which is not continuous at $\pm\theta_0$ unless $M = \beta_\infty$, that is when the bounded extremal problem is in fact a Nehari problem. More details on this can be found in [2].

5 Conclusion

Having analysed existence and uniqueness of solutions to bounded extremal problems, the next thing is how to compute them. For system–theoretic applications, the interest lies mostly with the situation where K is a union of intervals and $f \vee h$ is continuous.

To analyse the solution to Nehari problems from a practical perspective, let us recall briefly the elegant operator–theoretic approach (see e.g. [10], [14]).

Write $F = (f \vee h) w_{\frac{M}{\beta_\infty}}$. Problem 2 then consists of finding $v_0 \in H^\infty$ to meet

$$\inf_{v \in H^\infty} \|F - v\|_{L^\infty(\mathbb{T})}. \tag{20}$$

Let $\Gamma : H^2 \to (H^2)^\perp$ be the Hankel operator defined by

$$\Gamma(u) = P_{(H^2)^\perp}(Fu), \qquad u \in H^2,$$

where $P_{(H^2)^\perp} : L^2(\mathbb{T}) \to (H^2)^\perp$ is the orthogonal projection. It is a theorem of Nehari that the infimum in (20) is equal to $\|\Gamma\|$. Suppose that Γ attains its norm at a nonzero vector $u_0 \in H^2$, that is $\|\Gamma(u_0)\| = \|\Gamma\|\|u_0\|$. This will happen for example if Γ is compact, that is if $F \in H^\infty + C(\mathbb{T})$ by Hartman's theorem.

Then there is a unique best H^∞ approximant v_0 to F and it is given by

$$v_0(z) = F(z) - (\Gamma(u_0))(z)/u_0(z).$$

Moreover $|F(e^{i\theta}) - v_0(e^{i\theta})| = \|\Gamma\|$ almost everywhere.

Now in our setting, if we assume that $f \vee h$ lies in $H^\infty + C(\mathbb{T})$, then $F \in H^\infty + C(\mathbb{T})$ also, and the Hankel operator is compact. Thus, in the notation of problems 1 and 2, we get with $g_0 = v_0 w^{-1}_{\frac{M}{\beta_\infty}}$:

$$|f(e^{i\theta}) - g_0(e^{i\theta})| = \|\Gamma\|\beta_\infty/M \quad \text{a.e. on } K$$

and

$$|h(e^{i\theta}) - g_0(e^{i\theta})| = \|\Gamma\| \quad \text{a.e. on } \mathbb{T} \setminus K,$$

which forces $\|\Gamma\| = M$ since we know, from theorem 4, that g_0 is a solution to problem 1.

It should be noted that problem 2 is an implicit one, since the data depend on the solution through β_∞. However, the above equations show that the right value of β_∞ is the one that makes $\|\Gamma\|$ equal to M. This property allows one to use a dichotomy procedure to solve iteratively for β_∞ [2].

A difficulty with the above approach is that it requires the knowledge of the singular values of Γ, and also of the singular vector associated with the largest singular value. Such a computation can be effective only when Γ is of finite rank, that is when there exists a rational symbol. Using the same trick as in remark 3, we can reduce the problem to a Nehari extension with continuous symbol for which accurate rational approximants can be found. However, some care is required, for the nonlinear operator taking a continuous function to its best analytic approximation is not norm continuous in general [9].

Back to the point of view of robust identification of linear dynamical systems developed in the introduction, we may recall that the function f was known to us only through a discrete set of pointwise values on K. Using the results of the present paper then requires convergent interpolation techniques that are described in [2].

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