Traces of Hardy functions and reproducing kernel Hilbert spaces

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Abstract

We study Loewner–type interpolation problems on the unit disk, using reproducing kernel Hilbert spaces methods. The special case of traces of the Hardy space H_2 on a subset of the unit circle is considered as an application.

Keywords : Loewner interpolation problems, reproducing kernel Hilbert spaces, Hardy spaces.

1 Introduction

In this paper we study a Loewner–type interpolation problem in certain reproducing kernel Hilbert spaces of functions analytic in the unit disk $\mathbb{D}$.

Let us first briefly review the interpolation problem on a finite number of points in a reproducing kernel Hilbert space (see e.g.[15]).
Let E denote some set and let $K(z,w)$ be a positive function on $E \times E$ with values in $\mathbb{C}$ (all relevant notions pertaining to positive functions and reproducing kernel Hilbert spaces are recalled in section 2.1). Let $w_1, \ldots, w_n$ be n preassigned points in E and $f_1, \ldots, f_n$ be n preassigned values in $\mathbb{C}$. The problem is to find all functions f in $H(K)$, the reproducing kernel Hilbert space with reproducing kernel K, such that $f(w_i) = f_i$ for $i = 1, \ldots, n$. ¿From the positivity stems that the $n \times n$ matrix P with ij entry $K(w_i, w_j)$ is nonnegative. When $P > 0$ the functions f are described as follows : the solution to the interpolation problem of minimal norm in $H(K)$ is given by

$$f(z) = [K(z,w_1), \ldots, K(z,w_n)]P^{-1}[f_1, \ldots, f_n]^t$$

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where ^{t} denotes transpose. This is the unique solution in the linear span $\mathcal{M}$ of the functions $K(\cdot, w_i)$, $i = 1, \cdots n$. Any other solution is of the form $f + g$ where g is in the orthogonal complement of $\mathcal{M}$. The behaviour of the interpolant as the number of points goes to infinity is of interest, in particular if E has a boundary (in a sense to be precised) and if the w_i may be chosen on the boundary ; see [4] for an example.

Loewner–type interpolation problems arise when, instead of a finite set of values $\{f_i\}$ taken by f, we are given the restriction $f_{|_I}$ of f on a subset I of the boundary of E, and we want to characterize functions h of $H(K)$, if any, which satisfy $h_{|_I} = f_{|_I}$. Of course, the existence of such an $h \in H(K)$ depends both on K and on the assumed regularity of $f_{|_I}$.

We consider this kind of problem when $E = \mathbb{D}$, the open unit disk, I is a subset of $\mathbb{T}$ of positive Lebesgue measure, and $K(z, w)$ is supposed jointly analytic in z and w^* in $\mathbb{D}$. More precisely, for a sequence $\{\alpha_n\}_{n \geq 0}$ of real numbers, and for every z and w in $\mathbb{D}$, K is of the following form :

$$K(z, w) = \sum_{n=0}^{\infty} \alpha_n \, z^n \, w^{*n} \, . \tag{1}$$

Note that the elements of $H(K)$ are analytic in $\mathbb{D}$, but need not have boundary values on the unit circle $\mathbb{T}$. For $1 \leq p \leq \infty$, H_p will denote the Hardy space of the disk. The questions arise of :

1. The existence of boundary values on the unit circle $\mathbb{T}$ of the elements of $H(K)$.
2. The characterization of functions defined on I which may be boundary values of elements of $H(K)$ with some norm constraint.
3. Reconstructing a function in $H(K)$ from its boundary values on I.

In the case of the Hardy space of the half–plane, the second and third points have been studied by M. Rosenblum and J. Rovnyak in [17] and [18] using the Hilbert and Mellin transforms. A key tool in the work of Rosenblum and Rovnyak is a lemma on operator ranges (see [17], [18], [19]).

These two points have also been considered in [13] by M. Kreĭn and P. Nudel'man in the setting of a constrained dual extremal problem. Traces of H_p functions on a subset of the unit circle $\mathbb{T}$ without norm constraint have been treated in [16] where an asymptotic formula is given which allows one to reconstruct an H_p function from its restriction to a subset of positive measure of $\mathbb{T}$. Such traces, together with constrained dual extremal problems on subsets of $\mathbb{T}$, are also studied in [5]. The first author to consider such problems was K. Loewner in his 1934 paper [14]; we refer to [1] and [20] for more references and discussion.

The purpose of this paper is to prove the following theorem related to the two first points above :

Theorem 1 *Let $\{\alpha_n\}_{n\geq 0}$ be a bounded sequence of nonnegative numbers and let K be the associated reproducing kernel defined by* (1). *The following statements hold.*
(i) $H(K)$ is continuously included in the Hardy space H_2. The elements of $H(K)$ admit nontangential limits almost everywhere on the unit circle.
(ii) Let $M > 0$. A function $f \in L_2(I)$ is the trace on I of a function $h \in H(K)$ with $\|h\|_{H(K)} \leq M$ if and only if for every $\phi \in L_2(I)$ such that :

$$\sum_{n=0}^{\infty} \alpha_n \left| \int_I \phi(e^{it})\, e^{int}\, dt \right|^2 < +\infty\,, \tag{2}$$

we have :

$$\lim_{r\to 1} \int_I \int_I \left(M^2\, K(r\, e^{it}, r\, e^{is}) - f(e^{it})\, f(e^{is})^* \right)\, \phi(e^{it})\, \phi(e^{is})^*\, dt\, ds \geq 0\,, \tag{3}$$

or equivalently :

$$M^2 \sum_{n=0}^{\infty} \alpha_n \left| \int_I \phi(e^{it})\, e^{int}\, dt \right|^2 \geq \left| \int_I f(e^{it})\, \phi(e^{it})\, dt \right|^2 . \tag{4}$$

(iii) Finally, the norm of h in $H(K)$ is the smallest M for which the relation (4) *holds.*

The particular case where $\alpha_n = 1$ for all $n \in \mathbb{N}$ solves the following problem.

Problem 1 *Given an arc I of the unit circle $\mathbb{T}$, a function $f \in L_2(I)$, and $M > 0$, find necessary and sufficient conditions for f to be the trace of an H_2 function h such that $\|h\|_{L_2(\mathbb{T})} \leq M$.*

The same problem in the framework of an open half–plane was studied, as already mentioned, by M. Rosenblum and J. Rovnyak in [17].
Related to Problem 1 is the following question of harmonic analysis.

Problem 2 *Given an arc I of the unit circle $\mathbb{T}$ and $f \in L_2(I)$, find necessary and sufficient conditions on the Fourier coefficients of $\chi_I f$:*

$$a_n = \frac{1}{2\pi} \int_I f(e^{it})\, e^{-int}\, dt \ , \ \ n \in \mathbb{Z} \,,$$

for f to be the trace of an H_2 function.

We will obtain a number of necessary conditions, as corollaries of our solution to Problem 1.

2 Proof of Theorem 1

In this section we prove Theorem 1 using the theory of reproducing kernel Hilbert spaces. The section is divided into two parts : the first part consists of a review of some tools from the theory of reproducing kernel spaces. The second one is devoted to the proof of the theorem.

2.1 Preliminary results on reproducing kernel Hilbert spaces

We refer the reader to [3], [8], [21], and [22] for general information on reproducing kernel spaces.

A function $K(z, w)$ defined for z and w in a set E is *positive* if :
(i) it is hermitian : $K(z, w) = K(w, z)^*$ for every choice of $z, w \in E$,
(ii) for every positive integer r and every choice of points $w_1, \ldots, w_r$ in E, the $r \times r$ hermitian matrix with ij entry $K(w_i, w_j)$ is nonnegative.

A Hilbert space $\mathcal{H}$ of complex valued functions defined on E is called a *reproducing kernel Hilbert space* if there exists a function $K(z, w)$ defined on $E \times E$ with the following properties :

(i) for every choice of $w \in E$, the function $z \longmapsto K(z,w)$ belongs to $\mathcal{H}$, and
(ii) for every $h \in \mathcal{H}$:

$$[h, K(\cdot, w)]_{\mathcal{H}} = h(w).$$

By a result of Aronszjan and Moore [3], there is a one to one correspondence between positive functions on $E \times E$ and reproducing kernel Hilbert spaces of complex valued functions defined on E. We will denote by $H(K)$ the reproducing kernel Hilbert space with reproducing kernel K.
We now gather some results on reproducing kernel Hilbert spaces which will be needed in the sequel.

Theorem 2 ([22, 6,prop.12]) *Let E be some set and let K_1 and K_2 be two functions positive on $E \times E$. Let $K = K_1 + K_2$. Then $H(K_i)$ is contractively included in $H(K)$ for $i = 1, 2$.*

As a matter of fact, the spaces $H(K_1)$ and $H(K_2)$ are complementary in $H(K)$, see [6], [7]. As a corollary of this theorem, or directly, one proves the following result.

Lemma 1 ([21, cor.3.2]) *Let K be a positive function on the set $E \times E$ and let $H(K)$ be the associated reproducing kernel Hilbert space. A function f belongs to $H(K)$ if and only if the kernel*

$$K(z,w) - \frac{f(z)f(w)^*}{M^2}$$

is positive in $E \times E$ for some $M > 0$. The norm of f is the smallest such M.

We will also need the next result :

Theorem 3 ([22, 7,prop.19]) *If for every $z, w \in E$,*

$$K(z,w) = \sum_{n=0}^{\infty} f_n(z) f_n(w)^*$$

with $\sum_{n=0}^{\infty} |f_n(z)|^2 < \infty$ for $z \in E$, then

$$H(K) = \{f(z) = \sum_{n=0}^{\infty} a_n\, f_n(z)\, , \sum_{n=0}^{\infty} |a_n|^2 < \infty\}.$$

Moreover,

$$\|f\|^2_{H(K)} = \inf \sum_{n=0}^{\infty} |a_n|^2$$

over such representations of f.

As a consequence of Theorem 3 we obtain:

Lemma 2 *Let $\{\alpha_n\}_{n\in Q}$, $Q \subset \mathbb{N}$, be a sequence of strictly positive numbers such that $\rho = 1/\overline{\lim}|\alpha_n|^{1/2n} > 0$. Then, the function*

$$K(z,w) = \sum_{n\in Q} \alpha_n \, z^n \, w^{*n}$$

is positive for z and w of modulus less than ρ and :

$$H(K) \; = \{ f(z) = \sum_{n\in Q} a_n \, z^n \, , \; \sum_{n\in Q} \frac{|a_n|^2}{\alpha_n} < \infty \}.$$

Moreover, $\|f\|^2_{H(K)} = \sum_{n\in Q} \frac{|a_n|^2}{\alpha_n}$.

2.2 Proof of Theorem 1

We begin with :

Lemma 3 *The inequalities* (3) *and* (4) *are equivalent.*

Proof. First, note that

$$K(r\,e^{it}, r\,e^{is}) = \sum_{n=0}^{\infty} \alpha_n \, r^{2n} \, e^{int} \, e^{-ins} \, ,$$

and by the assumption on K, the series converges uniformely and absolutely in each disk $|z| \leq r$, where $r < 1$. Replacing K by its expression above in inequality (3) leads then to the desired result. ∎

We now turn to the proof of Theorem 1. Let $N = \sup \alpha_n$. The functions :

$$\frac{N}{1 - zw^*}$$

and

$$\frac{N}{1-zw^*} - K(z,w) = \sum_{n=0}^{\infty} (N-\alpha_n) z^n w^{*n}$$

are positive in $\mathbb{D}$. The function $\frac{N}{1-zw^*}$ is the reproducing kernel of the space H_2, with norm equal to $\sqrt{N}^{-1}$ times the usual H_2 norm. Thus, applying Theorem 2 to the decomposition

$$\frac{N}{1-zw^*} = \left(\frac{N}{1-zw^*} - K(z,w)\right) + K(z,w),$$

we obtain that the space $H(K)$ is continuously included in H_2. In particular, functions in $H(K)$ possess nontangential limits almost everywhere, which is (i).

To establish (ii), let first $f \in L_2(I)$ be such that $f = h|_I$ for some $h \in H(K)$ satisfying $\|h\|_{H(K)} \leq M$. Using Lemma 2, we may write $h(z) = \sum_{n \in Q} h_n z^n$, where Q denotes the subset of $\mathbb{N}$ for which $\alpha_n \neq 0$. We have :

$$\|h\|_{H(K)} = \sum_{n \in Q} \frac{|h_n|^2}{\alpha_n}.$$

For every $\phi \in L_2(I)$:

$$\begin{aligned}
\int_I f(e^{it})\,\phi(e^{it})\,dt &= \int_I \lim_{r\to 1} h(r\,e^{it})\,\phi(e^{it})\,dt \\
&= \lim_{r\to 1} \int_I h(r\,e^{it})\,\phi(e^{it})\,dt \\
&= \lim_{r\to 1} \int_I (\sum_{n\in Q} h_n r^n e^{int}) \phi(e^{it})\,dt \\
&= \lim_{r\to 1} \sum_{n\in Q} h_n\, r^n \int_I \phi(e^{it})\,e^{int}\,dt\,,
\end{aligned}$$

where we have used the fact that the functions $z \mapsto h(rz)$ converge in the H_2 norm to the function h as $r \uparrow 1$ and the dominated convergence theorem. Multiplying and dividing by $\sqrt{\alpha_n}$ and applying the Cauchy–Schwartz inequality give :

$$|\int_I f(e^{it})\,\phi(e^{it})\,dt|^2 \leq (\sum_{n\in Q} \frac{|h_n|^2}{\alpha_n}) \lim_{r\to 1} (\sum_{n\in Q} \alpha_n r^{2n} |\int_I \phi(e^{it}) e^{int}\,dt|^2)$$

from which we obtain inequality (4).

Conversely, assume that (3) (or equivalently (4)) holds for every ϕ satisfying (2). Then, the sesquilinear map K_1 defined by

$$K_1(\phi,\psi) = M^2 \sum_{n=0}^{\infty} \alpha_n \left(\int_I \phi(e^{it})\, e^{int}\, dt\right) \left(\int_I \psi(e^{it})\, e^{int}\, dt\right)^* - \left(\int_I f(e^{it})\, \phi(e^{it})\, dt\right) \left(\int_I f(e^{it})\, \psi(e^{it})\, dt\right)^*,$$

for ϕ and ψ in $L_2(I)$ satisfying (2), is positive. Let us denote by E the set of functions ϕ in $L_2(I)$ such that (2) holds. There exists a reproducing kernel Hilbert space (of functions from E into $\mathbb{C}$) with kernel K_1. Let K_0 and K_2 be defined respectively by

$$K_2(\phi,\psi) = \left(\int_I f(e^{it})\, \phi(e^{it})\, dt\right) \left(\int_I f(e^{it})\, \psi(e^{it})\, dt\right)^*$$

and

$$\begin{aligned} K_0(\phi,\psi) &= K_1(\phi,\psi) + K_2(\phi,\psi) \\ &= M^2 \sum_{n=0}^{\infty} \alpha_n \left(\int_I \phi(e^{it})\, e^{int}\, dt\right) \left(\int_I \psi(e^{it})\, e^{int}\, dt\right)^* . \end{aligned}$$

Theorem 3 gives :

$$H(K_0) = \{F(\phi) = M \sum_Q \sqrt{\alpha_n} \left(\int_I \phi(e^{it})\, e^{int}\, dt\right) a_n \ , \ \text{with} \ \sum_Q |a_n|^2 < \infty\}$$

and, since the $\{e^{int}\}_{n\geq 0}$ are linearly independent in $L_2(I)$, such a representation $\{a_n\}$ is unique for every $F \in H(K_0)$. Moreover, $\|F\|^2_{H(K_0)} = \sum_{n=0}^{\infty} |a_n|^2$. Hence, every $F \in H(K_0)$ can be written as

$$F(\phi) = M \int_I \phi(e^{it}) \left(\sum_{n=0}^{\infty} \sqrt{\alpha_n}\, a_n\, e^{int}\right) dt \, .$$

Let $u(e^{it}) = \sum_{n=0}^{\infty} \sqrt{\alpha_n}\, a_n\, e^{int}$. By Lemma 2, $u \in H(K)$ and $\|F\|_{H(K_0)} = \|u\|_{H(K)}$. The function F may be rewritten as

$$F(\phi) = M \int_I \phi(e^{it})\, u(e^{it})\, dt \, .$$

Thus,

$$H(K_0) = \{F(\phi) = M \int_I \phi(e^{it})\, u(e^{it})\, dt \ , \ u \in H(K)\,\}\,.$$

Now, by Theorem 2, $H(K_2) \subset H(K_0)$. Moreover

$$H(K_2) = \{F(\phi) = c \int_I f(e^{it})\, \phi(e^{it})\, dt \ , c \in \mathbb{C}\}\,.$$

Hence, there exists $u \in H(K)$ such that for every $c \in \mathbb{C}$ and for every $\phi \in E$:

$$c \int_I f(e^{it})\, \phi(e^{it})\, dt = M \int_I \phi(e^{it})\, u(e^{it})\, dt\,,$$

which implies that $c\, f = M\, u$ a.e. on I. Let $h = \frac{M}{c}\, u$: $h \in H(K)$ and $f = h$ a.e. on I. Moreover, the inclusion $H(K_2) \subset H(K_0)$ being contractive, $\|F\|_{H(K_0)} \leq \|F\|_{H(K_2)}$, which implies that $\|u\|_{H(K)} \leq |c|$ and finally, $\|h\|_{H(K)} \leq M$. This ends the proof of (ii).

Finally, claim (iii) is a consequence of Lemma 1. Indeed, let h be a solution of the problem, and let $F(\phi) = \int_I \phi(e^{it})\, h(e^{it})\, dt$. By (4) the function $K_1(\phi, \psi) = K_0(\phi, \psi) - F(\phi) F(\psi)^*$ is positive on E. Lemma 1 implies that $F \in H(K_0)$ and that the norm of F in $H(K_0)$ is equal to the smallest M such that K_1 is positive. This concludes the proof since $\|F\|_{H(K_0)} = \|h\|_{H(K)}$. ■

3 Solution to Problem 1 : Rosenblum–Rovnyak like characterization

This section deals with the particular Hardy case where $H(K) = H_2$ which corresponds to $\alpha_n = 1$, $\forall n \geq 0$. Then :

$$K(z, w) = \sum_{n=0}^{\infty} z^n\, w^{*n} = \frac{1}{1 - z\, w^*}\ , \ \forall z, w \in \mathbb{D}\,.$$

In this particular case, Theorem 1 leads to the following corollary which solves Problem 1 and from which we will deduce some necessary conditions related to Problem 2. It is analogous to results of [20] on Loewner type interpolation problems in the Schur and Carathéodory classes. In the sequel P_{H_2} will denote the orthogonal projection fron $L_2(\mathbb{T})$ onto H_2. We refer to [9], [10], [11], [12] for the theory of Hardy spaces.

Corollary 1 *A function $f \in L_2(I)$ is a solution to Problem 1 if and only if*

$$\lim_{r \to 1} \int_I \int_I \left(\frac{M^2}{1 - r^2 \, e^{it} \, e^{-is}} - f(e^{it}) \, f(e^{is})^* \right) \phi(e^{it}) \, \phi(e^{is})^* \, dt \, ds \geq 0 \qquad (5)$$

for every $\phi \in L_2(I)$, or equivalently if and only if

$$M^2 \sum_{n=0}^{\infty} \left| \int_I \phi(e^{it}) \, e^{int} \, dt \right|^2 \geq \left| \int_I \phi(e^{it}) \, f(e^{it}) \, dt \right|^2 ,$$

which can still be written as :

$$|(f, \phi)_{L_2(I)}| \leq M \, \|P_{H_2} \, \chi_I \, \phi\|_{L_2(\mathbb{T})} .$$

Note that a different but related characterization of traces of H_p functions is given in [5] where the following is established : Let $1 \leq p \leq \infty$ and assume that both I and $\mathbb{T} \setminus I$ have positive measure. Denote by $H_{p|_I}$ the space of traces on I of H_p functions. Construct for each $f \in L_p(I)$ a family (g_n) of H_p functions such that :
(i) $g_{n|_I}$ converges to f in $L_p(I)$,
(ii) g_n converges in H_p whenever $f \in H_{p|_I}$.
Then, $f \in H_{p|_I}$ if and only if $\|g_n\|_{L_p(\mathbb{T})}$ remains bounded. (Such a family (g_n) may be found in [16] and in [18]).
Now, in the case $p = 2$, this necessary and sufficient boundedness condition may be viewed as a consequence of corollary 1. Indeed, assume first that $f \in H_{2|_I}$. Then, for every $\phi \in L_2(I)$,

$$|(f, \phi)_{L_2(I)}| = \lim |(g_n, \phi)_{L_2(I)}| = \lim |(g_n, P_{H_2} \, \chi_I \, \phi)_{L_2(\mathbb{T})}|$$

$$\leq M \, \|P_{H_2} \, \chi_I \, \phi\|_{L_2(\mathbb{T})} ,$$

by corollary 1, so that, extracting a subsequence if necessary, $\|g_n\|_{L_2(\mathbb{T})} \leq M$ for all n. Conversely, assume that $\|g_n\|_{L_2(\mathbb{T})}$ remains bounded, and let $M \geq \|g_n\|_{L_2(\mathbb{T})}$ for all n. We have for every $\phi \in L_2(I)$:

$$|(f, \phi)_{L_2(I)}| = \lim |(g_n, P_{H_2} \, \chi_I \, \phi)_{L_2(\mathbb{T})}| \leq \lim \|g_n\|_{L_2(\mathbb{T})} \, \|P_{H_2} \, \chi_I \, \phi\|_{L_2(\mathbb{T})}$$

$$\leq M \, \|P_{H_2} \, \chi_I \, \phi\|_{L_2(\mathbb{T})} ,$$

and then, still by corollary 1, $f \in H_{2|_I}$.

Consider the operator which, to each function $\phi \in L^2(I)$, associates $P_{H_2}\, \chi_I\, \phi$. Denote by Ψ its restriction to H_2, which is the Toeplitz operator with symbol χ_I. It may be viewed as a semi–infinite matrix $(\Psi_{np})_{n,p\geq 0}$. When $I = (-\theta_0, \theta_0)$, we have :

$$\Psi_{np} = \frac{1}{\pi}\, \frac{\sin(n-p)\theta_0}{n-p}\,, \text{ for } n \neq p\,,\ \Psi_{pp} = \frac{\theta_0}{\pi}\,.$$

We used such a Toeplitz operator in [2] to construct the best normed constrained H_2 approximation of an $L_2(I)$ function.

We deduce the following proposition from Corollary 1. It provides some necessary conditions for $f \in L_2(I)$ to be the trace of an H_2 function, giving a partial answer to Problem 2.

Proposition 1 *Given an arc I of the unit circle $\mathbb{T}$ and $M > 0$, let $f \in L_2(I)$. If f is the trace of an H_2 function of norm less or equal to M, then necessarily :*

$$M^2 \sum_{n=0}^{\infty} |a_n|^2 \geq \left(\sum_{n=-\infty}^{+\infty} |a_n|^2 \right)^2 \tag{6}$$

which is equivalent to :

$$M\, \|\Psi\, f\|_{L_2(\mathbb{T})} \geq \|f\|^2_{L_2(I)}\,.$$

Moreover, for $p \geq 0$,

$$|a_p|^2 \leq M^2 \sum_{n=0}^{\infty} |\Psi_{np}|^2\,,$$

and, when $I = (-\theta_0, \theta_0)$,

$$|a_p|^2 \leq M^2 \sum_{n=0}^{\infty} \frac{\sin^2(n-p)\theta_0}{(n-p)^2}\ ,\ \forall p \in \mathbb{Z}\,. \tag{7}$$

Proof. Apply Corollary 1 with $\phi(e^{it}) = f(e^{it})$ and $\phi(e^{it}) = e^{ipt}$, $p \in \mathbb{Z}$, ∎

Other test functions ϕ may be of interest to find necessary conditions on the Fourier coefficients of $\chi_I\, f$ (in particular, $\phi = f^*$, $\phi = \chi_\Gamma$ for $\Gamma \subset I$,

$\phi(t) = f(t)e^{ikt}$ or $\phi(t) = f(t)^k$, for some fixed k).

Different choices of $\{\alpha_n\}$ may be considered. For instance, $\alpha_n = \frac{1}{1+n^2}$, $\forall n \geq 0$, gives :

$$K(z,w) = K_{\mathcal{D}_2}(z,w) = \sum_{n=0}^{\infty} \frac{z^n\, w^{*n}}{1+n^2}\,,$$

which is the reproducing kernel of the Hilbert (Hardy–Sobolev) space $\mathcal{D}_2$ of analytic functions with H_2 derivative. Observe that traces on I of $\mathcal{D}_2$ functions belong to the Sobolev space $W^{1,2}(I)$. Theorem 1 has in this case the following corollary :

Corollary 2 *A function $f \in L_2(I)$ is the trace of an $\mathcal{D}_2$ function h such that $\|h\|_{W^{1,2}(I)} \leq M$ if and only if :*

$$\lim_{r\to 1} \int_I \int_I \left(M^2\, K_{\mathcal{D}_2}(r\, e^{it}, r\, e^{is}) - f(e^{it})\, f(e^{is})^* \right)\, \phi(e^{it})\, \phi(e^{is})^*\, dt\, ds \geq 0\,,$$

for every $\phi \in W^{1,2}(I)$, or equivalently if and only if

$$M^2 \sum_{n=0}^{\infty} \frac{1}{1+n^2} \left| \int_I \phi(e^{it})\, e^{int}\, dt \right|^2 \geq \left| \int_I \phi(e^{it})\, f(e^{it})\, dt \right|^2 . \tag{8}$$

This result may be extended to any $\mathcal{D}_2^p$ space of analytic functions with p^{th}–derivative in H_2.

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