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Best Approximation in Hardy Spaces and by Polynomials, with Norm Constraints

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Abstract. Two related approximation problems are formulated and solved in Hardy spaces of the disc and annulus. With practical applications in mind, truncated versions of these problems are analysed, where the solutions are chosen to lie in finite-dimensional spaces of polynomials or rational functions, and are expressed in terms of truncated Toeplitz operators. The results are illustrated by numerical examples. The work has applications in systems identification and in inverse problems for PDEs.

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1. Introduction

Let $\mathbb{G} \subset \mathbb{C}$ be equal to either the disk $\mathbb{D}$ or the annulus $\mathbb{A}$ (or to any conformally equivalent domain with Dini-smooth boundary [25]) and $1 < p < \infty$. Let $I \subset \partial\mathbb{G}$ with positive Lebesgue measure, such that $J = \partial\mathbb{G} \setminus I$ also has positive Lebesgue measure.

In the Hardy spaces $H^p(\mathbb{G})$, whose definitions are recalled in Sect. 2, we consider for both cases the following best constrained boundary approximation question.

For a given function $f \in L^p(I)$ and prescribed numbers $c \in \mathbb{C}$, $M \geq 0$, find a solution $g_c = g(c, f, M; \mathbb{G}, I)$ to

$$\|f - g_{c|_I}\|_{L^p(I)} = \min_g \{\|f - g_{|_I}\|_{L^p(I)}\,,\ g \in H^p(\mathbb{G})\,,\ \|g_{|_J} - c\|_{L^p(J)} \leq M\}. \tag{1.1}$$

This is an abstract *bounded extremal problem*, related to those studied in [2,6–8,13–15,32] for various configurations. Namely, the simply connected situation where $\mathbb{G} = \mathbb{D}$ is considered for $p = 2$ in [2,6], for $1 \leq p < \infty$ in [7], and for $p = \infty$ in [8]. The doubly connected case $\mathbb{G} = \mathbb{A}$ is handled in [13,14,32] for $p = 2$ and in [15] for $1 < p < \infty$.

Numerous applications of this constrained approximation issue have been found in the areas of systems identification, parameter identification and inverse problems for PDEs. In particular, we mention: [9], where bounded extremal problems were applied to band-limited frequency-domain systems identification; [22], where inverse diffusion problems were studied; and [20, 21], where approximation problems on the annulus were applied to boundary inverse problems for 2D elliptic PDEs.

Below, we study some finite order discretization schemes for problem (1.1) in classes of polynomials and trigonometric polynomials (as model spaces of $H^p(\mathbb{G})$, cf. [24, Part B,Ch. 3]). Well-posedness properties will be recalled for $1 \leq p < \infty$, and constructive aspects will be developed, together with error estimates and convergence properties, including preliminary numerics in the Hilbertian case $p = 2$.

Further, a new and more subtle issue is to allow c to vary, so that we minimize jointly over $(g, c) \in H^p(\mathbb{G}) \times \mathbb{C}$, and this leads us to the next best-approximation issue.

For a given function $f \in L^p(I)$ and a prescribed $M \geq 0$, find a function $g_* \in H^p(\mathbb{G})$ and a constant $c_* \in \mathbb{C}$ such that $\|g_{*|_J} - c_*\|_{L^p(J)} \leq M$ and

$$\|f - g_{*|_I}\|_{L^p(I)} = \inf_{(g,c)} \{\|f - g_{|_I}\|_{L^p(I)}\,,\ g \in H^p(\mathbb{G})\,,\ c \in \mathbb{C}\,,\ \|g_{|_J} - c\|_{L^p(J)} \leq M\}. \tag{1.2}$$

Well-posedness will be established for such problems with $1 < p < \infty$, together with a density result of traces on $I \subset \partial\mathbb{G}$ of $H^p(\mathbb{G})$ functions. For these issues also, constructive aspects are discussed.

The following comment about geometries is pertinent: in the simply connected situation $\mathbb{G} = \mathbb{D}$, with $I \subset \mathbb{T}$, $J = \mathbb{T} \setminus I$, the associated Toeplitz operator has no eigenvalues; for $\mathbb{G} = \mathbb{A}$, where $\partial\mathbb{A}$ is made up of two circles, say $s\mathbb{T}$ $(0 < s < 1)$ and $\mathbb{T}$, two type of situations may occur, depending on whether I is equal to one of these circles or not. The second case is strongly related to that of the disk, because the Toeplitz operator again has no eigenvalues, while the first allows easier computations, because there is a basis of eigenvectors.

We will further study below some discretization properties of the Toeplitz operators involved in the resolution schemes when $p = 2$, in situations where $\mathbb{G} = \mathbb{D}$ and $I \subset \mathbb{T}$ and where $\mathbb{G} = \mathbb{A}$ and $I = \mathbb{T}$. We provide convergence results and error estimates of the computational algorithms, when the solutions are sought in finite-dimensional spaces of polynomials, and therefore expressed with truncated Toeplitz operators (Toeplitz matrices).

We begin in Sect. 2 by establishing the necessary notation and definitions. The analysis of Problem (1.2) in undertaken in Sect. 3. Then, in Sect. 4 we turn to truncated versions of Problem (1.1). In Sect. 5 an explicit solution is given in the case $\mathbb{G} = \mathbb{A}$ and $I = \mathbb{T}$, and this is illustrated by numerical simulations in Sect. 6. Finally, some conclusions and perspectives for future work are presented.

2. Notation, Definitions, Basic Properties

Let $\mathbb{G}$ be equal to the unit disk $\mathbb{D}$ or to the annulus $\mathbb{A}$.

For $1 \le p < \infty$, the Hardy spaces $H^p(\mathbb{G})$ are defined as the collection of functions g analytic in the domain $\mathbb{G}$ and bounded there in Hardy norm, see [17,27]:

$$\|g\|_{H^p} = \sup_{\varrho<r<1} \left[\frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p \, d\theta \right]^{1/p} < \infty,$$

with $\varrho = 0$ if $\mathbb{G} = \mathbb{D}$ and $\varrho = s$ if $\mathbb{G} = \mathbb{A}$. We could similarly define the Hardy spaces $H^p(\mathbb{C}\setminus s\overline{\mathbb{D}})$, which can be directly seen as the image of $H^p(\mathbb{D})$ functions under the isomorphism $g \mapsto zg(s/z)$. In the same way, the set $H^p_0(\mathbb{C} \setminus s\overline{\mathbb{D}})$ of functions in $H^p(\mathbb{C} \setminus s\overline{\mathbb{D}})$ that vanish at ∞, is the image under the same isomorphism of the subset $H^p_0(\mathbb{D}) \subset H^p(\mathbb{D})$ of functions that vanish at 0.

The space $H^p(\mathbb{A})$ is isomorphic to the direct sum of two Hardy spaces of simply connected domains:

$$H^p(\mathbb{A}) = H^p(\mathbb{D}) \oplus H^p_0(\mathbb{C}\setminus s\overline{\mathbb{D}}). \tag{2.1}$$

Every $H^p(\mathbb{G})$ function g admits a non-tangential limit (trace) $g_{|\partial\mathbb{G}}$ on $\partial\mathbb{G}$, which defines the subset $H^p(\partial\mathbb{G}) \subset L^p(\partial\mathbb{G})$.

The $H^p(\partial\mathbb{G})$ coincides with the closure of algebraic, respectively trigonometric, polynomials, for $\mathbb{G} = \mathbb{D}$, resp. $\mathbb{A}$ (the closure in $L^p(\partial\mathbb{G})$ of the set of all rational functions with no poles in $\overline{\mathbb{G}}$). The two spaces $H^p(\mathbb{G})$ and $H^p(\partial\mathbb{G})$ are isometrically isomorphic. We thus identify a function $g \in H^p(\mathbb{G})$ with its non-tangential limit (trace) $g_{|\partial\mathbb{G}}$, see [17,27], and we have $\|g\|_{H^p} = \|g_{|\partial\mathbb{G}}\|_{L^p(\partial\mathbb{G})}$. When $p = 2$, the above Hardy spaces are Hilbert spaces with respect to the $L^2(\partial\mathbb{G})$ inner product. In this case, the direct sum (2.1) is orthogonal, and Fourier bases are Hilbert bases.

Let P_{H^2} denote the orthogonal projection from $L^2(\partial\mathbb{G})$ onto $H^2(\mathbb{G})$. For $\Omega \subset \partial\mathbb{G}$ and $\varphi \in L^2(\Omega)$ we put for simplicity

$$\frac{1}{2\pi} \int_\Omega \varphi = \frac{1}{2\pi} \int_{\varrho e^{i\theta} \in \Omega} \varphi(\varrho e^{i\theta}) \, d\theta = \frac{1}{2\pi\varrho} \int_{z\in\Omega} \varphi(z) \, |dz| \, ,$$

with $\varrho = 1$ if $\Omega \subset \mathbb{T}$ (or in $\Omega \cap \mathbb{T}$) and $\varrho = s$ if $\Omega \subset s\mathbb{T}$ (or in $\Omega \cap s\mathbb{T}$).

In the sequel, for $f \in L^p(I)$ and $k \in L^p(J)$, $f \vee k$ will denote the function of $L^p(\partial\mathbb{G})$ equal to f on I and equal to k on J. Most of the time, f or k will be supposed equal to 0. We use the notation $\chi_\Omega g$, where χ_Ω is the characteristic function of Ω, when g is defined on the whole $\partial\mathbb{G}$.

3. Analysis of Problem (1.2)

For a given $M \ge 0$, we introduce the approximation subsets of $H^p(\mathbb{G})$, denoted by $\mathcal{B}_{M,c}$, $c \in \mathbb{C}$ and $\mathcal{B}_M$ as follows. For $c \in \mathbb{C}$, put

$$\mathcal{B}_{M,c} = \{ \, g \in H^p(\mathbb{G}) \, , \; \|g_{|J} - c\|_{L^p(J)} \le M \},$$
$$\mathcal{B}_M = \{ \, g \in H^p(\mathbb{G}) \, , \; \|g_{|J} - c\|_{L^p(J)} \le M \text{ for some } c \in \mathbb{C} \} = \bigcup_{c\in\mathbb{C}} \mathcal{B}_{M,c}.$$

For simplicity, we omit in the above notation the dependence with respect to $p, \mathbb{G}$ and I. Observe that for every $M \geq 0$ and $c \in \mathbb{C}$, the constant function $g \equiv c$ always belongs to $\mathcal{B}_{M,c}$.

3.1. Well-Posedness

The existence and uniqueness of Problem (1.2) is given by the following theorem.

Theorem 3.1. *Let $1 < p < \infty$. For every $M > 0$ and $f \in L^p(I)$, there exist a unique function $g_* = g_*(f, M; p, \mathbb{G}, I) \in H^p(\mathbb{G})$ and a constant $c_* = c_*(f, M; p, \mathbb{G}, I) \in \mathbb{C}$, such that $\|g_{*|_J} - c_*\|_{L^p(J)} \leq M$ and*

$$\|f - g_{*|_I}\|_{L^p(I)} = \inf_{(g,c)} \{\|f - g_{|_I}\|_{L^p(I)}\,,\ g \in \mathcal{B}_{M,c}\,,\ c \in \mathbb{C}\}.$$

Moreover, if $f \notin \mathcal{B}_{M|_I}$, then the solution (g_*, c_*) is unique and saturates the constraint: $\|g_{*|_J} - c_*\|_{L^p(J)} = M$.

To prove Theorem 3.1, we need a preliminary density result.

Proposition 3.2. *Let $1 < p < \infty$ and let $\mathbb{G} = \mathbb{D}$ or $\mathbb{G} = \mathbb{A}$. If $J = \partial\mathbb{G} \setminus I$ has positive Lebesgue measure, then $H^p(\mathbb{G})_{|_I}$ is dense in $L^p(I)$.*

Proof. The result for $\mathbb{G} = \mathbb{D}$ is given in [8, Prop. 1]. A similar duality proof works for $\mathbb{G} = \mathbb{A}$: namely we write q for the conjugate index to p, and suppose that $f \in L^q(I)$ annihilates $H^p(\mathbb{A})_{|_I}$. It is convenient in this proof to use harmonic measure m for a point $t \in \mathbb{A}$, rather than Lebesgue measure, as in [1, Sec. 1]: we have $dm(z) = \frac{1}{2\pi i} v(z) dz$, where v is meromorphic on a neighbourhood of $\overline{\mathbb{A}}$, with one zero in $\mathbb{A}$, a pole at t, and no other zeroes or poles in $\overline{\mathbb{A}}$: in particular, density in $L^p(I)$ does not depend on whether we use m or Lebesgue measure. We have

$$\int_{\partial\mathbb{A}} (f \vee 0) g \, dm = 0 \qquad \text{for all} \quad g \in H^p(\mathbb{A}).$$

But the dm-annihilator of $H^p(\mathbb{A})$ in $L^q(\partial\mathbb{A})$ is $v^{-1} H^q(\mathbb{A})$ (the proof of [1, Thm. 1.7] for $p = 2$ works more generally). However, since $f \vee 0$ vanishes on a set of positive measure, it follows from [1, Cor. 1.19] that vf, and hence f, is zero a.e. on I. This establishes the density. □

Proof of Theorem 3.1. Introduce the following operators A and B:

$$\begin{aligned} A \,:\, H^p(\mathbb{G}) \times \mathbb{C} &\longrightarrow L^p(I) \\ (g, c) \quad &\longmapsto g_{|_I} \end{aligned}$$

$$\begin{aligned} B \,:\, H^p(\mathbb{G}) \times \mathbb{C} &\longrightarrow L^p(J) \\ (g, c) \quad &\longmapsto g_{|_J} - c. \end{aligned}$$

In view of [15, Lem. 2.1], the next result holds for $1 < p < \infty$. If A and B are bounded linear operators with dense ranges, which are also coprime in the sense that there exists $\delta > 0$ such that for all $(g, c) \in H^p(\mathbb{G}) \times \mathbb{C}$,

$$\|A(g, c)\|_{L^2(I)} + \|B(g, c)\|_{L^p(J)} \geq \delta \|(g, c)\|_{H^p(\mathbb{G}) \times \mathbb{C}}\,, \tag{3.1}$$

then, if $f \notin \mathcal{B}_{M_{|_I}}$, there exists a unique solution $(g_*, c_*) \in \mathcal{B}_M \times \mathbb{C}$ to Problem (1.2), and the conclusions of Theorem 3.1 regarding uniqueness and constraint saturation hold true.

Let us then prove that the above assumptions are satisfied here. First, observe that A and B have dense ranges from Proposition 3.2. Next, we claim that A and B satisfy (3.1), or equivalently that

$$\|g_{|_I}\|_{L^p(I)} + \|g_{|_J} - c\|_{L^p(J)} \geq \delta(\|g_{|_J}\|_{L^p(\partial\mathbb{G})} + |c|).$$

Assume to the contrary that there exists a sequence $(g_n, c_n) \in H^p(\mathbb{G}) \times \mathbb{C}$, such that

$$\|g_n\|_{L^p(\partial\mathbb{G})} + |c_n| = 1, \tag{3.2}$$

while

$$\|g_{n|_I}\|_{L^p(I)} + \|g_{n|_J} - c_n\|_{L^p(J)} \to 0.$$

Because (g_n) is bounded in $H^p(\mathbb{G})$, we may pass to a subsequence and suppose that it is weakly convergent to a function $g \in H^p(\mathbb{G})$; at the same time we may suppose that $c_n \to c$ for some $c \in \mathbb{C}$. Now $g = 0$ on I and so $g = 0$ everywhere on $\partial\mathbb{G}$, by uniqueness results for $H^p(\mathbb{G})$ functions on subsets of positive measure of the boundary. Also, $g = c$ on J, so $c = 0$. Finally, we obtain that $\|g_n\|_{L^p(\partial\mathbb{G})} \to 0$ and $c_n \to 0$, which is a contradiction to (3.2). From [15, Lem. 2.1], these properties give the proof in the above situation. If $f \in \mathcal{B}_{M_{|_I}}$,then the best approximation $g_* = f$ is still unique, although there may exist a set of complex numbers c_* such that $\|f_{|_J} - c_*\|_{L^p(J)} \leq M$ (see Lemma 3.3). □

3.2. Expression of Solutions to Problem (1.2)

Now, let p be equal to 2. Let us compute the solution (g_*, c_*).

3.2.1. General Situation, $f \notin \mathcal{B}_{M_{|_I}}$.

Assuming that $f \notin \mathcal{B}_{M_{|_I}}$, $M > 0$; the solution (g_*, c_*) is then given by the implicit equation [14]:

$$(A^* A - \gamma B^* B)(g_*, c_*) = A^* f, \tag{3.3}$$

for the unique parameter $\gamma < 0$ such that $\|g_{*|_J} - c_*\|_{L^2(J)} = M$ and the adjoint operators A^*, B^* of A, B that are given from their definition by:

$$\begin{aligned} A^* \, &: L^2(I) \longrightarrow H^2(\mathbb{G}) \times \mathbb{C} \\ & \quad h \quad \longmapsto (P_{H^2}(h \vee 0), 0) \\ B^* \, &: L^2(J) \longrightarrow H^2(\mathbb{G}) \times \mathbb{C} \\ & \quad \varphi \quad \longmapsto \left(P_{H^2}(0 \vee \varphi), -\frac{1}{2\pi}\int_J \varphi\right). \end{aligned}$$

Then, for $(g, c) \in H^2(\mathbb{G}) \times \mathbb{C}$,

$$\begin{aligned} A^*A(g,c) &= (P_{H^2}\chi_I g_{|_I}, 0), \text{ and } B^*B(g,c) \\ &= \left(P_{H^2}\chi_J(g_{|_J} - c), -\frac{1}{2\pi}\int_J (g_{|_J} - c)\right). \end{aligned}$$

From Eq. (3.3), we get:

$$(P_{H^2}\chi_I g_*, 0) - \gamma \left(P_{H^2}\chi_J (g_{*|_J} - c_*), -\frac{1}{2\pi}\int_J (g_{*|_J} - c_*) \right) = (P_{H^2}(f \vee 0), 0) \,,$$

whence

$$\begin{cases} \frac{1}{2\pi}\int_J (g_{*|_J} - c_*) = 0 \\ P_{H^2}\chi_I g_* - \gamma P_{H^2}\chi_J (g_{*|_J} - c_*) = P_{H^2}(f \vee 0) \,, \end{cases} \tag{3.4}$$

for the unique $\gamma < 0$ s.t. $\|g_{*|_J} - c_*\|_{L^2(J)} = M$. In particular, we get that c_* is equal to the normalized mean value of g_* on J:

$$c_* = \frac{1}{|J|}\int_J g_{*|_J}. \tag{3.5}$$

Introduce the Toeplitz operator $T^J = P_{H^2}\chi_J$ of symbol χ_J on $H^2(\mathbb{G})$, defined by:

$$\begin{array}{rcl} T^J : H^2(\mathbb{G}) & \longrightarrow & H^2(\mathbb{G}) \\ g & \longmapsto & P_{H^2}(\chi_J g) \,. \end{array} \tag{3.6}$$

We shall discuss such Toeplitz operators in detail later, but for now we merely mention the easily-verified fact that T^J is self-adjoint and its spectrum is contained in the interval $[0, 1]$.

Now the second relation in (3.4) is thus equivalent to:

$$(Id - (\gamma+1)T^J)g_* = P_{H^2}(f \vee (-\gamma c_*)) \,, \tag{3.7}$$

hence, with (3.5),

$$(Id - (\gamma+1)T^J)g_* + \frac{\gamma}{|J|}T^J\left(\int_J g_{*|_J}\right) = P_{H^2}(f \vee 0).$$

Further, (3.4) is to the effect that

$$\frac{1}{2\pi}\int_I (f - g_{*|_I}) = \frac{1}{2\pi}\int_J (g_{*|_J} - c_*) = 0 \quad \text{whence} \quad \int_{\partial\mathbb{G}} (g_* - f \vee c_*) = 0.$$

Whenever $\mathbb{G} = \mathbb{A}$, $I = \mathbb{T}$ and $J = s\mathbb{T}$,

$$\frac{1}{2\pi}\int_{\mathbb{T}} (f - g_{*|_{\mathbb{T}}}) = \frac{1}{2\pi}\int_{s\mathbb{T}} (g_{*|_{s\mathbb{T}}} - c_*) = 0 \,,$$

and because $g_* \in H^2(\mathbb{A})$, see Theorem 5.1, we have

$$\frac{1}{2\pi}\int_{\mathbb{T}} g_{*|_{\mathbb{T}}} = \frac{1}{2\pi}\int_{s\mathbb{T}} g_{*|_{s\mathbb{T}}} \,, \quad \text{hence} \quad c_* = \frac{1}{2\pi}\int_{\mathbb{T}} g_{*|_{\mathbb{T}}} = \frac{1}{2\pi}\int_{\mathbb{T}} f.$$

3.2.2. Approximation Class $\mathcal{B}_M$. We begin with the following lemma.

Lemma 3.3. *Let $M > 0$ and $f_J \in L^2(J)$. Then, there exists $c \in \mathbb{C}$ such that $\|f_J - c\|_{L^2(J)} \leq M$ if, and only if,*

$$\frac{1}{2\pi|J|}\left|\int_J f_J\right|^2 \geq \|f_J\|^2_{L^2(J)} - M^2. \tag{3.8}$$

Proof. Let $\mathbf{1}_J$ be the function defined on J identically equal to 1. Since the orthogonal projection of f_J onto $\{\mathbf{1}_J\, c\, :\, c \in \mathbb{C}\}$ is equal to $\frac{2\pi}{|J|}\langle f_J, \mathbf{1}_J\rangle_{L^2(J)}\, \mathbf{1}_J$, we have that

$$\left\| f_J - \frac{2\pi}{|J|}\langle f_J, \mathbf{1}_J\rangle_{L^2(J)}\, \mathbf{1}_J \right\|^2_{L^2(J)} = \|f_J\|^2_{L^2(J)} - \frac{2\pi}{|J|}\left|\langle f_J, \mathbf{1}_J\rangle_{L^2(J)}\right|^2$$
$$= \|f_J\|^2_{L^2(J)} - \frac{1}{2\pi|J|}\left|\int_J f_J\right|^2,$$

from which the result follows. □

Recall that $F \in \mathcal{B}_M$ if and only if $F \in H^2(\mathbb{G})$ and there exists a constant $c \in \mathbb{C}$ such that $\|F_{|_J} - c\|_{L^2(J)} \leq M$, whence, in view of Lemma 3.3, if and only if $F_{|_J}$ satisfies (3.8).

Note that for $F \in H^2(\mathbb{G})$, if $\int_J F_{|_J} = 0$, then either $\|F_{|_J}\|_{L^2(J)} \leq M$, whence $f \in \mathcal{B}_{M,0}$, or $F \notin \mathcal{B}_M$ (from the proof of Lemma 3.3).

Assume that $f \in \mathcal{B}_{M|_I}$, in Problem (1.2). In this situation, and only in this situation, one can and has to choose $\gamma = 0$ in Eq. (3.7), to the effect that $g_* \equiv F$, for the function $F \in \mathcal{B}_M$ such that $F_{|_I} = f$. The fact that γ can be made as small as possible truly characterizes the traces on I of functions in $\mathcal{B}_M$, as was discussed in [7] for $\mathbb{G} = \mathbb{D}$.

Though there may exist several values for c_* (the criterion is now equal to 0), the one given by (3.5) achieves a minimal value for the constraint on J. Indeed, the proof of Lemma 3.3 is to the effect that whenever (3.8) holds, the constraint value $\|f_{|_J} - c\|_{L^2(J)}$ is minimal for

$$c = c_* = \frac{1}{|J|}\int_J f_{|_J} \text{ for which } \|f_{|_J} - c_*\|^2_{L^2(J)} = \|f_{|_J}\|^2_{L^2(J)} - \frac{|J|}{2\pi}\,|c_*|^2.$$

This constraint on J however is no longer saturated, in this case.

Note that expressions such as $\|f_{|_J} - c_*\|^2_{L^2(J)}$ (allowing J to vary) are closely related to the BMO norm of f.

3.3. Degenerate Situation $M = 0$

For $M = 0$ and $c \in \mathbb{C}$, $\mathcal{B}_{0,c} = \{g \equiv c \text{ on } \mathbb{G}\} = \{c\}$, whence $\mathcal{B}_0 = \mathbb{C}$. With $f \in L^2(I)$, Problem (1.2) becomes

$$\inf_{(g,c)}\{\|f - g_{|_I}\|_{L^p(I)}\,,\ g \in \mathcal{B}_{0,c}\,,\ c \in \mathbb{C}\} = \inf_{c\in\mathbb{C}}\|f - c\|_{L^p(I)} = \|f - c_*\|_{L^p(I)}$$

and $g_* \equiv c_*$. Using the same argument as in Sect. 3.2.2, the solution for $p = 2$ is equal to

$$c_* = \frac{1}{|I|}\int_I f.$$

Note that for $p \neq 2$, one can establish from [10] or from [15] the implicit relation:

$$\frac{1}{|I|}\int_I |f - c_*|^{p-2}\,(f - c_*) = 0.$$

Situations where $f \in \mathcal{B}_0 = \mathbb{C}$ then reduce to $f \equiv c_f \in \mathbb{C}$, whence $g_* \equiv c_* \equiv c_f$.

4. Polynomials Approximation: Truncated Problem (1.1)

In this section, we focus on the discretization of Problem (1.1). With an appropriate choice of c this leads to a discretization of Problem (1.2).

4.1. Analysis of Problem (1.1)

We recall that $\mathbb{G}$ denotes the unit disc $\mathbb{D}$ or the annulus $\mathbb{A}$, $I \subset \partial\mathbb{G}$ and $J = \partial\mathbb{G}\backslash I$ are subsets of $\partial\mathbb{G}$ such that I and J have positive Lebesgue measure.

Such problems of minimization have been settled in [2,7] in the unit disc and for a more general constraint $\|g_{|_J} - h\|_{L^p(J)} \leq M$ for a given function $h \in L^p(J)$ and in [13,32] for the annulus with $I \subset \mathbb{T}$.

Let $1 < p < \infty$. Given a function $f \in L^p(I)$, a complex number c and a positive number M, there exists a solution $g_c \in H^p(\mathbb{G})$ to Problem (1.1):

$$\|f - g_{c|_I}\|_{L^p(I)} = \min_g\{\|f - g_{|_I}\|_{L^p(I)}\ ,\ g \in \mathcal{B}_{M,c}\}.$$

Moreover, if f, c and M are such that $f \notin \mathcal{B}_{M,c|_I}$, then the solution g_c is unique and the constraint is saturated: $\|g_{c|_J} - c\|_{L^p(J)} = M$.

When $p = 2$, if f does not lie in $\mathcal{B}_{M,c|_I}$, then the solution g_c to Problem (1.1) is given by the implicit equation

$$(Id - (\gamma + 1)T^J)g_c = P_{H^2}(f \vee (-\gamma c)), \tag{4.1}$$

with $\gamma < 0$ such that $\|g_{c|_J} - c\|_{L^2(J)} = M$ and T^J the Toeplitz operator with symbol χ_J defined by (3.6).

Observe that g_c given by (4.1) above and the solution g_* to Problem (1.2) given by (3.7) admit similar expressions ($g_* = g_{c_*}$).

Remark 4.1. It is well known and easy to verify (see, for example, [11,26] for more general results) that in the case $\mathbb{G} = \mathbb{D}$ for any nontrivial $\Omega \subset \mathbb{T}$ the spectrum of T^Ω is $[0, 1]$ and there is no point spectrum. Note that 0 and 1 are in the continuous spectrum of T^Ω (T^Ω and $Id - T^\Omega$ are injective and have a dense range respectively). This follows from uniqueness results for $H^2(\mathbb{D})$ functions on subsets of positive measure of the boundary, see [1,17] and the self-adjointness of T^Ω.

In the case $\mathbb{G} = \mathbb{A}$, the spectrum of T^Ω is again $[0, 1]$ whenever either Ω or its complement has a non-null intersection with both components of $\partial\mathbb{A}$ (see [1,4]), and there is no point spectrum.

The only case remaining is when $\mathbb{G} = \mathbb{A}$ and $\Omega = \mathbb{T}$, respectively $\Omega = s\mathbb{T}$; then the spectrum of T^Ω consists of $\{0, 1\}$ and the simple eigenvalues (of finite multiplicity), see [1,32]:

$$\left\{\frac{1}{1 + s^{2k}}, k \in \mathbb{Z}\right\}\ ,\ \text{respectively}\ \left\{\frac{s^{2k}}{1 + s^{2k}}, k \in \mathbb{Z}\right\}\ ,$$

that lie in $(0, 1)$ and accumulate only at 0 and 1. The operator T^Ω has an orthonormal basis of eigenvectors, namely the functions

$$\left\{\frac{z^k}{1 + s^{2k}}, k \in \mathbb{Z}\right\}.$$

In view of studying convergence properties and numerical aspects of the truncation of the solution of (1.1), we suggest a new approach to this problem. It consists in considering the problem for functions in classes of polynomials.

4.2. Analysis of Truncated Problem (1.1): well-posedness

Let $N \in \mathbb{N}$ and $\mathcal{P}_N$ be the subspace of polynomials or trigonometric polynomials, respectively, of $H^p(\mathbb{G})$ defined by

$$\mathcal{P}_N = \begin{cases} \operatorname{span}\left\{z^k, 0 \le k \le N\right\} & \text{if } \mathbb{G} = \mathbb{D}, \\ \operatorname{span}\left\{z^k, -N \le k \le N\right\} & \text{if } \mathbb{G} = \mathbb{A}. \end{cases}$$

Remark 4.2. If $\mathbb{G} = \mathbb{D}$, then

$$\mathcal{P}_N = K_\Theta = H^p(\mathbb{D}) \cap (\Theta H^p(\mathbb{D}))^\perp = H^p(\mathbb{D}) \cap \Theta H^p_0(\mathbb{C} \setminus \overline{\mathbb{D}}),$$

the model space associated with the finite Blaschke product $\Theta(z) = z^{N+1}$ (see [24, Part B,Ch. 3] and Sect. 7.2 for $\mathbb{G} = \mathbb{A}$, $p = 2$).

Put $\mathcal{B}_{M,c,N} = \mathcal{B}_{M,c} \cap \mathcal{P}_N$. Let (BEP_N) be the following problem: for $f \in L^p(I)$, $M > 0$ and $c \in \mathbb{C}$, we seek $g_N \in \mathcal{B}_{M,c,N}$ such that

$$\|f - g_{N|_I}\|_{L^p(I)} = \min_{p_N} \left\{ \|f - p_{N|_I}\|_{L^p(I)}, p_N \in \mathcal{B}_{M,c,N} \right\}. \qquad (BEP_N)$$

When $p = 2$, we solve problem (BEP_N) by giving an expression of the solution g_N comparable to the implicit equation (4.1) satisfied by the solution g_c of (1.1). As (BEP_N) is the discretization of (1.1), a truncation of the Toeplitz operator will naturally appear. This operator is called a truncated Toeplitz operator and was discussed in [28]. On the Fourier basis, it coincides with a Toeplitz matrix of size $(N+1) \times (N+1)$ if $\mathbb{G} = \mathbb{D}$, or $(2N+1) \times (2N+1)$ if $\mathbb{G} = \mathbb{A}$. It is denoted by T_N^Ω, for $\Omega \subset \partial\mathbb{G}$ and defined by

$$\begin{aligned} T_N^\Omega : \mathcal{P}_N &\longrightarrow \quad \mathcal{P}_N \\ p_N &\longmapsto P_N\left(\chi_\Omega p_N\right), \end{aligned}$$

where P_N is the orthogonal projection from $L^2(\partial\mathbb{G})$ onto $\mathcal{P}_N$. Note that the (point) spectrum of the truncated Toeplitz operator T_N^Ω is included into $(0,1)$ and the invertibility of T_N^Ω and $Id - T_N^\Omega = T_N^{\mathbb{G}\setminus\Omega}$ is due to the finite dimension of $\mathcal{P}_N$.

Well-posedness of Problem (BEP_N) is then ensured by the next result for $1 < p < \infty$. Let Q_N be the projection from $L^p(I)$ onto $\mathcal{P}_{N|_I}$.

Theorem 4.3. *For every $f \in L^p(I)$, $c \in \mathbb{C}$ and $M > 0$, there exists a unique function $g_N = g_N(f, M; \mathbb{G}, I) \in \mathcal{P}_N$ such that*

$$\|f - g_{N|_I}\|_{L^p(I)} = \min_{p_N} \{ \|f - p_{N|_I}\|_{L^p(I)}, p_N \in \mathcal{B}_{M,c,N} \}.$$

Moreover, if $Q_N f \notin \mathcal{B}_{M,c,N|_I}$, then the constraint is saturated: $\|g_{N|_J} - c\|_{L^p(J)} = M$.

Proof. Observe that $\mathcal{B}_{M,c,N|_I}$ is a closed and convex set of $L^p(I)$. The existence and uniqueness of g_N follows from the projection theorem on a closed and convex set (see [10]). Now, suppose that $Q_N(f) \notin \mathcal{B}_{M,c,N|_I}$ and that

$\|g_{N|_J} - c\|_{L^p(J)} < M$. Let q_N be the element of $\mathcal{P}_N$ such that $Q_N(f) = q_{N|_I}$. One can find $\lambda \in (0,1)$ such that

$$\|\lambda q_{N|_J} + (1-\lambda) g_{N|_J}\|_{L^p(J)} \le M.$$

Since $\|f - Q_N(f)\|_{L^p(I)} \le \|f - g_{N|_I}\|_{L^p(I)}$, we have that

$$\|f - \lambda Q_N(f) - (1-\lambda) g_{N|_I}\|_{L^p(I)} < \|f - g_{N|_I}\|_{L^p(I)},$$

which contradicts the minimality of g_N. □

4.3. Convergence Properties of Solutions to (BEP_N)

In this subsection, we establish convergence properties of the solution g_N and the error estimate β_N to g and β, respectively (related to Problem (1.1)).

Proposition 4.4. *Let $M > 0$, $c \in \mathbb{C}$ and $f \in L^p(I)$, such that $f \notin \mathcal{B}_{M,c|_I}$, and $N \in \mathbb{N}$. Let g and g_N respectively denote the associated solutions to (1.1) and (BEP_N). We define $\beta(M) = \|f - g_{|_I}\|_{L^p(I)}$ and for $N \in \mathbb{N}$, $\beta_N(M) = \|f - g_{N|_I}\|_{L^p(I)}$ the approximation errors on I associated to the constraint M on J, such that $M = \|g_{|_J} - c\|_{L^p(J)} = \|g_{N|_J} - c\|_{L^p(J)}$. Then, $(\beta_N(M))_{N\ge 0}$ converges and decreases to $\beta(M)$ as N tends to $+\infty$ and*

$$\|g - g_N\|_{L^p(\partial\mathbb{G})} \to 0, \ \ \text{as } N \to \infty.$$

Proof. Assume first that $c = 0$. The decay of $(\beta_N(M))_{N\ge 0}$ with N and the inequality $\beta_N(M) \ge \beta(M)$ both follow immediately from the increasing of the class of approximants. Let $\varepsilon > 0$ and g_ε be the solution to (1.1) associated to f and such that $\|g_{\varepsilon|_J}\|_{L^p(J)} \le M - \varepsilon$, whence $\|g_{\varepsilon|_J}\|_{L^p(J)} = M - \varepsilon$, see Sect. 4.1. Since β depends continuously on M, as a convex function of $M > 0$, it holds that

$$\|f - g_{\varepsilon|_I}\|_{L^p(I)} = \beta(M - \varepsilon) \le \beta + \delta_\varepsilon,$$

for some δ_ε which goes to 0 with ε. Since $\bigcup_{N\ge 0} \mathcal{P}_N$ is dense in $H^p(\mathbb{G})$, there exists $g_N^\varepsilon \in \mathcal{P}_N$ such that

$$\|g_\varepsilon - g_N^\varepsilon\|_{L^p(\partial\mathbb{G})} \le \varepsilon.$$

Hence, we have that $\|f - g^\varepsilon_{N|_I}\|_{L^p(I)} \le \beta(M) + \delta_\varepsilon + \varepsilon$ while $\|g^\varepsilon_{N|_J}\|_{L^p(J)} \le M$. But necessarily, we have that

$$\beta_N(M) \le \|f - g^\varepsilon_{N|_I}\|_{L^p(I)} \le \beta(M) + \delta_\varepsilon + \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain that $\beta_N(M) \to \beta(M)$, as N tends to $+\infty$, which proves the claim. Now, since $(g_N)_N$ is bounded in $L^p(\partial\mathbb{G})$ norm uniformly in N by $2\|f\|_{L^p(I)} + M$, we get that every subsequence of $(g_N)_{N\in\mathbb{N}}$ converges weakly to some function, say $h \in H^p(\mathbb{G})$. This implies that

$$\|f - h_{|_I}\|_{L^p(I)} \le \liminf \|f - g_{N|_I}\|_{L^p(I)} = \beta(M)$$

while $\|h_{|_J}\|_{L^p(J)} \le M$ from [12, Prop. III.12]. Hence, $h = g$, and because this holds for every subsequence, we get that $(g_N)_{N\in\mathbb{N}}$ weakly converges to g in $H^p(\mathbb{G})$; in particular, the sequence $(f - g_{N|_I})_{N\in\mathbb{N}}$ weakly converges in $L^p(I)$ to $f - g_{|_I}$. As

$$\|f \vee 0 - g_N\|_{L^p(\partial\mathbb{G})} = \beta_N(M) + \|g_{N|_J}\|_{L^p(J)},$$

we get that

$$\|f \vee 0 - g_N\|_{L^p(\partial\mathbb{G})} \xrightarrow[N\to+\infty]{} \|f \vee 0 - g\|_{L^p(\partial\mathbb{G})} = \beta(M) + \|g_{|_J}\|_{L^p(J)}$$

Applying [12, Prop. III.30], it follows that

$$\|g - g_N\|_{L^p(\partial\mathbb{G})} \to 0, \text{ as } N \to \infty.$$

For $c \neq 0$, the proof is similar, working with $g - c$ and $g_N - c$. □

4.4. Expression of Solutions to Problem (BEP_N)

In the sequel, we let $p = 2$. In this case, $Q_N : L^2(I) \to \mathcal{P}_{N|_I}$ is an orthogonal projection. Since for $p_N \in \mathcal{P}_N$, we have that

$$\|f - p_{N|_I}\|^2_{L^2(I)} = \|Q_N(f) - p_{N|_I}\|^2_{L^2(I)} + \|f - Q_N(f)\|^2_{L^2(I)},$$

we now consider the Problem (BEP_N) rewritten as follows

$$\|Q_N(f) - g_{N|_I}\|_{L^2(I)} = \min\{\|Q_N(f) - g_{|_I}\|_{L^2(I)}\,,\ g \in \mathcal{B}_{M,c,N}\}.$$

The next lemma gives an explicit expression of $Q_N(f)$, for $f \in L^2(I)$.

Lemma 4.5. *Let $f \in L^2(I)$. Then,*

$$Q_N(f) = (R_I^*)^{-1}\, P_N(f \vee 0) = R_I\,(T_N^I)^{-1}\, P_N(f \vee 0),$$

where R_I denotes the projection from $\mathcal{P}_N$ onto $\mathcal{P}_{N|_I}$ (restriction map).

Proof. Let $f \in L^2(I)$. Then, $f - Q_N(f)$ is orthogonal to $\mathcal{P}_{N|_I}$ for the inner product on $L^2(I)$ which is equivalent to

$$\langle f - Q_N(f), R_I(q_N)\rangle_{L^2(I)} = 0, \text{ for all } q_N \in \mathcal{P}_N.$$

Hence, for $q_N \in \mathcal{P}_N$,

$$\begin{aligned}\langle {R_I}^* Q_N(f), q_N\rangle_{L^2(\partial\mathbb{G})} = \langle Q_N(f), R_I(q_N)\rangle_{L^2(I)} &= \langle f, R_I(q_N)\rangle_{L^2(I)} \\ &= \langle P_N(f \vee 0), q_N\rangle_{L^2(\partial\mathbb{G})}.\end{aligned}$$

It follows that $Q_N(f) = R_I^{*-1}\, P_N(f \vee 0)$. Now, let $p_N \in \mathcal{P}_N$ be such that $Q_N(f) = R_I(p_N)$. Note that for $q_N, k_N \in \mathcal{P}_N$, we have

$$\begin{aligned}&\langle R_I^* R_I(q_N), k_N\rangle_{L^2(\partial\mathbb{G})} = \langle R_I(q_N), R_I(k_N)\rangle_{L^2(I)} \\ &\quad = \langle \chi_I q_N, k_N\rangle_{L^2(\partial\mathbb{G})} = \langle P_N(\chi_I q_N), k_N\rangle_{L^2(\partial\mathbb{G})}\,,\end{aligned}$$

whence $R_I^* R_I = T_N^I$, by definition. Since $R_I^* Q_N(f) = R_I^* R_I(p_N)$, we obtain from what precedes that

$$Q_N(f) = R_I(p_N) = R_I(T_N^I)^{-1} P_N(f \vee 0).$$

□

We obtain the following proposition.

Proposition 4.6. *Let $f \in L^2(I)$, $c \in \mathbb{C}$ and $M > 0$. The solution g_N to (BEP_N) is given by*

$$g_N = (Id - (\gamma_N + 1)T_N^J)^{-1}\, P_N(f \vee (-\gamma_N c))\,, \tag{4.2}$$

where $\gamma_N \le 0$. More precisely, γ_N is equal to 0 if and only if $Q_N(f) \in \mathcal{B}_{M,c,N|_I}$ and if $Q_N(f) \notin \mathcal{B}_{M,c,N|_I}$, then $\gamma_N < 0$ and g_N saturates the constraint:

$$\|g_{N|_J} - c\|_{L^2(J)} = M.$$

Proof. If $Q_N(f) \in \mathcal{B}_{M,c,N|_I}$, then the solution g_N is such that $R_I(g_N) = Q_N(f)$ and is the minimum of the function $p_N \in \mathcal{P}_N \longmapsto \|Q_N(f) - p_{N|_I}\|_{L^2(I)}$. By Lemma 4.5, one can have that

$$g_N = (Id - T_N^J)^{-1} P_N(f \vee 0) = (T_N^I)^{-1} P_N(f \vee 0). \tag{4.3}$$

Suppose now that $Q_N(f) \notin \mathcal{B}_{M,c,N|_I}$. Applying [14, Thm 2.1], it follows that there exists a unique solution g_N to (BEP_N) given by

$$(R_I^* R_I - \gamma_N R_J^* R_J) g_N = R_I^*(Q_N(f)) - \gamma_N P_N(\chi_J c) = P_N(f \vee (-\gamma_N c)), \tag{4.4}$$

where $\gamma_N < 0$ and R_J is the restriction map from $\mathcal{P}_N$ onto $\mathcal{P}_{N|_J}$. Since we have from the proof of Lemma 4.5 that $R_I^* R_I = T_N^I$, by symmetry, we also have $R_J^* R_J$ equal to the truncated Toeplitz operator T_N^J. So, Eq. (4.4) can be rewritten in terms of truncated Toeplitz operators as follows

$$(T_N^I - \gamma_N T_N^J) g_N = (Id - (\gamma_N + 1) T_N^J) g_N = P_N(f \vee (-\gamma_N c)),$$

where $\gamma_N < 0$ is such that $\|g_{N|_J} - c\|_{L^2(J)} = M$. The solution g_N of (BEP_N) is given by (4.2).

Note that if $\gamma_N = 0$, we find again Eq. (4.3). □

Observe that a similar result holds for truncated versions of Problem 1.2, with $c = c_N = 1/|J| \int_J g_{N|_J}$ in equation (3.5), see Sect. 3.2.1.

Remark 4.7. For $N \in \mathbb{N}$, we define the functions m_N and m from $(-\infty, 0]$ to $[0, +\infty)$ as follows:

$$\begin{aligned} m_N(t) &= \|(Id - (t+1)T_N^J)^{-1} P_N(f \vee 0)_{|_J} - c\|_{L^2(J)}\,, \\ m(t) &= \|(Id - (t+1)T^J)^{-1} P_{H^2}(f \vee 0)_{|_J} - c\|_{L^2(J)}. \end{aligned} \tag{4.5}$$

Then the saturation of the constraint in the bounded extremal problems (1.1), (1.2) and (BEP_N) implies that

$$m_N(\gamma_N) = m(\gamma) = M. \tag{4.6}$$

Note that the operators R_I and R_J have the same role as A and B appearing in Sect. 3. Likewise, we have that, for $\Omega = I$ or J,

$$\begin{aligned} R_I^* : \mathcal{P}_{N|_\Omega} &\longrightarrow \mathcal{P}_N \\ p_{N|_\Omega} &\longmapsto P_N(p_{N|_\Omega} \vee 0) = P_N(\chi_\Omega p_N). \end{aligned}$$

Observe further that $T_N^\Omega = P_N T^\Omega_{|_{\mathcal{P}_N}}$ while for $g \in H^2(\mathbb{G})$,

$$P_N T^\Omega g = T_N^\Omega P_N g + P_N T^\Omega (g - P_N g) = T_N^\Omega P_N g + P_N[(P_N \chi_\Omega) R_{N_\infty} g]\,,$$

if we let $R_{N_\infty} = Id - P_N$ on $H^2(\mathbb{G})$ (see also Sect. 5.2 for $\mathbb{G} = \mathbb{A}$). The finite dimensional truncated problem (BEP_N) could also be approached and solved using arguments from convex optimization (cf. [16]).

5. Solution to (BEP_N) in the Annulus

Now, we assume that $\mathbb{G}$ is the annulus $\mathbb{A}$ defined by $\mathbb{D}\backslash s\overline{\mathbb{D}}$ with boundary $\partial\mathbb{A} = \mathbb{T} \cup s\mathbb{T}$. We also suppose that I is equal to the unit circle $\mathbb{T}$ and J to $s\mathbb{T}$.

5.1. Explicit Expressions in the Annulus

We construct the solution g_N to (BEP_N) with $c = 0$, for this particular configuration, for which explicit expressions of P_N, $T_N^{s\mathbb{T}}$ and g_N will be obtained.

We recall the following characterization from [27] of functions in $H^2(\mathbb{A})_{|\partial\mathbb{A}}$.

Theorem 5.1. *Let $x \in L^2(\partial\mathbb{A})$ such that*

$$x_{|_\mathbb{T}}(e^{i\theta}) = \sum_{k\in\mathbb{Z}} a_k e^{ik\theta} \quad \textit{and} \quad x_{|_{s\mathbb{T}}}(se^{i\theta}) = \sum_{k\in\mathbb{Z}} b_k s^k e^{ik\theta}$$

almost everywhere on $\mathbb{T}$ and $s\mathbb{T}$ respectively. Then, $x \in H^2(\mathbb{A})_{|\partial\mathbb{A}}$ if and only if $b_k = a_k$.

Remark 5.2. If $x \in H^2(\mathbb{A})_{|\partial\mathbb{A}}$, then by Theorem 5.1, we have that $a_k = b_k$,

$$P_N x(z) = \sum_{k=-N}^{N} a_k z^k,$$

and $P_N x$ is the truncation at order N of the Laurent expansion of x.

The next lemma computes the orthogonal projection P_N from $L^2(\partial\mathbb{A})$ onto $\mathcal{P}_N$.

Lemma 5.3. *Let $x \in L^2(\partial\mathbb{A})$ with Fourier series on $\mathbb{T}$ and on $s\mathbb{T}$ given by*

$$x_{|_\mathbb{T}}(e^{i\theta}) = \sum_{k\in\mathbb{Z}} a_k e^{ik\theta} \quad \textit{and} \quad x_{|_{s\mathbb{T}}}(se^{i\theta}) = \sum_{k\in\mathbb{Z}} b_k s^k e^{ik\theta},$$

almost everywhere on $\mathbb{T}$ and $s\mathbb{T}$ respectively. Then, for $z \in \mathbb{A}$,

$$P_N x(z) = \sum_{k=-N}^{N} \frac{a_k + s^{2k} b_k}{1 + s^{2k}} z^k. \tag{5.1}$$

Proof. We have that P_N is the truncation at order N of the Laurent expansion of the projection onto $H^2(\mathbb{A})$ of x. Indeed, if P_{H^2} denotes the orthogonal projection from $L^2(\partial\mathbb{A})$ onto $H^2(\mathbb{A})$, then $P_N x = P_N(P_{H^2} x)$ since $(x - P_{H^2} x) \in (H^2(\mathbb{A}))^\perp \subset \mathcal{P}_N^\perp$. By [32, Lem. 4.1], we have for $z \in \mathbb{A}$ the Laurent expansion:

$$P_{H^2} x(z) = \sum_{k\in\mathbb{Z}} \frac{a_k + s^{2k} b_k}{1 + s^{2k}} z^k\,, \text{ for } z \in \mathbb{A},$$

whose truncation at order N is then given by (5.1), see Remark 5.2. □

The next proposition gives an explicit expression of the solution g_N to (BEP_N).

Proposition 5.4. *Let $f \in L^2(\mathbb{T})$ with Fourier series $\sum_{k\in\mathbb{Z}} a_k e^{ik\theta}$. Then, for $z \in \mathbb{A}$, the solution g_N to (BEP_N) in $\mathbb{A}$ is given by*

$$g_N(z) = \sum_{k=-N}^{N} \frac{a_k}{1-\gamma_N s^{2k}} z^k,$$

for a unique $\gamma_N \le 0$. If $Q_N(f) \notin \mathcal{B}_{M,0,N_{|\mathbb{T}}}$, then $\gamma_N < 0$ is such that

$$\sum_{k=-N}^{N} \frac{|a_k|^2 s^{2k}}{(1-\gamma_N s^{2k})^2} = M^2.$$

Proof. Every $p_N \in \mathcal{P}_N$ can be written for $z \in \mathbb{A}$ as

$$p_N(z) = \sum_{k=-N}^{N} a_k z^k.$$

Then,

$$T_N^{\mathbb{T}} p_N(z) = \sum_{k=-N}^{N} \frac{a_k}{1+s^{2k}} z^k \quad \text{and} \quad T_N^{s\mathbb{T}} p_N(z) = \sum_{k=-N}^{N} \frac{a_k s^{2k}}{1+s^{2k}} z^k.$$

Indeed, since $(\chi_{\mathbb{T}} p_N)_{|_{\mathbb{T}}}(e^{i\theta}) = \sum_{k=-N}^{N} a_k e^{ik\theta}$ and $(\chi_{\mathbb{T}} p_N)_{|_{s\mathbb{T}}}(se^{i\theta}) = 0$ almost everywhere on $\mathbb{T}$ and $s\mathbb{T}$ respectively, the result follows from Lemma 5.3. Similar arguments hold for the expression of $T_N^{s\mathbb{T}}$. Proposition 4.6 and expression (4.2) of the solution to (BEP_N) leads to the conclusion. □

Remark 5.5. The orthonormal basis $(e_k)_{-N\le k\le N}$ of $\mathcal{P}_N$ where $e_k(z) = z^k(1+s^{2k})^{-1/2}$ for $z \in \mathbb{A}$ is a basis of eigenvectors of $T_N^{\mathbb{T}}$ and $T_N^{s\mathbb{T}}$. Moreover, the matrix of $T_N^{\mathbb{T}}$ and $T_N^{s\mathbb{T}}$ respectively in the basis $(e_k)_{-N\le k\le N}$ are diagonal and so, if q_N is given by $q_N(z) = \sum_{k=-N}^{N} b_k z^k$, we have for $z \in \mathbb{A}$:

$$(T_N^{\mathbb{T}})^{-1} q_N(z) = \sum_{k=-N}^{N} b_k(1+s^{2k})z^k\,,\ (T_N^{s\mathbb{T}})^{-1} q_N(z) = \sum_{k=-N}^{N} b_k \frac{(1+s^{2k})}{s^{2k}} z^k.$$

Likewise, if the Fourier series of $f \in L^2(\mathbb{T})$ is given by $\sum_{k\in\mathbb{Z}} a_k e^{ik\theta}$, then we have that for all $N \in \mathbb{N}$,

$$Q_N(f)(e^{i\theta}) = \sum_{k=-N}^{N} a_k e^{ik\theta}, \ \text{for almost every } e^{i\theta} \in \mathbb{T}.$$

Further, whenever $f \in H^2(\mathbb{A})_{|_{\partial\mathbb{A}}}$, it holds from Remark 5.2 that $Q_N(f_{|_{\mathbb{T}}}) = (P_N f)_{|_{\mathbb{T}}}$.

We will denote by $\mathcal{C}_M^2$ the set $\{h \in H^2(\mathbb{A}), \|h_{|_{s\mathbb{T}}}\|_{L^2(s\mathbb{T})} \le M\}$ (which coincides with the approximation class $\mathcal{B}_{M,0}$ for $p = 2$, see Sect. 3).

Lemma 5.6. *If $f \in L^2(\mathbb{T})$ but $f \notin \mathcal{C}^2_{M_{|_{\mathbb{T}}}}$, there exists $N_0 \in \mathbb{N}$ such that for $N \ge N_0$, $Q_N f \notin \mathcal{B}_{M,0,N|_{\mathbb{T}}}$, while for $N < N_0$, $Q_N f \in \mathcal{B}_{M,0,N|_{\mathbb{T}}} \subset \mathcal{B}_{M,0,N_0-1|_{\mathbb{T}}}$.*

Proof. Assume $f \notin \mathcal{C}^2_{M_{|_{\mathbb{T}}}}$. In any cases $Q_N f \in \mathcal{P}_{N_{|_{\mathbb{T}}}} \subset H^2(\mathbb{A})_{|_{\mathbb{T}}}$ is such that $\|f - Q_N(f)\|_{L^2(\mathbb{T})} \to 0$ as $N \to +\infty$. Let $p_N \in \mathcal{P}_N$, $p_{N_{|_{\mathbb{T}}}} = Q_N(f)$.

- Suppose first that $f \notin H^2(\mathbb{A})_{|\mathbb{T}}$; then it follows as a consequence of the density result in Proposition 3.2 in that particular setting, see also [13, Prop. 4.1], that $\|p_N\|_{L^2(s\mathbb{T})} \to \infty$ as $N \to \infty$, whence there exist N_0 such that $\|p_{N_0}\|_{L^2(s\mathbb{T})} > M$.
- Suppose now that $f \in H^2(\mathbb{A})_{|\mathbb{T}}$, and $f = F_{|\mathbb{T}}$ for $F \in H^2(\mathbb{A})$ with $\|F\|_{L^2(s\mathbb{T})} > M$. In this case, $Q_N(f) = (P_N F)_{|\mathbb{T}} = p_{N|\mathbb{T}}$ (see Remark 5.5), and because $\|F - p_N\|_{L^2(s\mathbb{T})} \to 0$, there exists N_0 such that $\|p_{N_0}\|_{L^2(s\mathbb{T})} > M$.

In both cases, one can choose N_0 to be the smallest such integer. In particular, if p_N has Fourier coefficients (a_k), $k = -N, \cdots, N$, the quantity $\|p_n\|^2_{L^2(s\mathbb{T})} = \sum_{k=-n}^{n} |a_k|^2 s^{2k}$ is increasing with n for $0 \leq n \leq N$, and we get the conclusion. We also get that necessarily $a_{n_0} \neq 0$ for some $n_0 \in \mathbb{Z}$ with $|n_0| = N_0 \leq N$. □

5.2. Error Estimates

Now, we are interested in establishing some error estimates between the solution g to Problem (1.1) and g_N to (BEP_N) with $c = 0$ for the same given constraint $M > 0$. In the sequel, we will mention the dependence of g and g_N on the Lagrange parameters γ and γ_N respectively, that also depend on the constraint M. Let $R_{N_\infty}(\sum_{k\in\mathbb{Z}} a_k e^{ik\theta})(z) = \sum_{|k|>N} a_k z^k$, on $L^2(\mathbb{T})$.

Proposition 5.7. *Let $f \in L^2(\mathbb{T})$ with Fourier series $\sum_{k\in\mathbb{Z}} a_k e^{ik\theta}$ and $f \notin \mathcal{C}^2_{M|_\mathbb{T}}$. Then, the sequence of parameters $(\gamma_N)_{N\in\mathbb{N}}$ is non-increasing and converges to γ and there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$, we have that, for some constant $C_0 > 0$,*

$$0 \leq \gamma_N - \gamma \leq C_0\, s^{2N}\, \|R_{N_\infty}(f)\|^2_{L^2(\mathbb{T})}. \tag{5.2}$$

We may take $C_0 = \max(1, 1/\gamma^2)\,(s^{-n_0} - \gamma s^{n_0})^4/2\,|a_{n_0}|^2$, for some $n_0 \in \mathbb{Z}$ with $|n_0| = N_0$ such that $a_{n_0} \neq 0$ (or simply, if $a_0 \neq 0$, $C_0 = \max(1, 1/\gamma^2)$ $(1-\gamma)^4/2\,|a_0|^2$).

Proof. Let $f \notin \mathcal{C}^2_{M|_\mathbb{T}}$. From [2,14], and the results recalled in Sect. 4.1, we know that $\gamma \neq 0$ and Lemma 5.6 together with Proposition 4.6 imply the existence of $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$, $\gamma_N < 0$. Let $N \geq N_0$. From (4.5) together with Proposition 5.4 and Remark 5.5, we have for $t < 0$ that

$$m_N^2(t) = \sum_{k=-N}^{N} \frac{|a_k|^2 s^{2k}}{(1 - t s^{2k})^2} \quad \text{and} \quad m_{N+1}^2(t) = \sum_{k=-(N+1)}^{N+1} \frac{|a_k|^2 s^{2k}}{(1 - t s^{2k})^2}.$$

The functions m_N^2 and m_{N+1}^2 are continuous and non-increasing, as are their inverse functions. Since Theorem 4.3 ensures that g_N saturates the constraint for every $N \geq N_0$, (4.5) implies that

$$\begin{aligned} M^2 = m^2(\gamma) = m_N^2(\gamma_N) = \sum_{k=-N}^{N} \frac{|a_k|^2 s^{2k}}{(1 - \gamma_N s^{2k})^2} &= m_{N+1}^2(\gamma_{N+1}) \\ \geq \sum_{k=-N}^{N} \frac{|a_k|^2 s^{2k}}{(1 - \gamma_{N+1} s^{2k})^2} &= m_N^2(\gamma_{N+1}). \end{aligned}$$

Applying the inverse function to m_N^2 to the previous inequality, it follows that $(\gamma_N)_{N\in\mathbb{N}}$ is non-increasing and in particular, $\gamma_N = 0$ for $0 \le N < N_0$ (see Proposition 4.6). Now, equality (4.6) gives that

$$\sum_{k=-N}^{N} \frac{|a_k|^2 s^{4k}[(1-\gamma s^{2k})^2 - (1-\gamma_N s^{2k})^2]}{(1-\gamma_N s^{2k})^2(1-\gamma s^{2k})^2} = \sum_{|k|>N} \frac{|a_k|^2 s^{2k}}{(1-\gamma s^{2k})^2},$$

which leads to

$$(\gamma_N - \gamma) \sum_{k=-N}^{N} \frac{|a_k|^2 s^{4k}(2-(\gamma_N+\gamma)s^{2k})}{(1-\gamma_N s^{2k})^2(1-\gamma s^{2k})^2} = \sum_{|k|>N} \frac{|a_k|^2 s^{2k}}{(1-\gamma s^{2k})^2}. \tag{5.3}$$

Since $\gamma_N, \gamma < 0$, equality (5.3) implies that $\gamma_N > \gamma$.

It follows that

$$\sum_{k=-N}^{N} \frac{|a_k|^2 s^{4k}(2-(\gamma_N+\gamma)s^{2k})}{(1-\gamma_N s^{2k})^2(1-\gamma s^{2k})^2} > \sum_{k=-N}^{N} \frac{2|a_k|^2 s^{4k}}{(1-\gamma s^{2k})^4} \ge \frac{2\,|a_{n_0}|^2}{(s^{-n_0}-\gamma s^{n_0})^4}$$

(recall that we choose $N \ge N_0$ in order to ensure that $|a_{n_0}| \ne 0$ for $|n_0| = N_0$).

Clearly, we have that

$$\sum_{|k|>N} \frac{|a_k|^2 s^{2k}}{(1-\gamma s^{2k})^2} = \sum_{k=-\infty}^{-N-1} \frac{|a_k|^2 s^{2k}}{(1-\gamma s^{2k})^2} + \sum_{k=N+1}^{+\infty} \frac{|a_k|^2 s^{2k}}{(1-\gamma s^{2k})^2}$$

$$\le \frac{1}{\gamma^2} s^{2N} \sum_{k=-\infty}^{-N-1} |a_k|^2 + s^{2N} \sum_{k=N+1}^{+\infty} |a_k|^2 \le s^{2N} \max\left(1, \frac{1}{\gamma^2}\right) \|R_{N_\infty}(f)\|^2_{L^2(\mathbb{T})}.$$

Combining the previous inequalities with (5.3) completes the proof. □

In the next Corollary, the index N_0 is the same as the one appearing in Proposition 5.7, which only depends on f and M.

Corollary 5.8. *Let $f \in L^2(\mathbb{T})$ with Fourier series $\sum_{k\in\mathbb{Z}} a_k e^{ik\theta}$ be such that $f \notin \mathcal{C}^2_{M|_\mathbb{T}}$. Then, there exist C_1, $C_2 > 0$ such that for all $N \ge N_0$,*

$$\|g_N - g\|_{L^2(\partial\mathbb{A})} \le C_1 \|R_{N_\infty}(f)\|^2_{L^2(\mathbb{T})} + C_2\, \|R_{N_\infty}(f)\|_{L^2(\mathbb{T})}.$$

Indeed, we may take $C_1 = C_0\|f\|_{L^2(\mathbb{T})}$ and $C_2 = \max(1, 1/|\gamma|)$.

In other words, $\|g_N - g\|_{L^2(\partial\mathbb{A})} = O\left(\|R_{N_\infty}(f)\|_{L^2(\mathbb{T})}\right)$, as $N \to +\infty$ and the solution g_N to (BEP_N) converges in $L^2(\partial\mathbb{A})$ norm to the solution g to Problem (1.1).

Proof. By Proposition 5.4 and [32, Prop. 4.3], we have that

$$\|g_N - g\|^2_{L^2(\partial\mathbb{A})}$$
$$= (\gamma_N - \gamma)^2 \sum_{k=-N}^{N} \frac{s^{4k}(1+s^{2k})|a_k|^2}{(1-\gamma_N s^{2k})^2(1-\gamma s^{2k})^2} + \sum_{|k|>N} \frac{|a_k|^2(1+s^{2k})}{(1-\gamma s^{2k})^2}.$$

For $-N \le k \le N$, we write that

$$\frac{(1+s^{2k})s^{4k}}{(1-\gamma_N s^{2k})^2(1-\gamma s^{2k})^2} = \frac{1+s^{2k}}{(1-\gamma_N s^{2k})^2(s^{-2k}-\gamma)^2}.$$

For $0 \le k \le N$, we have that

$$\frac{1+s^{2k}}{(1-\gamma_N s^{2k})^2(s^{-2k}-\gamma)^2} \le \frac{2}{(1-\gamma)^2} \le \frac{2s^{-4N}}{(1-\gamma)^2} \le 2s^{-4N},$$

and for $-N \le k \le -1$, we have that

$$\frac{1+s^{2k}}{(1-\gamma_N s^{2k})^2(s^{-2k}-\gamma)^2} \le \frac{1+s^{-2N}}{s^{2N}-\gamma} \le 2s^{-4N}.$$

Thus, we obtain that

$$\sum_{k=-N}^{N} \frac{s^{4k}(1+s^{2k})|a_k|^2}{(1-\gamma_N s^{2k})^2(1-\gamma s^{2k})^2} \le 2s^{-4N}\|f\|^2_{L^2(\mathbb{T})}. \tag{5.4}$$

Now for $k > N$, we have that

$$\frac{1+s^{2k}}{(1-\gamma s^{2k})^2} = \frac{1}{(1-\gamma s^{2k})^2} + \frac{1}{(s^{-k}-\gamma s^{k})^2} \le 1+s^{2N},$$

and

$$\sum_{k=N+1}^{+\infty} \frac{|a_k|^2(1+s^{2k})}{(1-\gamma s^{2k})^2} \le (1+s^{2N}) \sum_{k>N} |a_k|^2.$$

For $k < -N$, we have that

$$\frac{1+s^{2k}}{(1-\gamma s^{2k})^2} \le \frac{2}{(s^{-k}-\gamma s^{k})^2} \le \frac{2}{\gamma^2} s^{2N}.$$

As $s < 1$, one can write that

$$\sum_{|k|>N} \frac{|a_k|^2(1+s^{2k})}{(1-\gamma s^{2k})^2} \le 2 \max\left(1, \frac{1}{\gamma^2}\right) \sum_{|k|>N} |a_k|^2.$$

It follows from Proposition 5.7 that

$$\begin{aligned}
&\|g_N - g\|^2_{L^2(\partial\mathbb{A})} \\
&\le 2\,C_0^2\,\|R_{N_\infty}(f)\|^4_{L^2(\mathbb{T})} \times \|f\|^2_{L^2(\mathbb{T})} + 2 \max\left(1, \frac{1}{\gamma^2}\right) \|R_{N_\infty}(f)\|^2_{L^2(\mathbb{T})} \\
&\le 2\,C_1^2\|R_{N_\infty}(f)\|^4_{L^2(\mathbb{T})} + 2\,C_2^2\|R_{N_\infty}(f)\|^2_{L^2(\mathbb{T})},
\end{aligned}$$

with $C_1 = C_0\|f\|_{L^2(\mathbb{T})}$ and $C_2 = \max(1, 1/|\gamma|)$. □

Further, one directly deduces from (5.4) and Proposition 5.7 that as $N \to \infty$:

$$\|g_N - P_N g\|_{L^2(\partial\mathbb{A})} \le \sqrt{2}\,C_0\,\|R_{N_\infty}(f)\|^2_{L^2(\mathbb{T})} \times \|f\|_{L^2(\mathbb{T})} = O\left(\|R_{N_\infty}(f)\|^2_{L^2(\mathbb{T})}\right).$$

6. Numerical Illustrations

In order to illustrate the considerations and results of Sects. 4, 5 for $\mathbb{G} = \mathbb{A}$ and $I = \mathbb{T}$, we let $f = f_\varepsilon \in L^2(\mathbb{T}) \setminus H^2(\mathbb{A})_{|\mathbb{T}}$ be explicitly defined as a perturbation of some function $f_0 \in H^2(\mathbb{A})$:

$$f(e^{i\theta}) = f_{0|\mathbb{T}}(e^{i\theta}) + \frac{\varepsilon/d}{e^{i\theta} - d},$$

for some (small) $\varepsilon > 0$, $d \in \mathbb{A}$, and

$$f_0(z) = \frac{1}{z-a} + \frac{1}{z-b} \in H^2(\mathbb{A}), \text{ with } a \in s\mathbb{D}, \ b \in \mathbb{C} \setminus \overline{\mathbb{D}}.$$

For $N \in \mathbb{N}$, $N \geq 1$, we then have:

$$Q_N f = f_{N|\mathbb{T}}, \text{ with } f_N(z) = f_{0,N}(z) + \varepsilon \sum_{p=-1}^{-N} \frac{z^p}{d^{p+2}} \in \mathcal{P}_N,$$

$$\text{and } f_{0,N}(z) = P_N f_0(z) = -\sum_{p=0}^{N} \frac{z^p}{b^{p+1}} + \sum_{p=-1}^{-N} \frac{z^p}{a^{p+1}},$$

while

$$Q_N(f_{0|\mathbb{T}}) = f_{0,N|\mathbb{T}} = (P_N f_0)_{|\mathbb{T}}.$$

We fix $s = 1/3$ and the annulus $\mathbb{A}$, and take $a = 1/5$, $b = 5/3$, $d = 11/30$ which determine f_0 and f. All the computations and illustrations are made with *Maple 15*.

Because $\|f_{0|_{s\mathbb{T}}}\|_{L^2(s\mathbb{T})} \simeq 3.8$, we choose $M_r = 4$ as a reference value for the constraint M, so that $f_0 \in \mathcal{B}_{M_r,0}$. We thus expect the solution g_N to (BEP_N) to also provide a reasonable approximation to the $H^2(\mathbb{A})$-function f_0 (not only to f), for large enough N. Indeed, the choice $M = M_r$, together with the saturation of the constraint by g_N if $f_N \notin \mathcal{B}_{M_r,0,N}$, will ensure that $M_r = \|g_{N|_{s\mathbb{T}}}\|_{L^2(s\mathbb{T})} > \|f_{0|_{s\mathbb{T}}}\|_{L^2(s\mathbb{T})}$, whence

$$\begin{aligned} \|f_{0,N|\mathbb{T}} - g_{N|\mathbb{T}}\|_{L^2(\mathbb{T})} &\leq \|f_{0,N|\mathbb{T}} - f_{N|\mathbb{T}}\|_{L^2(\mathbb{T})} + \|f_{N|\mathbb{T}} - g_{N|\mathbb{T}}\|_{L^2(\mathbb{T})} \\ &\leq 2\,\|f_{0,N|\mathbb{T}} - f_{N|\mathbb{T}}\|_{L^2(\mathbb{T})}, \end{aligned}$$

because $f_{0,N} \in \mathcal{B}_{M_r,0,N}$.

Table 1 relates different values of ε and M to the corresponding integer $N_0 = N_0(M, \varepsilon)$ for which $f_{N_0} \notin \mathcal{B}_{M,0,N_0}$ and $f_{N_0-1} \in \mathcal{B}_{M,0,N_0-1}$, see Lemma 5.6.

Table 1. Smallest $N_0 = N_0(M, \varepsilon) \in \mathbb{N}$ such that $f_{N_0} \notin \mathcal{B}_{M,0,N_0}$, for different values of M and of $\varepsilon \in [10^{-2}, 10^{-1}]$.

	$\varepsilon =$	10^{-2}	2.10^{-2}	3.10^{-2}	5.10^{-2}	6.10^{-2}	10^{-1}
$M = 4.5$	$N_0 =$	26	17	10	4	3	2
$M = M_r = 4$	$N_0 =$	10	4	3	2	2	2
$M = 3.81$	$N_0 =$	3	3	2	2	2	1
$M = 3.5$	$N_0 =$	2	2	2	2	1	1

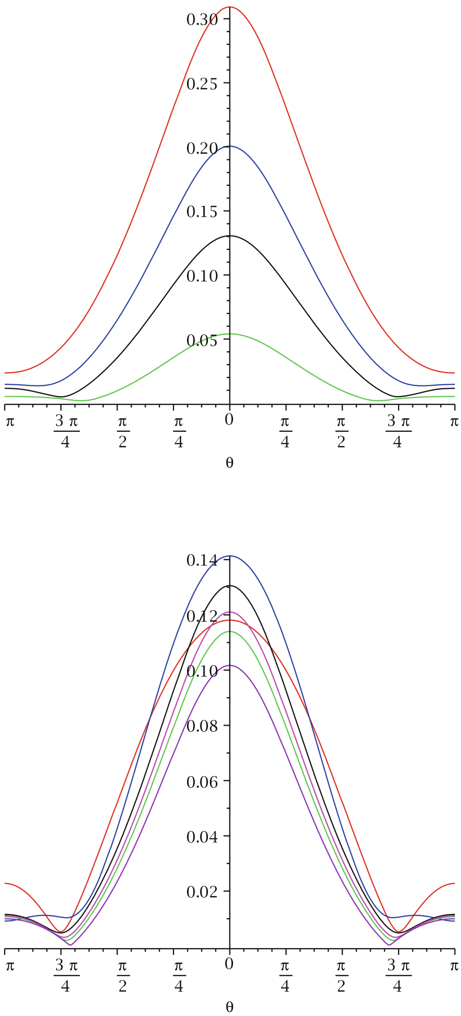
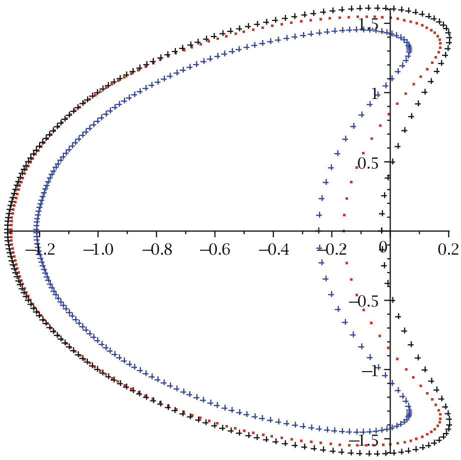


FIGURE 1. $|(f_{N|_{\mathbb{T}}} - g_{N|_{\mathbb{T}}})(e^{i\theta})|$, w.r.t. $\theta \in [0, 2\pi]$, with $M = M_r = 4$; *top* for $N = 10$ and ε varying; *bottom* for $\varepsilon = 5.10^{-2}$ and N varying.

We next compute the solutions g_N to (BEP_N) associated to f, $c = 0$, and a number of values of ε, N, and M.

In Fig. 1, we took $M = M_r = 4$. The left-hand plot corresponds to the pointwise error $|(f_{N|_{\mathbb{T}}} - g_{N|_{\mathbb{T}}})(e^{i\theta})|$, $e^{i\theta} \in \mathbb{T}$, for $N = 10$ and some values of ε between 10^{-1} and 10^{-2}. The right-hand plot shows the same quantity, for $\varepsilon = 5.10^{-2}$ and some values of N between 3 and 50. Fig. 1 shows that $\varepsilon = 5.10^{-2}$ and $N = 10$ ensure a small enough pointwise approximation error, and still permit f_N and $f_{0,N}$ to be numerically distinct on $\mathbb{T}$; see also Fig. 2, left.



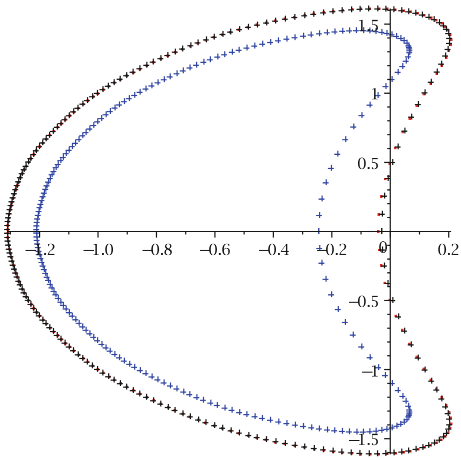


FIGURE 2. $f_{N|_{\mathbb{T}}}$ (*black crosses*), $f_{0,N|_{\mathbb{T}}}$ (*blue crosses*) and $g_{N|_{\mathbb{T}}}$ (*red dots*) for $N = 10$, $\varepsilon = 5.10^{-2}$; *top* $M = M_r = 4$, *bottom* $M = 4.5$ (color figure online).

Numerical computations of the quadratic error (the criterion) with $M = M_r$ give that:

- for $N = 10$, $\|f_{N|_{\mathbb{T}}} - g_{N|_{\mathbb{T}}}\|_{L^2(\mathbb{T})} \leq .18$ and has the expected increasing behaviour with $\varepsilon \in [10^{-2}, 10^{-1}]$; further, $\|f - f_{0,N|_{\mathbb{T}}}\|_{L^2(\mathbb{T})} = C\,\varepsilon$, with $C \simeq 2.9$ (in accordance with the value of the parameter d);
- for $\varepsilon = 5.10^{-2}$, $\|f_{N|_{\mathbb{T}}} - g_{N|_{\mathbb{T}}}\|_{L^2(\mathbb{T})}$ is contained within $(.05, .08)$ when $N = 3, \ldots, 50$.

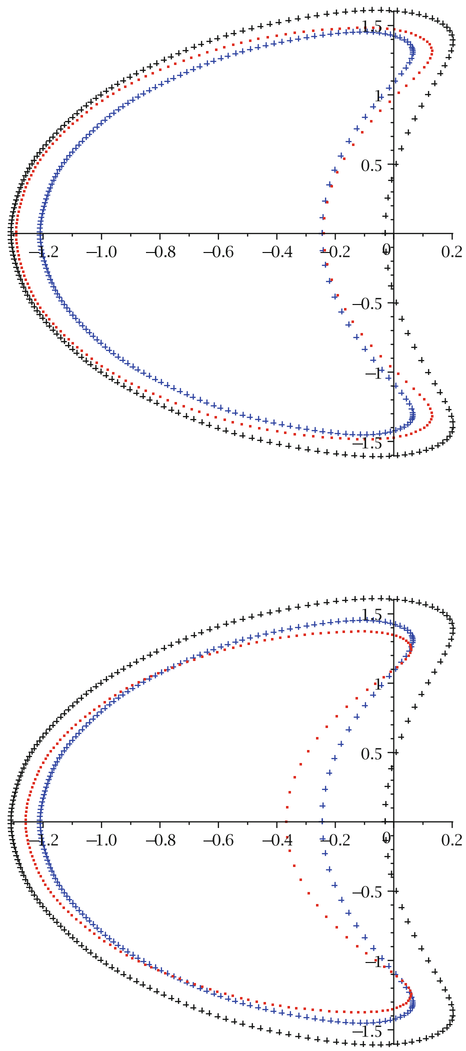


FIGURE 3. $f_{N|_{\mathbb{T}}}$ (*black crosses*), $f_{0,N|_{\mathbb{T}}}$ (*blue crosses*) and $g_{N|_{\mathbb{T}}}$ (*red dots*) for $N = 10$, $\varepsilon = 5.10^{-2}$; *top* $M = 3.81$; *bottom* $M = 3.5$ (color figure online).

We now fix $N = 10$, $\varepsilon = 5.10^{-2}$, whence $f_N \notin \mathcal{B}_{M,0,N}$ for $M \le M_r$ and the other values of M considered in Figs. 2, 3, see Table 1 to the effect that $N > N_0$. In Fig. 1, the top plots are associated to $\varepsilon = 10^{-1}$ (red), 7.10^{-2} (blue), 5.10^{-2} (black), 3.10^{-2} (green), while the bottom ones correspond to $N = 3$ (red), 4 (blue), 10 (black), 20 (green), 30 (pink), 50 (violet). Figs. 2, 3 show Nyquist plots (real and imaginary parts) of the functions f_N, $f_{N,0}$ and g_N on $\mathbb{T}$, for different values of M. In Fig. 2, the fact that the error between $f_{N|_{\mathbb{T}}}$ and $g_{N|_{\mathbb{T}}}$ decreases with M corresponds to the fact that on the right-hand plot, for $M = 4.5$, the two functions are not distinct. For smaller values of M, in Figs. 3, $g_{N|_{\mathbb{T}}}$ becomes closer to $f_{0,N|_{\mathbb{T}}}$.

The influence of ε and of the parameters a, b, d remain to be numerically studied, together with more sophisticated functions (or data) on $\mathbb{T}$.

7. Conclusion

7.1. Slepian Functions

Questions concerning the computation of the solutions to (BEP_N) are naturally connected to the existence of a basis of functions in $\mathcal{P}_N$ which concentrate their energy over I.Indeed, one can seek a basis of functions in $L^2(\partial\mathbb{G})$ mostly concentrated on I: these functions, when they exist, are called Slepian functions, following a series of articles by D. Slepian, among which we mention [30] for $\mathbb{G} = \mathbb{D}$.

In this context, already related to applications in signal processing or 2D inverse recovery problems,a criterion to compute Slepian functions (see [30,31]) consists in maximizing the ratio

$$\frac{\|g_{|_I}\|_{L^2(I)}}{\|g_{|_{\partial\mathbb{G}}}\|_{L^2(\partial\mathbb{G})}},$$

among the (finite-dimensional) space of "band-limited" functions (polynomials $g \in \mathcal{P}_N$, with fixed N, in this discrete situation). A family of such functions coincide with the eigenfunctions of the truncated Toeplitz operator T_N^I. One of their most interesting features here would be that they form an orthogonal basis of functions both on $\mathcal{P}_{N|_{\partial\mathbb{G}}}$ (in $L^2(\partial\mathbb{G})$) and on $\mathcal{P}_{N|_I}$ (in $L^2(I)$). Note that in the annular setting of $H^2(\mathbb{A})$ ($\mathbb{G} = \mathbb{A}$) and for the diagonal situation where $I = \mathbb{T} \subset \partial\mathbb{A}$ coincides with one of the two connected components of $\partial\mathbb{A}$, the Fourier basis provides an example of such Slepian functions in infinite dimension.

More generally, such functions have also been studied [23,29,31] when:

- $\mathbb{G}$ is the unit ball in $\mathbb{R}^3$ with I a polar cap contained in $\partial\mathbb{G} = \mathbb{S}$, and the Slepian functions are sought among spherical polynomials of prescribed degree N (spherical harmonic basis); their computation however is not so easy for large N and requires additional considerations, some of which are developed in [23,29].
- $\mathbb{G}$ is a half-plane and I an interval of $\partial\mathbb{G} = \mathbb{R}$, the minimization class being a set of functions $f \in L^2(\partial\mathbb{G})$ whose Fourier transforms are compactly supported, with a prescribed support (the "bandwidth"), while the Slepian functions are the so-called prolate spheroidal wave functions), see [31].

7.2. Model Spaces

Following Remark 4.2, the space $\mathcal{P}_N$ can be decomposed as follows for $\mathbb{G} = \mathbb{A}$ and $p = 2$:

$$\begin{aligned}\mathcal{P}_N &= \operatorname{span}\left\{z^k, -N \le k \le N\right\} \\ &= \operatorname{span}\left\{z^k, 0 \le k \le N\right\} \oplus \operatorname{span}\left\{z^k, -N \le k \le -1\right\},\end{aligned}$$

and so $\mathcal{P}_N$ can be seen as the orthogonal sum $K_\Theta \oplus K_{\widetilde{\Theta}}$ where K_Θ is the model space of $H^2(\mathbb{D})$ associated with the Blaschke product $\Theta(z) = z^{N+1}$ (see [24, Part B,Ch. 3]) and

$$K_{\widetilde{\Theta}} = H_0^2(\mathbb{C}\backslash s\overline{\mathbb{D}}) \cap \left(\widetilde{\Theta} H_0^2(\mathbb{C}\backslash s\overline{\mathbb{D}})\right)^{\perp} = H_0^2(\mathbb{C}\backslash s\overline{\mathbb{D}}) \cap \widetilde{\Theta} H^2(s\mathbb{D})$$

is the model space of $H_0^2(\mathbb{C}\backslash s\overline{\mathbb{D}})$ associated with $\widetilde{\Theta}(z) = z^{-N}$. The special (simple) form of Θ above leads to an orthogonal decomposition in $L^2(\partial\mathbb{A})$ (the second $\oplus$):

$$H^2(\mathbb{A}) = (K_\Theta \oplus K_{\widetilde{\Theta}}) \oplus (\Theta H^2(\mathbb{D}) \oplus \widetilde{\Theta} H_0^2(\mathbb{C}\backslash s\overline{\mathbb{D}}))$$

whereas this need not hold for more general inner functions Θ (even for Blaschke products). One may also consider other model spaces, determined by infinite Blaschke products or more general inner functions Θ, for $1 < p < \infty$ as well.

7.3. Other Related Issues

In the Hardy spaces $H^2(\mathbb{G})$ and the model spaces (polynomials) $\mathcal{P}_N$, reproducing kernels allow one to get integral representations of the projections P_{H^2} and P_N, whence of the solutions g and g_N to the bounded extremal problems, from the available boundary data. Such Carleman integral formulas are established in [7] for $\mathbb{G} = \mathbb{D}$ and $I \subset \partial\mathbb{G} = \mathbb{T}$. This is of interest by itself and for further numerics, whose analysis will be undertaken in a subsequent work.

On the same line, the reproducing kernel Hilbert space structure leads to characterizations of traces on $I \subset \partial\mathbb{G}$ of functions belonging to $H^2(\mathbb{G})$ with bounded norm, see Sect. 3.2.2 for related questions, and [3] for $\mathbb{G} = \mathbb{D}$. Similar issues could be handled in $\mathcal{P}_N$ or in other model spaces, with their reproducing kernels.

Let us mention that we did not consider here the cases $p = 1$ and $p = \infty$, where some of the above properties may still be true (the situation $p = \infty$ in particular is algorithmically tractable and involves Hankel operators, see [8]). One could further consider different constraints on $J = \partial\mathbb{G} \setminus I$, such as a $L^p(J)$-norm constraint involving the real part of the approximant only, as in [22], or constraints expressed with a different norm.

Another possible extension is to the Hardy classes of gradients of harmonic functions in balls or spherical shells of $\mathbb{R}^3$, see [5], with related Toeplitz operators and spherical polynomials.

We finally mention [18, 19], where bounded extremal problems in Hardy spaces of pseudo-analytic functions have been studied in $\mathbb{G} = \mathbb{D}$ and $\mathbb{A}$; this has applications in the analysis of inverse problems in tokamak fusion reactors (plasma boundary recovery) and its discretization should be further studied.

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