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Analytic extensions and Cauchy-type inverse problems on annular domains: stability results

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Abstract — We consider the Cauchy issue of recovering boundary values on the inner circle of a two-dimensional annulus from available overdetermined data on the outer circle, for solutions to the Laplace equation. Using tools from complex analysis and Hardy classes, we establish stability properties and error estimates.

1. INTRODUCTION

Let $\mathcal{G} \subset \mathbb{R}^2$ be a doubly-connected domain with smooth boundary $\partial\mathcal{G} = \Gamma_i \cup \Gamma_e$ made of two Jordan closed curves Γ_i, Γ_e such that $\Gamma_i \cap \Gamma_e = \emptyset$.

Consider the following inverse problem: given two functions u_b and Φ defined on Γ_e, or a number of their pointwise measurements, with $\Phi \not\equiv 0$, find a function φ, such that a solution u to

$$\begin{cases} \Delta u = 0 & \text{in } \mathcal{G} \\ u = u_b & \text{on } \Gamma_e \\ \partial_n u = \Phi & \text{on } \Gamma_e \end{cases} \tag{1}$$

also satisfies

$$\partial_n u + \varphi u = 0 \quad \text{on } \Gamma_i, \tag{2}$$

where ∂_n stands for the partial derivative w.r.t. the outer normal unit vector to $\partial\mathcal{G}$.

Let us now explain the physical motivation of such an inverse problem. Among data extension issues in elliptic inverse problems there arises the task

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of recovering either Dirichlet or Neumann boundary data (such as temperature field, or electric potential), or a Robin type exchange coefficient (which may represent thermal exchange, corrosion effects, etc.), or geometrical singularities (cracks, sources), from overdetermined measurements on part of the boundary of a domain.

In the thermal framework, u_b and Φ above correspond to the measured temperature and to the imposed heat flux on the outer boundary of some plane section of a tube, while φ is the Robin exchange coefficient to be recovered on the associated inner boundary.

This amounts to solving a Cauchy problem from available data on part of the boundary. This problem is known to be ill-posed since the work of Hadamard, its most critical feature being the lack of continuity of the solution — whenever it exists. (This is the case for compatible data, which means that the overdetermined data is indeed the trace and normal derivative of the solution of a single harmonic function.) Therefore great care is required when studying or solving such a problem.

Sufficient conditions on the available data together with a priori hypotheses on the missing data may however be provided for continuity and stability properties, and also error estimates, to hold. This is the topic of the present work where we mainly consider the issue of recovering a Robin coefficient — or the Dirichlet or Neumann boundary data — on the inner boundary of a two-dimensional annulus, or of a conformally equivalent doubly-connected domain, from overdetermined data on the outer boundary. Note that the case of cylindrical 3D objects, like tubes or pipes, reduces to this one.

Stability results and error estimates for the inverse Robin problem (with suitable norms) will be established as consequences of boundedness properties for functions of weighted Hardy classes — more specifically, by means of the recovery of functions in Hardy classes of certain domains (in this work, doubly-connected domains) from their restrictions to subsets of the boundary (see [7] and the survey article [10] for more on this approach). Our results can be viewed as an extension of those established in [2, 7], which hold on part of connected Lipschitz or Hölder smooth boundaries. One can also refer to [13, 16] for local or global estimates in the case of square domains.

The overview of the present article is as follows. The next section, Section 2, is devoted to notation and preliminary well-posedness results. Our main stability results and errors estimates for the inverse problem are then stated in Section 3. Using harmonic conjugation, we then introduce in Subsection 4.1 some analytic functions associated with the problem, which in fact belong to Hardy spaces of an annulus, defined in Subsection 4.2, for which norm estimations on the boundary are discussed in Subsection 4.3. This allows us to give a proof of our results in Section 5 and some conclusions are given in Section 6.

This paper is mainly concerned with stability results. In a later paper, we shall present further details of the Hardy class approximation scheme, which require modest amounts of operator theory; we shall also present some numerical results.

2. NOTATION AND PRELIMINARY RESULTS

Let $\mathcal{G} \subset \mathbb{R}^2 \simeq \mathbb{C}$ be a bounded domain, with Lebesgue measure ν. We let $L^2(\mathcal{G})$ be the Hilbert space of square-integrable functions on $\mathcal{G}$ (w.r.t. ν) while, for $m \in \mathbb{N}$ the Sobolev spaces $W^{m,2}(\mathcal{G})$ (of real or more generally complex valued functions) are defined as usual by [6, 12]:

$$W^{m,2}(\mathcal{G}) = \left\{ f \in L^2(\mathcal{G}),\ \|f\|^2_{W^{m,2}(\mathcal{G})} = \sum_{0 \leq |p| \leq m} \int_{\mathcal{G}} |D^p f(\xi)|^2 \, d\nu(\xi) < \infty \right\},$$

with, as usual,

$$|p| = p_1 + p_2, \qquad D^p f = \frac{\partial^{p_1+p_2}}{\partial x_1^{p_1} \partial x_2^{p_2}} \, f.$$

They become Hilbert spaces with the related inner product. Whenever $\partial\mathcal{G}$ is smooth enough ($C^{n+1,\beta}$, say, $0 < \beta < 1$), the following characterization of Sobolev spaces $W^{n+1/2,2}(\partial\mathcal{G})$ also holds, for $n \in \mathbb{N}$:

$$W^{n+1/2,2}(\partial\mathcal{G}) = \{ f \in L^2(\partial\mathcal{G}) \text{ s.t. } \exists F \in W^{n+1,2}(\mathcal{G}) \,:\, F_{|\partial\mathcal{G}} = f \},$$

with the norm $\|f\|_{W^{n+1/2,2}(\partial\mathcal{G})} = \inf \{ \|F\|_{W^{n+1,2}(\mathcal{G})},\ F_{|\partial\mathcal{G}} = f \}$. Further, there exist constants $k_{n,\mathcal{G}}$ and $K_{n,\mathcal{G}}$ such that for all $f \in W^{n+1,2}(\mathcal{G})$ we have:

$$k_{n,\mathcal{G}} \|f_{|\partial\mathcal{G}}\|_{W^{n,2}(\partial\mathcal{G})} \leq \|f_{|\partial\mathcal{G}}\|_{W^{n+1/2,2}(\partial\mathcal{G})} \leq K_{n,\mathcal{G}} \|f\|_{W^{n+1,2}(\mathcal{G})}.$$

Concerning the associated Neumann to Dirichlet direct problem of finding the solution u and its trace u_b on $\mathbb{T}$ when Φ and φ are given in (1), (2), we have the following existence and regularity Theorem 1 which, as well as the following results, requires a number of classical *prior assumptions*, see [2, 7, 9, 13, 16]. They are of three kinds.

Let $n \geq 0$. The first class of assumptions concerns the smoothness of the boundary of the domain $\mathcal{G}$; this physically means that the initial (non-corroded) domain should be smooth, which is not a so severe restriction.

$(H_{\mathcal{G},n})$: Assume that Γ_i, Γ_e are both of class $C^{n,\beta}$, $0 < \beta < 1$, $n \geq 1$.

The second set of hypotheses concern the imposed flux Φ on Γ_e, which should be smooth enough. In order to guarantee without too many technicalities that the solution does not vanish in $\mathcal{G}$, we simply assume Φ to be without change of sign. Note however that this could be expected to hold even if Φ had variable sign, provided that it did not oscillate too much, as in [2] for the simply-connected case. Although these are additional physical restrictions, they can be guaranteed to hold on Γ_e, where Φ is chosen.

$(H_{\Phi,n})$: $\Phi \in W^{n,2}(\Gamma_e)$, $\Phi \geq 0, \Phi \not\equiv 0$.

The last hypotheses concern the unknown Robin coefficient φ, which has to be smooth and bounded from below and above. The smoothness requirement is indeed a restrictive condition, technically needed for the inverse problem to make sense (see Theorem 2). Boundedness, however, corresponds to physical limitations on the exchange process on Γ_i, which in particular may not turn out to be perfectly insulating or conducting. Let $c_s, C_s > 0$ and introduce the following class of "admissible" Robin coefficients on Γ_i:

$$A^{(n)} = A^{(n)}(\Gamma_i, n, c_s, C_s) = \{\varphi \in C^n(\Gamma_i),\ |\varphi^{(k)}| \leq C_s,\ 0 \leq k \leq n, \text{ and } \varphi \geq c_s\}.$$

$(H_{\varphi,n})$: $\varphi \in A^{(n)}$, for some constants $c_s,\ C_s > 0$.

Theorem 1 ([8, 9]). *Let $n \geq 0$ and assume that $(H_{\mathcal{G},n})$, $(H_{\Phi,n})$, $(H_{\varphi,n})$ are satisfied by $\mathcal{G}$, Φ and φ, respectively. Then there exists a unique function $u \in W^{n+1,2}(\bar{\mathcal{G}})$, which is a solution to* (1), (2).

Further, there exist constants $m > 0$ and $\kappa > 0$ (depending on the class $A^{(n)}$) such that for all $\varphi \in A^{(n)}$ and $\Phi \in W^{n,2}(\Gamma_e)$,

$$u \geq m > 0 \quad \text{on } \Gamma_i, \tag{3}$$

and

$$\|u\|_{W^{n+1,2}(\partial\mathcal{G})} \leq \kappa. \tag{4}$$

The proofs of the above results, see [8, Lemma 2], [9, Theorem 2], rely on shift and Sobolev embedding theorems, together with the Hopf maximum principle [12, 18, 23].

The next identifiability property [8, Theorem 1] ensures the uniqueness of solutions φ to the inverse problem, which is a necessary prerequisite for the stability issue to make sense.

Theorem 2 ([8]). *Suppose that Φ satisfies $(H_{\Phi,0})$ and φ_1, φ_2 satisfy $(H_{\varphi,0})$. Let u_1 and u_2 be the associated solutions. If $u_{1|\mathcal{K}} = u_{2|\mathcal{K}}$ on some open subset $\mathcal{K} \neq \emptyset$ of Γ_e, then $\varphi_1 = \varphi_2$.*

3. STABILITY RESULTS AND ERROR ESTIMATES

Let $\mathbb{D}$ be the unit disc and G be the annulus $G = \mathbb{D} \setminus \overline{s\mathbb{D}}$ for some fixed s with $0 < s < 1$. We consider in this particular framework the inverse problem (1), (2) which is then stated as follows.

Given two functions u_b and Φ, with $\Phi \not\equiv 0$, find a function φ, such that a solution u to

$$\begin{cases} \Delta u = 0 & \text{in } G \\ u = u_b & \text{on } \mathbb{T} \\ \partial_n u = \Phi & \text{on } \mathbb{T} \end{cases} \tag{5}$$

also satisfies

$$\partial_n u + \varphi u = 0 \text{ on } s\mathbb{T}, \tag{6}$$

where ∂_n stands for the partial derivative w.r.t. the outer normal unit vector to $\mathbb{T}$.

We are now in a position to state our main results concerning the Cauchy problem (5), (6) in the annulus G.

Theorem 3. *Suppose that $n \geq 1$, that Φ_1 and Φ_2 satisfy $(H_{\Phi,n})$ on $\mathbb{T}$ and φ_1 and φ_2 satisfy $(H_{\varphi,n})$ on $s\mathbb{T}$. Let u_1, u_2 be the associated solutions to* (5), (6), *and assume that:*

$$\|u_1 - u_2\|_{L^2(\mathbb{T})} \leq \varepsilon, \qquad \|\Phi_1 - \Phi_2\|_{\mathbb{L}^2(\mathbb{T})} \leq \varepsilon, \tag{7}$$

for some $\varepsilon > 0$. Then, there exists a constant $K = K(s, A^{(n)}) > 0$ such that the following estimate holds:

$$\|\varphi_1 - \varphi_2\|_{\mathbb{L}^2(s\mathbb{T})} \leq K/|\log \varepsilon|^n. \tag{8}$$

This still holds in more general situations of a smooth doubly-connected domain $\mathcal{G} \subset \mathbb{R}^2$ as in (1), (2).

Corollary 4. *Let $n \geq 1$, assume that Γ_i, Γ_e satisfy $(H_{\mathcal{G},n})$, that Φ_1 and Φ_2 satisfy $(H_{\Phi,n})$ and that φ_1 and φ_2 satisfy $(H_{\varphi,n})$. Let u_1, u_2 be the associated solutions to* (1), (2), *and assume that:*

$$\|u_1 - u_2\|_{L^2(\Gamma_e)} \leq \varepsilon, \qquad \|\Phi_1 - \Phi_2\|_{\mathbb{L}^2(\Gamma_e)} \leq \varepsilon, \tag{9}$$

for some $\varepsilon > 0$. Then, there exists a constant $K = K(s, A^{(n)}) > 0$ such that the following estimate holds:

$$\|\varphi_1 - \varphi_2\|_{\mathbb{L}^2(\Gamma_i)} \leq K/|\log \varepsilon|^n. \tag{10}$$

In the uniform norm, we have the following:

Corollary 5. *Let $n \geq 2$, let Φ_1 and Φ_2 satisfy $(H_{\Phi,n})$ on $\mathbb{T}$ and φ_1 and φ_2 satisfy $(H_{\varphi,n})$ on $s\mathbb{T}$. Let u_1, u_2 be the associated solutions to* (5), (6), *and assume that:*

$$\|u_1 - u_2\|_{L^\infty(\mathbb{T})} \leq \varepsilon, \qquad \|\Phi_1 - \Phi_2\|_{L^\infty(\mathbb{T})} \leq \varepsilon, \tag{11}$$

for some $\varepsilon > 0$. Then, there exists a constant $K = K(s, A^{(n)}) > 0$ such that the following estimate holds:

$$\|\varphi_1 - \varphi_2\|_{\mathbb{L}^\infty(s\mathbb{T})} \leq K/|\log \varepsilon|^{n-1}. \tag{12}$$

Remark 6. Note that the (proofs of the) above results also contain estimates of the errors on inner boundary data $\|u_1 - u_2\|_{W^{1,p}(s\mathbb{T})}$ and $\|\partial_n u_1 - \partial_n u_2\|_{L^p(s\mathbb{T})}$ for $p = 2, \infty$, which can be extended to higher order Sobolev spaces.

4. HARDY SPACES OF ANNULUS

4.1. Harmonic conjugation

Let $G = \mathbb{D} \setminus \overline{s\mathbb{D}} \subset \mathbb{C} \simeq \mathbb{R}^2$, $0 < s < 1$, be an annulus, its boundary $\partial G = s\mathbb{T} \cup \mathbb{T}$ being equipped with the Lebesgue measure normalized so that the circles $\mathbb{T}$ and $s\mathbb{T}$ each have unit measure.

Let $\Phi \in L^2(\mathbb{T})$ and assume that $\varphi \in A^{(0)}$. From Theorem 1, $u_{|\partial G} \in W^{1,2}(\partial G)$. There exists a locally single-valued function v harmonic in G such that $\partial_\theta v = \partial_n u$ on ∂G, where ∂_θ stands for the tangential partial derivative on ∂G, from the Cauchy–Riemann equations.

Note that v is given on $\mathbb{T}$ up to a constant by

$$v_{|\mathbb{T}}(e^{i\theta}) = \int_{\theta_0}^{\theta} \Phi(e^{i\tau})\, d\tau,$$

for an arbitrary $e^{i\theta_0} \in \mathbb{T}$, a quantity which is available from (5). Thus, $f = u + iv$ is analytic (and many-valued) in G; it is given on $\mathbb{T}$ by

$$f(e^{i\theta}) = u_b(e^{i\theta}) + i \int_{\theta_0}^{\theta} \Phi(e^{i\tau})\, d\tau. \tag{13}$$

Also, on $s\mathbb{T}$,

$$\varphi = -\frac{\partial_\theta v}{u} = -\frac{\partial_\theta \operatorname{Im} f}{\operatorname{Re} f}, \tag{14}$$

which gives the link to be used between φ and f, in order to recover φ from approximants to f on the outer part $\mathbb{T}$ of the boundary ∂G or to establish stability results as continuity properties of the map $(u, \Phi) \to \varphi$ (see Section 3).

However, since the annulus is not simply-connected, it may not be possible to define f globally in G as a single-valued function. Indeed, one can see from Green's formula applied to the solution u of (5), (6) and to any constant function in G, that

$$\int_{\partial G} \partial_n u = \int_{\mathbb{T}} \Phi - \int_{s\mathbb{T}} \partial_\theta v = 0. \tag{15}$$

necessarily holds. Thus, if $\int_{\mathbb{T}} \Phi\, d\theta \neq 0$, then v, and hence also f, is multiply-valued in G, see [1].

But since u is locally in G the real part of the analytic function $f = u + iv$, we may lift the local definition of f to the simply-connected Riemann surface $R = \{\sigma \in \mathbb{C} : \log s < \operatorname{Re} \sigma < 0\}$, by means of the covering mapping $h : R \to G$, $h(\sigma) = e^{\sigma}$. That is, there is an analytic function $g : R \to \mathbb{C}$ such that locally $f = g \circ h^{-1}$.

Now $g(\sigma + 2\pi i) - g(\sigma)$ is an analytic function in R whose real part is zero, and it is therefore equal to a (purely imaginary) constant, ic say. Thus $g(\sigma) - (2\pi)^{-1} c\sigma$ is a $2\pi i$-periodic function of σ. We conclude that there is a single-valued analytic function F defined on G such that

$$F(z) = f(z) - (2\pi)^{-1} c \log z, \tag{16}$$

whence $u(z) = \operatorname{Re} F(z) + (2\pi)^{-1} c \log |z|$. Now (15) implies that

$$c = \int_0^{2\pi} \Phi(e^{i\theta})\, d\theta. \tag{17}$$

Indeed, if u has the standard representation of a harmonic function in an annulus, namely

$$u(re^{i\theta}) = \sum_{k \in \mathbb{Z},\ k \neq 0} r^k (a_k \cos k\theta + b_k \sin k\theta) + a_0 + b_0 \ln r,$$

then $c = 2\pi b_0$ and v is given by

$$v(re^{i\theta}) = \sum_{k \in \mathbb{Z},\ k \neq 0} r^k (a_k \sin k\theta - b_k \cos k\theta) + b_0 \theta,$$

and

$$F(z) = \sum_{k \in \mathbb{Z},\ k \neq 0} (a_k - i b_k) z^k + a_0.$$

On examining the Fourier coefficients of the functions involved, we see immediately that, if $u \in W^{m,2}(\partial G)$ for some m, then v, f and F also lie in $W^{m,2}(\partial G)$, and the Hilbert transform is a contraction with respect to each of these norms.

4.2. Weighted Hardy classes of circular domains

Let G be a *circular domain*, that is, a domain consisting of the open unit disc from which a finite number of pairwise disjoint closed discs have been removed:

$$G = \mathbb{D} \setminus \bigcup_{j=1}^{N} (a_j + r_j \overline{\mathbb{D}}), \tag{18}$$

with the obvious inequalities satisfied by the a_j and r_j for $j = 1, \dots, N$. We write $D_j = a_j + r_j \mathbb{D}$ for $1 \leq j \leq n$. Let Γ denote the boundary of G. We normalize the Lebesgue measure on Γ so that each circle Γ_j composing it is given unit measure.

The Hardy spaces $H^p(G)$ on a circular domain G were defined by Rudin [24] in terms of analytic functions f such that $|f(z)|^p$ has a *harmonic majorant* on G, that is, a real harmonic function $u(z)$ such that $|f(z)|^p \leq u(z)$ on G.

It is also possible to define the Hardy spaces $H^p(\partial G)$ for $1 \leq p < \infty$ as the closure in $L^p(\partial G)$ of the set R_G of rational functions whose poles lie in the complement of $\overline{G}$. This approach, similar to one in [3], was taken in [11]. The spaces $H^p(G)$ and $H^p(\partial G)$ are then isomorphic in a natural way, and so we identify the two spaces.

Below, we stick to the most completely analysed example of the annulus $G = \mathbb{D} \setminus s\mathbb{D}$ for some fixed s, $0 < s < 1$, and to the Hilbert case where $p = 2$. Here again, ∂G is equipped with the Lebesgue measure normalized so that the circles $\mathbb{T}$ and $s\mathbb{T}$ each have unit measure.

The space $H^2(\partial G)$ has a canonical orthonormal basis consisting of the functions

$$e_n(z) := (z^n/\sqrt{1+s^{2n}}\,)_{n\in\mathbb{Z}},$$

and it can be written as an orthogonal direct sum

$$H^2(\partial G) = H^2(\mathbb{D}) \oplus H_0^2(\mathbb{C}\setminus s\overline{\mathbb{D}}\,) \tag{19}$$

of elementary Hardy spaces, by taking the closed linear spans of $(e_n)_{n\geq 0}$ and $(e_n)_{n<0}$ respectively. Here $H_0^2(\mathbb{C}\setminus s\overline{\mathbb{D}})$ is the Hardy space of functions analytic on the complement of $s\overline{\mathbb{D}}$, with L^2 boundary values, and vanishing at infinity. It should be noted that a similar decomposition applies to general spaces $H^p(\partial G)$, but the direct sum is no longer orthogonal in the case $p\neq 2$, see [15, Theorem 10.12].

Given sequences $(w_n)_{n\in\mathbb{Z}}$ and $(\mu_n)_{n\in\mathbb{Z}}$ of positive numbers, we introduce $H^2_{w,\mu}(\partial G)$ to be the weighted Hardy space of the annulus G, with the norm

$$\|g\|^2_{H^2_{w,\mu}(\partial G)} = \sum_{n\in\mathbb{Z}} |g_n|^2[w_n + s^{2n}\mu_n],$$

for functions $g(z) = \sum_{n\in\mathbb{Z}} g_n z^n$, $z\in G$. Provided that

$$\inf_{n\in\mathbb{Z}} \frac{w_n + s^{2n}\mu_n}{1+s^{2n}} > 0, \tag{20}$$

and because the sequence of functions $(z^n/\sqrt{1+s^{2n}}\,)_{n\in\mathbb{Z}}$ is an orthonormal basis of $H^2(\partial G)$, the space $H^2_{w,\mu}(\partial G)$ embeds continuously in the unweighted space $H^2(\partial G)$, and thus its elements possess boundary values on $\mathbb{T}$ and $s\mathbb{T}$, as follows. Let $L^2_w(\mathbb{T})\subset L^2(\mathbb{T})$ and $L^2_\mu(s\mathbb{T})\subset L^2(s\mathbb{T})$ respectively be the spaces of functions $g = \sum_{n\in\mathbb{Z}} g_n z^n$ such that $\|g\|^2_{L^2_w(\mathbb{T})} = \sum_{n\in\mathbb{Z}} |g_n|^2 w_n < \infty$ and $\|g\|^2_{L^2_\mu(s\mathbb{T})} = \sum_{n\in\mathbb{Z}} |g_n|^2 s^{2n}\mu_n < \infty$, respectively. Functions belonging to $H^2_{w,\mu}(\partial G)$ thus admit traces on $\mathbb{T}$ and $s\mathbb{T}$ that belong to $L^2_w(\mathbb{T})$ and $L^2_\mu(s\mathbb{T})$, respectively.

More generally, we have a continuous embedding from $H^2_{w,\mu}(\partial G)$ into $H^2_{w',\mu'}(\partial G)$ if and only if

$$\inf_{n\in\mathbb{Z}} \frac{w_n + s^{2n}\mu_n}{w'_n + s^{2n}\mu'_n} > 0. \tag{21}$$

We write $L^2_{w,\mu}(\partial G)\subset L^2(\partial G)$ for the space of those functions defined on ∂G such that their restrictions to $\mathbb{T}$ and $s\mathbb{T}$ lie in $L^2_w(\mathbb{T})$ and $L^2_\mu(s\mathbb{T})$, respectively. Assumption (21) is also necessary and sufficient to ensure that $L^2_{w,\mu}(\partial G)$ is continuously embedded into $L^2_{w',\mu'}(\partial G)'$.

We write $P_{L^2_w(\mathbb{T})}g = \chi_{\mathbb{T}}g$ for the function in $L^2_{w,\mu}(\partial G)$ that coincides with g on $\mathbb{T}$ and vanishes on $s\mathbb{T}$. The definition of $P_{L^2_\mu(s\mathbb{T})}$ is analogous.

For $m\geq 1$, introduce $H^{m,2}(\partial G) = H^2(\partial G)\cap W^{m,2}(\partial G)$, the Hardy–Sobolev space of the annulus G, with the $W^{m,2}(\partial G)$ norm:

$$\|g\|^2_{H^{m,2}(\partial G)} = \|g\|^2_{W^{m,2}(\partial G)} = \sum_{n\in\mathbb{Z}} |g_n|^2[w_{m,n} + \mu_{m,n}s^{2n}],$$

for functions $g \in H^{m,2}(\partial G)$, $g(z) = \sum_{n\in\mathbb{Z}} g_n z^n$, $z \in G$, and

$$\begin{cases} w_{m,n} = 1 + n^2 + n^2(n-1)^2 + \cdots + n^2(n-1)^2 \cdots (n-m+1)^2, \\ \mu_{m,n} = 1 + n^2 s^{-2} + \cdots + n^2(n-1)^2 \cdots (n-m+1)^2 s^{-2m}. \end{cases} \tag{22}$$

For consistency of notation, we shall also write $H^{0,2}(\partial G)$ for $H^2(G)$ and $W^{0,2}(\partial G)$ for $L^2(\partial G)$.

4.3. Norm estimations in Hardy spaces

We now establish $L^2_{w',\mu'}$ estimates for functions in rather general weighted Hardy classes $H^2_{w,\mu}(\partial G)$. These results are strongly linked to those of [5, 7], which hold in Hardy–Sobolev spaces of the unit disc, the estimates concerning the norm on subsets of the connected boundary $\mathbb{T}$. In the present case, estimates are obtained in Hardy spaces of circular domains, on one of the two components of the boundary ∂G.

Let $B^{m,2}$ denote the unit ball of $H^{m,2}(\partial G)$, and $B^2_{w,\mu}$ the unit ball of $H^2_{w,\mu}(\partial G)$.

Let (w_n), (μ_n), and (w'_n), (μ'_n) be sequences of positive numbers: in order to guarantee boundary values of $H^2_{w',\mu'}(G)$ functions we shall suppose that (w'_n), (μ'_n) satisfy (20); recall that in this case, $H^2_{w',\mu'}(\partial G) \subset H^2(\partial G)$. We shall also assume that

$$\frac{s^{2n}\mu'_n}{w_n + s^{2n}\mu_n} \le \delta(n) \quad \text{for } n < 0, \\ \text{where } \delta(n) \text{ decreases to 0 as } n \to -\infty, \tag{23}$$

together with

$$\sup_{n\in\mathbb{Z}} \frac{\mu'_n}{w'_n} \le \varrho, \qquad \text{for some constant } \varrho > 0. \tag{24}$$

The following result uses the one-sided condition in (23) in order to deduce results about a function's behaviour on $s\mathbb{T}$ from its behaviour on $\mathbb{T}$. This may be contrasted with a later result, Theorem 10, where we use the two-sided conditions (20) and (21) in order to deduce the convergence to zero on ∂G of a sequence of functions from convergence (in another norm) on a subset of ∂G.

Theorem 7. *Assume that hypotheses* (20), (23) *and* (24) *are satisfied for sequences* (w_n), (μ_n) *and* (w'_n), (μ'_n) *of positive numbers. Let* $g \in H^2_{w,\mu}(\partial G)$ *be such that* $g \in B^2_{w,\mu}$ *and* $\|g\|_{L^2_{w'}(\mathbb{T})} \le \varepsilon$ *for some* $\varepsilon > 0$. *Then*

$$\|g\|_{L^2_{\mu'}(s\mathbb{T})} \le \left(\delta\Big(-1 - \Big\lfloor\Big|\frac{\log\varepsilon}{2\log s}\Big|\Big\rfloor\Big) + \varrho\varepsilon\right)^{1/2} \le \left(\delta\Big(-\Big|\frac{\log\varepsilon}{2\log s}\Big|\Big) + \varrho\varepsilon\right)^{1/2}.$$

In particular, if there are constants $c > 0$ *and* $\alpha > 0$ *such that* $\delta(n) \le c|n|^{-\alpha}$, *then there exists a constant* $C > 0$ *such that*

$$\|g\|_{L^2_{\mu'}(s\mathbb{T})} \le C/|\log\varepsilon|^{\alpha/2}.$$

Proof. We estimate the quantity

$$\|g\|^2_{L^2_{\mu'}(s\mathbb{T})} = \sum_{n\le -N} s^{2n}\mu'_n|g_n|^2 + \sum_{n=-N+1}^{\infty} s^{2n}\mu'_n|g_n|^2 = \sigma_1+\sigma_2, \quad \text{say.}$$

Because $g\in B^2_{w,\mu}$, we see that

$$\sum_{n\le -N} |g_n|^2(w_n+s^{2n}\mu_n)\le 1.$$

Hence $\sigma_1 \le \sup_{n\le -N}\delta(n)=\delta(-N)$. Moreover, $\sum_{n\in\mathbb{Z}} w'_n|g_n|^2\le\varepsilon^2$, and hence $\sigma_2\le s^{-2(N-1)}\varrho\varepsilon^2$. Choosing $N=1+\lfloor|\log\varepsilon/(2\log s)|\rfloor$, we have

$$\|g\|^2_{L^2_\mu(s\mathbb{T})}\le\delta(-N)+\varrho\varepsilon,$$

and the result follows. $\square$

Corollary 8. *Let m and k be integers with $m>k\ge 0$. Then there exists a constant C, depending only on m, k and s, such that whenever $g\in B^{m,2}$ with $\|g\|_{W^{k,2}(\mathbb{T})}\le\varepsilon$ for some $\varepsilon>0$, we have*

$$\|g\|_{W^{k,2}(s\mathbb{T})}\le C/|\log\varepsilon|^{m-k}.$$

Proof. This follows from Theorem 7, on taking the weights $w_n=w_{m,n}$, $\mu_n=\mu_{m,n}$ and $w'_n=w_{k,n}$, $\mu'_n=\mu_{k,n}$. We then have Condition (20) for both (w_n,μ_n) and (w'_n,μ'_n) because all the weights are greater than or equal to 1. Condition (23) holds with $\delta(n)$ of order $|n|^{-2(m-k)}$ for $n<0$, since μ_n grows as $|n|^{2m}$ and μ'_n grows as $|n|^{2k}$ as $n\to-\infty$. Finally, we also have (24) directly from the definition of the weights given in (22), with $\varrho=s^{-2k}$. $\square$

Remark 9. The estimate of Corollary 8 can be shown to be sharp, by considering for instance functions of the form $g(z)=\delta z^p$, where $\delta>0$ and p is a negative integer. Given $\varepsilon>0$, choose p as large in absolute value as possible such that

$$\|g\|_{W^{m,2}(\mathbb{T})}/\|g\|_{W^{k,2}(\mathbb{T})}\le 1/\varepsilon,$$

and then choose δ such that $\|g\|_{W^{k,2}(\mathbb{T})}=\varepsilon$, and hence $g\in B^{m,2}$. It is easy to see that p is asymptotic to $-\log\varepsilon/\log s$, as $\varepsilon\to 0$. But $\|g\|_{W^{k,2}(s\mathbb{T})}$ is now asymptotic to $c_1\delta|p|^k s^p$, which is asymptotic to $c_2|\log\varepsilon|^{k-m}$, where c_1 and c_2 are constants depending on m, k and s, but not p, δ or ε. The result follows.

Whenever the norm of the $H^2_{w,\mu}(\partial G)$ function is known to be "small" on only a subset I of ∂G, we can still conclude that its norm on the whole boundary ∂G remains small (in a certain sense to be made precise). The following result requires a stronger assumption than (21), but it is satisfied in the situation we are presenting.

Theorem 10. *Let (w'_n), (μ'_n) be weight sequences satisfying (20), and suppose additionally that (w_n), (μ_n) is another weight sequence such that*

$$\sup_{|n|\geq N} \frac{w'_n + s^{2n}\mu'_n}{w_n + s^{2n}\mu_n} \to 0$$

as $N \to \infty$. Let $I \subset \partial G$ be a subset with strictly positive measure, and suppose that (g_p) is a sequence of functions in $B^2_{w,\mu} \subset H^2_{w,\mu}(\partial G)$ such that $\|g_p\|_{L^2(I)} \to 0$. Then $\|g_p\|_{H^2_{w',\mu'}(\partial G)} \to 0$.

Proof. We claim that $B^2_{w,\mu}$ is a compact subset of $H^2_{w',\mu'}(\partial G)$. It is closed, since if (g_p) is a sequence in $B^2_{w,\mu}$ which converges to g in $H^2_{w',\mu'}(\partial G)$, then for every $N > 0$ the Fourier coefficients $g_{p,n}$ of g_p satisfy

$$\sum_{n=-N}^{N} |g_{p,n}|^2 (w_n + s^{2n}\mu_n) \leq 1,$$

and thus the same holds for g. Now taking the limit as $N \to \infty$ shows that $g \in B^2_{w,\mu}$, which is therefore closed. It is also a bounded subset and for every $g \in B^2_{w,\mu}$ we have that

$$\sum_{|n|\geq N} |g_n|^2 (w'_n + s^{2n}\mu'_n) \leq \sup_{|n|\geq N} \frac{w'_n + s^{2n}\mu'_n}{w_n + s^{2n}\mu_n} \to 0$$

as $N \to \infty$ (uniformly for $g \in B^2_{w,\mu}$). We deduce easily that $B^2_{w,\mu}$ is a totally bounded subset of $H^2_{w',\mu'}(\partial G)$, and hence compact (cf. [14, ch. IV, Ex. 13] for a compactness criterion in ℓ^2, which is easily adapted here).

Now, let (g_p) satisfy the assumptions of the proposition. Either $\|g_p\|_{H^2_{w',\mu'}(\partial G)} \to 0$, or, after extracting a subsequence and relabelling, we may suppose that (g_p) converges in $H^2_{w',\mu'}(\partial G)$ norm to some function g lying in $B^2_{w,\mu}$, a compact set; however, g necessarily vanishes on I. Now I is a uniqueness set in $H^2(\partial G)$ (this can be seen using the fact that either $I \cap \mathbb{T}$ or $I \cap s\mathbb{T}$ has positive measure, and g also lies in $H^2(E)$ for some suitable simply-connected domain $E \subset G$ with a smooth boundary). Hence g vanishes identically, and this is a contradiction. $\square$

Corollary 11. *Let m and k be integers with $m > k \geq 0$, and let $I \subset \partial G$ be a compact subset with nonempty interior. Let (g_p) be a sequence of functions in $B^{m,2} \subset H^{m,2}(\partial G)$ such that $\|g_p\|_{L^2(I)} \to 0$. Then $\|g_p\|_{H^{k,2}(\partial G)} \to 0$.*

Proof. It is easily verified that the weights $w_n = w_{m,n}$, $\mu_n = \mu_{m,n}$ and $w'_n = w_{k,n}$, $\mu'_n = \mu_{k,n}$ satisfy the conditions of Theorem 10. $\square$

5. PROOF OF THE MAIN RESULTS

Related to the direct problem of finding the solution u and its trace u_b on $\mathbb{T}$ when Φ and φ are given in (5), (6), we have the following result:

Lemma 12. *Let $\Phi \in L^2(\mathbb{T})$ and assume that $\varphi \in A^{(0)}$. There exists a function $f \in H^{1,2}(\partial G)$ such that the solution u to (5), (6) satisfies $u = \operatorname{Re} f$ in $\overline{G}$.*

Moreover, there exists a single-valued function $F \in H^{1,2}(\partial G)$ such that $f = F + c \log z$ in $\overline{G}$, for c defined by (17).

More generally, let $n \geq 0$, $\Phi \in W^{n,2}(\mathbb{T})$, $\varphi \in A^{(n)}$. Then $u = \operatorname{Re} f$, for some $f \in H^{n+1,2}(\partial G)$.

The proof of this assertion is contained in Section 4.1 for $n = 0$ and may be established as a corollary for $n \geq 1$.

We are now in a position to prove our main results.

Proof of Theorem 3. We define $u = u_1 - u_2$, $\Phi = \Phi_1 - \Phi_2$. Introduce now as in (13) the functions f_i analytic in G such that $u_i = \operatorname{Re} f_i$. Together with the regularity results of Theorem 1, Lemma 12 implies that $f_j \in H^{n+1,2}(\partial G)$.

The following Hardy–Littlewood–Pólya inequality holds for all $u \in W^{n+1,2}(\mathbb{T})$ [17]:

$$\|D^1 u\|_{L^2(\mathbb{T})}^{n+1} = \|u'\|_{L^2(\mathbb{T})}^{n+1} \leq \|u\|_{W^{n+1,2}(\mathbb{T})} \|u\|_{L^2(\mathbb{T})}^{n}. \tag{25}$$

Hence

$$\|u\|_{W^{1,2}(\mathbb{T})}^2 \leq (\kappa \|u\|_{L^2(\mathbb{T})}^n)^{2/(n+1)} + \|u\|_{L^2(\mathbb{T})}^2$$

from (4) and for some $\kappa \geq 1$. Next, if $f = f_1 - f_2$, then

$$\|f\|_{W^{1,2}(\mathbb{T})}^2 \leq \|u\|_{W^{1,2}(\mathbb{T})}^2 + \|\Phi\|_{L^2(\mathbb{T})}^2 \leq c\varepsilon^{2n/(n+1)}$$

for $c = (\kappa)^{2/(n+1)} + 2$, as soon as $\varepsilon < 1$.

Further, from (4) in Theorem 1 and Lemma 12, there exists $\kappa' \geq \kappa$ (depending on s and the class $A^{(n)}$) such that $f/\kappa' \in B^{n+1,2}$. Now, let $k = \max(\sqrt{c}, \kappa') \geq 1$. We have:

$$\|f/k\|_{W^{1,2}(\mathbb{T})} \leq (c\varepsilon^{2n/(n+1)})^{1/2}/k \leq \varepsilon^{n/(n+1)},$$

while $f/k \in B^{n+1,2}$. In view of Corollary 8, this leads to

$$\|f/k\|_{W^{1,2}(s\mathbb{T})} \leq \frac{C}{|\log \varepsilon^{n/(n+1)}|^n} \leq \frac{eC}{|\log \varepsilon|^n}, \tag{26}$$

for some $C > 0$ (depending on s), noting that $((n+1)/n)^n < e$.

Let now $\varphi = \varphi_1 - \varphi_2$. On $s\mathbb{T}$, we have that $\partial_n u_i + \varphi_i u_i = 0$, whence $\partial_n u + \varphi u_1 + \varphi_2 u = 0$ and

$$\varphi u_1 = -\partial_n u - \varphi_2 u \quad \text{on } s\mathbb{T}.$$

Now by hypothesis, and from (3) in Theorem 1, we have

$$\|\varphi\|_{\mathbb{L}^2(s\mathbb{T})} \leq (\sqrt{2}/m) \max(1, C_s) \|u\|_{W^{1,2}(s\mathbb{T})}. \tag{27}$$

Finally, since $\|u\|_{W^{1,2}(s\mathbb{T})} \le \|f\|_{W^{1,2}(s\mathbb{T})}$, we conclude from (26) and (27) that:

$$\|\varphi\|_{\mathbb{L}^2(s\mathbb{T})} \le \frac{e\sqrt{2}\, Ck \max(1, C_s)}{m|\log \varepsilon|^n}.$$

□

Proof of Corollary 4. It is classical that there exists a conformal transformation $C_{\mathcal{G}} : G \to \mathcal{G}$, for appropriate value of $0 < s < 1$, [19, Theorem 17.1a]. Further, as in the case of simply-connected domains D and of conformal mappings from the disc, the smoothness assumptions on $\partial\mathcal{G}$ allow to get an appropriate C^n extension of $C_{\mathcal{G}}$: $\partial G \to \partial\mathcal{G}$, [4, 22]. Problems (5), (6) and (1), (2) are then linked each to the other by $C_{\mathcal{G}}$, and the result follows, as in [7]. Note that the constant K in (10) depends on the choice of $C_{\mathcal{G}}$. □

Proof of Corollary 5. We employ the notation already used in the proof of Theorem 3, and follow the same steps with the appropriate modifications. Here again, the Hardy–Littlewood–Pólya inequality [17] is to the effect that, for $0 < k < N$, one has

$$\|u^{(k)}\|^N_{L^2(\mathbb{T})} \le \|u^{(N)}\|^k_{L^2(\mathbb{T})} \|u\|^{N-k}_{L^2(\mathbb{T})}$$

for all $u \in W^{N,2}(\mathbb{T})$. Applying these for $k = 1$ and $k = 2$ with $N = n + 1$, we arrive at inequalities of the form

$$\|u\|_{L^2(\mathbb{T})} \le a_0\varepsilon, \quad \|u'\|_{L^2(\mathbb{T})} \le a_1\varepsilon^{n/(n+1)}, \quad \|u''\|_{L^2(\mathbb{T})} \le a_2\varepsilon^{(n-1)/(n+1)}$$

for $u = u_1 - u_2$, with appropriate constants $a_0, a_1, a_2 > 0$, and similarly for $\Phi = \Phi_1 - \Phi_2$. The conclusion is that, provided that $0 < \varepsilon < 1$, we have

$$\|f\|_{W^{2,2}(\mathbb{T})} \le c\varepsilon^{(n-1)/(n+1)}$$

for some $c > 0$. We are now in position to apply the same arguments as in the proof of Theorem 3, which, using Theorem 1 and Lemma 12 to show that $u = \operatorname{Re} f$, where $f \in H^{n+1,2}(\partial G)$, and then Corollary 8 to pass from a bound on $\mathbb{T}$ to a bound on $s\mathbb{T}$, bring us to an inequality corresponding to (26), which now has the form

$$\|f/k\|_{W^{2,2}(s\mathbb{T})} \le \frac{C}{|\log \varepsilon^{(n-1)/(n+1)}|^{n-1}} \le \frac{e^2 C}{|\log \varepsilon|^{n-1}} \tag{28}$$

for some $C, k > 0$. We have by hypothesis and from (3) that $\varphi = \varphi_1 - \varphi_2$ satisfies

$$\|\varphi\|_{\mathbb{L}^\infty(s\mathbb{T})} \le (1/m) \max(1, C_s) \|u\|_{W^{1,\infty}(s\mathbb{T})}. \tag{29}$$

and, further,

$$\|u\|_{W^{1,\infty}(s\mathbb{T})} \le \|u\|_{W^{2,2}(s\mathbb{T})} \le \|f\|_{W^{2,2}(s\mathbb{T})}.$$

This finally leads to:

$$\|\varphi\|_{\mathbb{L}^\infty(s\mathbb{T})} \le \frac{e^2 C\, k \max(1, C_s)}{m|\log \varepsilon|^{n-1}}.$$

□

6. CONCLUSION

We have mainly derived error estimates for the Robin coefficient on the inner boundary of a doubly connected domain, in terms of the error on both Dirichlet and Neumann data on the whole outer boundary.

Notice that an additional stability property follows from Corollary 11 which is to the effect that if the error in Dirichlet and Neumann data is small on a part of the outer boundary, than so will be the error in the Robin coefficient on the inner one. However, no estimate is available so far.

Observe further that the results of Corollary 4 are expected to hold for doubly-connected domains $\mathcal{G}$ with only piecewise $C^{n,\beta}$ boundary, since the derivative of the conformal maps still has a suitable behaviour up to the boundary in this case, [4]. Note that conformal mappings can also be used for solving geometrical 2D inverse problems and to express them as data recovery ones, see e.g. [21].

Computational issues, robustness and convergence properties of the algorithm will be more deeply studied in a companion paper [20] (see [9] for the simply connected situation of the disk).

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