A constrained approximation problem arising in parameter identification

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Abstract

We pose and solve an extremal problem in the Hardy class H^2 of the disc, involving the best approximation of a function on a subarc of the circle by an H^2 function, subject to a constraint on its imaginary part on the complementary arc. A constructive algorithm is presented for the computation of such a best approximant, and the method is illustrated by a numerical example. The whole problem is motivated by boundary parameter identification problems arising in non-destructive control.

Keywords: Constrained approximation, extremal problem, parameter identification, inverse problem, Laplace's equation.

AMS Subject Classifications: 41A29, 46E20, 35R30, 30E10.

1 Introduction

The approximation of complex-valued functions by holomorphic functions, subject to appropriate constraints, is a technique that has many applications in Systems Theory, Signal Processing and the theory of Inverse Problems (geometrical issues or parameter identification) for Partial Differential Equations. We refer the reader to [10] for a recent survey of this general area, whereas in this article we focus on two closely-related approximation problems of this type together with their applications.

The first such problem concerns the identification of linear systems from measurements of their transfer functions. This general area is treated in the books

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[11, 16], as well as the survey [15], but we are concerned here with the case when the data is *band-limited*. More precisely, the input u and output y of our linear system are related by an unknown transfer function G which is holomorphic and bounded on the right half-plane and such that

$$Y(s) = G(s)\,U(s)\,,$$

where Y and U are the Laplace transforms of y and u respectively. By means of a conformal transformation, we may work on the unit disc $\mathbb{D}$ rather than the half-plane, and suppose that by physical measurements it is possible to obtain corrupted values of G on some subset I of the unit circle. This can be achieved by harmonic identification, the set I corresponding to the bandwidth of the system. We are thus provided with a function f defined on I which is close to G in an appropriate norm. The strategy behind band-limited identification [1, 5, 6, 13] is to approximate f on I by a holomorphic function g defined on $\mathbb{D}$ in order to obtain a model for G. The problem as stated is not, in general, well-posed, and it is necessary to impose some constraint on the norm of g on the subset J of the circle complementary to I in order to resolve it in a satisfactory manner and to get a stable model.

Similar considerations apply in the analysis of inverse problems arising in many applications of non-destructive control such as tomography and medical imaging, fault detection and localisation [4, 9]. Here we seek to identify an unknown holomorphic function G, of which the imaginary part should be a prescribed harmonic function ϕ representing, for example, the current (or heat) flux at the boundary of a simply-connected domain with Hölder-smooth boundary (again, a conformal transformation allows us to transfer the problem to $\mathbb{D}$). Hence, both $\operatorname{Re} G$ and $\operatorname{Im} G$ satisfy Laplace's equation together with boundary conditions that are fully-determined on a given subset I but not on its complement J. Thus, identifying G (and hence ϕ) requires us to approximate the given function f defined on I by a holomorphic function g, subject to a constraint that acts on $\operatorname{Im} g$ on J.

It is on this second extremal problem that we focus in this paper, and we make this mathematically precise by working in the Hardy space H^2 of the disc. It turns out that, in this setting, the two approximation problems outlined above, namely the first with a constraint on the norm and the second which involves a constraint on the imaginary part, are closely related, and a knowledge of the solution to the first gives information towards the solution of the second in a most transparent way.

We state the second problem in a well-posed form and show that it has a unique solution, which we characterize in terms of linear operators. By well-established approximation procedures, this solution is made constructive, and expressed purely in terms of Linear Algebra. Finally, we include a numerical example to illustrate the techniques involved.

1.1 Notation

We take I to be a compact subset of the unit circle $\mathbb{T} \subset \mathbb{C}$ such that I and $J = \mathbb{T} \setminus I$ have positive Lebesgue measure.
By H^2 we denote the Hardy space on the unit disc $\mathbb{D}$, and by $\overline{H}^2$ the set of functions $\overline{f}$ for $f \in H^2$, where $\overline{f}(z) = \overline{f(z)}$. The orthogonal complement of H^2 in $L^2(\mathbb{T})$ is written as $\overline{H}_0^2$; it coincides with the subset of $\overline{H}^2$ consisting of functions that vanish at infinity.
We write P for the orthogonal projection $P : L^2(\mathbb{T}) \to H^2$ and χ_I for the characteristic function of I.
The usual inner product in $L^2(\mathbb{T})$ is denoted by $\langle f, g \rangle$ for $f,\ g \in L^2(\mathbb{T})$, and we shall write $\langle f, g \rangle_I$ to mean $\langle f\chi_I, g\chi_I \rangle$. The associated norms on $L^2(\mathbb{T})$ and $L^2(I)$ are written as $\|\ \|$ and $\|\ \|_I$, respectively, while $\|\ \|_J^2 = \|\ \|^2 - \|\ \|_I^2$ on $L^2(J)$.
Whenever f_1 is defined on I and f_2 on J, we write $f_1 \vee f_2$ for the function equal to f_1 on I and f_2 on J.

1.2 Statement of the problem

The following problem, which has applications in frequency domain system identification, was solved in [1].

Problem 1 *Given $f \in L^2(I)$, $\psi \in L^2(J)$ and $M > 0$, find $g_1 \in H^2$ such that $\|g_1 - \psi\|_J \leq M$ and*

$$\|g_1 - f\|_I = \min\{\|g - f\|_I \mid g \in H^2 \text{ satisfying } \|g - \psi\|_J \leq M\}.$$

Here we address a more difficult version of the problem, which has further applications.

Problem 2 *Given $f \in L^2(I)$, $\phi \in L^2(J)$ and $M \geq 0$, find $g_2 \in H^2$ such that $\|\operatorname{Im} g_2 - \phi\|_J \leq M$ and*

$$\|g_2 - f\|_I = \min\{\|g - f\|_I \mid g \in H^2 \text{ satisfying } \|\operatorname{Im} g - \phi\|_J \leq M\}. \qquad (1)$$

As outlined above, boundary parameter identification problems for the 2D Laplace equation may be efficiently formulated as Problem 2 rather than Problem 1, since the constraint can be used to impose the condition that the imaginary part of g should be close to some known function ϕ on J which may be available through boundary measurements, see [4, 9].

Note that in Problem 2 we can retain the case $M = 0$, which is not of interest in Problem 1 because J is a set of uniqueness for H^2.

2 Existence and uniqueness

To relate the real and imaginary parts of an analytic function, we shall make use of the *Hilbert transform*, of which the basic properties can be found in [2, pp. 104–106]. Let u be a real function in $L^2(\mathbb{T})$; then the Hilbert transform of u is the unique real function v in $L^2(\mathbb{T})$ with vanishing Fourier coefficient $\hat{v}(0) = 0$ such that $u + iv$ has a holomorphic extension to the disc. It is known that the Hilbert transform is a contraction in the L^2 norm.
The following lemma will be of use to us several times.

Lemma 3 *Let I and J be fixed subsets of $\mathbb{T}$, as above. Then there is an absolute constant $C > 0$ such that, for all $h \in H^2$,*

$$\|h\| \leq C \max\{\|h\|_I, \|\operatorname{Im} h\|_J\}.$$

Proof. Without loss of generality, $\max\{\|h\|_I, \|\operatorname{Im} h\|_J\} = 1$. Hence $\|\operatorname{Im} h\| \leq \sqrt{2}$. Since $\operatorname{Im} h - i \operatorname{Re} h$ is holomorphic in $\mathbb{D}$, we see that $\operatorname{Re} h$ is the Hilbert transform of $-\operatorname{Im} h$ plus an unknown real additive constant α, say. As the Hilbert transform is a contraction on H^2, we have $\|\operatorname{Re} h - \alpha\| \leq \sqrt{2}$. But then $\|h - \alpha\|_I \leq 2\sqrt{2}$ and $\|h\|_I \leq 1$, so $\|\alpha\|_I \leq 1 + 2\sqrt{2}$. This gives a bound on α, and thus on $\|\operatorname{Re} h\|$, independently of h, as required. ∎

Proposition 4 *Problem 2 admits a unique solution $g_2 \in H^2$ if and only if $\|\operatorname{Im} \phi\|_J \leq M$. Moreover, g_2 saturates the constraint:*

$$\|\operatorname{Im} g_2 - \phi\|_J = M,$$

unless f is already the trace on I of an H^2 function such that $\|\operatorname{Im} f - \phi\|_J < M$.

Proof. From

$$\|\operatorname{Im} g - \phi\|_J \leq M \iff \|\operatorname{Im} g - \operatorname{Re} \phi\|_J \leq \sqrt{M^2 - \|\operatorname{Im} \phi\|_J^2}, \tag{2}$$

it follows that, if $\|\operatorname{Im} \phi\|_J > M$, then Problem 2 has no solution.
Assume now that $\|\operatorname{Im} \phi\|_J \leq M$. In view of the above inequality, this is equivalent to a version of Problem 2 with the hypothesis that ϕ is real-valued. We need first to check that Problem 2 is well-posed. Let F be any function in H^2 such that $\operatorname{Im} F = 0 \vee \phi$ on $\mathbb{T}$. Clearly the set of functions meeting the constraint is non-null, and we may in addition restrict to the subset S of those functions which approximate at least as well as F, i.e.,

$$S = \{g \in H^2 : \|g - f\|_I \leq \|F - f\|_I, \quad \|\operatorname{Im} g - \phi\|_J \leq M\}. \tag{3}$$

The set S is clearly closed and convex. It is also bounded in H^2, since if $g \in S$, we can bound $\|g\|_I$ and $\|\operatorname{Im} g\|_J$, and then use Lemma 3.

The existence of an optimum is now ensured by the best approximation projection from the (uniformly convex) Hilbert space $L^2(I)$ onto the closed convex set $S_{|_I}$, see [7, 3.II.1], or alternatively by a weak compactness argument. Likewise, the uniqueness follows from the strict convexity of the norm. Thus, for $\|\operatorname{Im}\phi\|_J \leq M$, Problem 2 has a unique solution, which we denote by g_2.
We also see that g_2 saturates the constraint, i.e., $\|\operatorname{Im} g_2 - \phi\|_J = M$, unless f is the trace of g_2. For otherwise, we would see that $g_2 + h$ also satisfied the constraint for sufficiently small $h \in H^2$; then we could choose h such that

$$\|(g_2 + h) - f\|_I < \|g_2 - f\|_I,$$

which would contradict the optimality of g_2. ■

Observe finally that, if $\|\operatorname{Im}\phi\|_J = M$, then $\|\operatorname{Im} g - \phi\|_J \leq M$ precisely when $\operatorname{Im} g = \operatorname{Re}\phi$ on J. There are many such functions, given by the set

$$\mathcal{G} = \{g \in H^2 \ , \ \operatorname{Im} g = \chi_I r_I + \chi_J \operatorname{Re}\phi = r_I \vee \operatorname{Re}\phi \ , \ r_I \in L^2(I) \text{ real-valued}\}$$

$$= \{g(z) = \frac{i}{2\pi}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}(r_I \vee \operatorname{Re}\phi)(e^{i\theta})d\theta \ , \ r_I \in L^2(I) \text{ real-valued}\}\,,$$

up to some real-valued additive constant.

Thus, we assume from now on that $\|\operatorname{Im}\phi\|_J \leq M$ which, in view of equation (2), reduces to the hypothesis that ϕ *is real-valued.*

3 Construction of the solution

In order to find a formula for the solution of Problem 2, we first characterize it in terms of a solution to Problem 1 at least when $M > 0$. Up to now, only suboptimal solutions to Problem 2 were available by solving versions of Problem 1 associated with exponential transformations of the data (see [4]).

Proposition 5 *If $M > 0$, then the unique solution g_2 to Problem 2 coincides with that of Problem 1 associated with both:*
(i) f, $\psi = i\,\phi + \chi_J \operatorname{Re} g_2$, and M, and:
(ii) f, $\psi = 2\,i\,\phi + \chi_J\,\overline{g_2}$, and $2\,M$.

Proof. We establish point (i). The unique solution $g_2 \in H^2$ to Problem 2 satisfies $\|\operatorname{Im} g_2 - \phi\|_J = M$ and (1). Using

$$i(\operatorname{Im} g_2 - \phi) = g_2 - \operatorname{Re} g_2 - i\phi$$

we get that $\|\operatorname{Im} g_2 - \phi\|_J \leq M$ is equivalent to $\|g_2 - \psi\|_J \leq M$. In [3] it is shown that Problem 1 associated to f, ψ, and M has a unique solution, which we denote by g_1. Because g_2 belongs to the set of solutions to Problem 1, it holds that:

$$\|g_1 - f\|_I \le \|g_2 - f\|_I\,. \tag{4}$$

Moreover,

$$M^2 \ge \|g_1 - \psi\|_J^2 = \|\operatorname{Re} g_1 - \operatorname{Re} g_2\|_J^2 + \|\operatorname{Im} g_1 - \phi\|_J^2,$$

whence

$$\|\operatorname{Im} g_1 - \phi\|_J \le M\,,$$

which implies that g_1 belongs to the set of solutions to Problem 2. In view of (4), since the solution g_2 to Problem 2 is unique, it necessarily holds that $g_1 = g_2$. This establishes (i). The same arguments lead to point (ii), after writing $2\,i(\operatorname{Im} g_2 - \phi) = g_2 - \overline{g_2} - 2\,i\,\phi$. ■

Making use of some results from [3] about Problem 1, Proposition 5 also provides alternative proofs for some properties of the solution to Problem 2, like the saturation of the constraint. These results allow to obtain a formula for g_2. We shall see in the next section that this gives a constructive scheme for solving Problem 2 via Problem 1, for which a resolution algorithm is already available and implemented.

Let $T_\lambda : H^2 \to H^2$ denote the real-linear operator given by

$$T_\lambda g = P(g\chi_I) + i\lambda P(\operatorname{Im} g\chi_J). \tag{5}$$

Theorem 6 *The operator $T_\lambda : H^2 \to H^2$ is invertible for every $\lambda > 0$, and hence the equation $T_\lambda g = h$ has a unique solution for every $h \in H^2$.*

Proof. For any $g \in H^2$, we have

$$\begin{aligned} \operatorname{Re}\langle T_\lambda g, g\rangle &= \operatorname{Re}\langle g\chi_I, g\rangle + \operatorname{Re} i\lambda\langle \operatorname{Im} g\chi_J, g\rangle \\ &= \|g\|_I^2 + \lambda\|\operatorname{Im} g\|_J^2. \end{aligned}$$

By virtue of Lemma 3, there is a constant $\alpha > 0$, depending on λ but not on g, such that

$$\operatorname{Re}\langle T_\lambda g, g\rangle \ge \alpha\|g\|^2 \qquad \text{for all } g \in H^2.$$

That is, T_λ is a *monotone* operator [8]. The rest follows from standard properties of monotone operators. Since $\operatorname{Re}\langle T_\lambda g, g\rangle \ge \alpha\|g\|^2$, we conclude that $\|T_\lambda g\| \ge \alpha\|g\|$, so T_λ has closed range and zero kernel; the range is also dense, because there is no nonzero function g orthogonal to it, since $\langle T_\lambda g, g\rangle \ne 0$. Hence T_λ is invertible. ■

Let $\mathcal{T} : H^2 \to H^2$ be the Toeplitz operator defined by

$$\mathcal{T} g = P \chi_J g .$$

Then the characterization result [3, thm 4] ensures that, if f is not already the trace on I of an H^2 function g with $\|g - \psi\|_J \leq M$, the unique solution g_1 to Problem 1 is given by the implicit equation

$$g_1 = g_1(\ell) = (1 + \ell \mathcal{T})^{-1} P (f \vee (\ell + 1) \psi) , \tag{6}$$

for the unique $\ell > -1$ such that $\|g_1 - \psi\|_J = M$. See also [14] for some extensions of this result. Note that, since $\mathcal{T}$ has spectrum $[0, 1]$, the operator $1 + \ell \mathcal{T}$ is indeed invertible for $\ell > -1$. Moreover, if f is the trace on I of an H^2 function g with $\|g - \psi\|_J \leq M$, then of course $g_1 = g = g_1(-1)$.

Theorem 7 *Let $M > 0$ and suppose that $f \in L^2(I)$ is not already the trace of an H^2 function g with $\| \operatorname{Im} g - \phi\|_J \leq M$. Then the unique solution g_2 to Problem 2 is given by*

$$T_\lambda \, g_2 = P (f \vee i \, \lambda \, \phi) , \tag{7}$$

for the unique $\lambda > 0$ such that $\| \operatorname{Im} g_2 - \phi\|_J = M$. Moreover, if f is the trace on I of an H^2 function g with $\| \operatorname{Im} g - \phi\|_J \leq M$, then $g_2 = g$ (and $T_0 \, g_2 = P f$). The correspondence between λ and M determines a mapping which is a continuous bijection onto its range.

Proof. It follows from Proposition 5 and (6) that the solution g_2 to Problem 2 satisfies the implicit equation:

$$(1 + \ell \mathcal{T}) \, g_2 = P (\chi_I \, f) + (\ell + 1) \, P(\chi_J \, [\operatorname{Re} g_2 + i \, \phi]) . \tag{8}$$

for some $\ell > -1$ such that $\| \operatorname{Im} g_2 - \phi\|_J = M$. Hence, recalling the definition of $\mathcal{T}$,

$$\begin{aligned} P (f \vee i \, (\ell + 1) \, \phi) &= (1 + \ell \mathcal{T}) \, g_2 - (\ell + 1) \, P(\chi_J \operatorname{Re} g_2) \\ &= P (\chi_I \, g_2) + i \, (\ell + 1) \, P(\chi_J \operatorname{Im} g_2) , \end{aligned}$$

whence, by definition of T_λ,

$$P (f \vee i \, (\ell + 1) \, \phi) = T_{\ell+1} \, g_2 .$$

Putting $\lambda = \ell + 1$, this establishes the formula (7).
Next, the mapping from λ to M is injective, since g_2 is unique, by Proposition 4, and if we have both

$$T_\lambda \, g_2 = P (f \vee i \, \lambda \, \phi) \text{ and } T_\mu \, g_2 = P (f \vee i \, \mu \, \phi) ,$$

for some $\mu \neq \lambda$, then by subtraction we have

$$i(\lambda - \mu) P(\operatorname{Im} g_2 \chi_J) = i(\lambda - \mu) P(\phi \chi_J) ,$$

from which $(\operatorname{Im} g_2 - \phi)\chi_J$ is a function in $\overline{H}_0^2$ vanishing on I, and hence identically zero. This is a contradiction, whence $\mu = \lambda$.
Finally, the mapping from λ to M is clearly continuous on $(0, \infty)$, and monotonic decreasing, and hence so is its inverse. ■

The case $M = 0$ of Problem 2 is somewhat exceptional, and not used in practical examples, since measured data will always be corrupted by noise. However, as the following result shows, it can be handled by a limiting argument.

Proposition 8 *Let $g_2^0 = g_2(f, \phi, 0) \in H^2$ denote the solution to Problem 2 with ϕ real and $M = 0$, and for $n = 1, 2, \ldots$, let $g_2^{(n)} = g_2(f, \phi, 1/n) \in H^2$ denote the solution for $M = 1/n$. Then $\|g_2^{(n)} - g_2^0\| \to 0$ as $n \to \infty$.*

Proof. Clearly $\|g_2^{(n)} - f\|_I \leq \|g_2^0 - f\|_I$, since g_2^0 satisfies the constraints of Problem 2 for $M = 1/n$. Moreover, $\|\operatorname{Im} g_2^{(n)} - \phi\|_J \leq 1/n$. This establishes that the sequence $(g_2^{(n)})$ is bounded in H^2, by Lemma 3. It therefore has a subsequence tending to a weak limit point g', which can only be g_2^0, because $\|g' - f\|_I \leq \|g_2^0 - f\|_I$ and $\operatorname{Im} g' = \operatorname{Im} g_2^0 = \phi$ on J. Since the weak limit point is unique, and the space H^2 is separable (and hence bounded sets are metrizable in its weak topology [8, III.25]), the whole sequence $(g_2^{(n)})$ converges weakly to g_2^0.
The next observation is that $g_2^{(n)} - f \to g_2^0 - f$ weakly in $L^2(I)$ and $\|g_2^{(n)} - f\|_I \leq \|g_2^0 - f\|_I$, which implies norm convergence on I (see, for example, [16, pp. 152–153]).
We thus have $\|g_2^{(n)} - g_0\|_I \to 0$ and, because $\operatorname{Im} g_2^0 = \phi$ on J, $\|\operatorname{Im} g_2^{(n)} - \operatorname{Im} g_2^0\|_J \to 0$. Finally, by Lemma 3, we can conclude that $\|g_2^{(n)} - g_2^0\| \to 0$, as asserted. ■

4 Constructive resolution schemes

A fairly standard algorithm for inverting an arbitrary monotone operator T can be given as follows:

Proposition 9 *Suppose that $T : H \to H$ is a bounded real-linear map on a real inner-product space such that for some $\alpha > 0$ we have $\langle Tg, g \rangle \geq \alpha \|g\|^2$ for all $g \in H$. Let $\gamma = \alpha / \|T\|^2$. Then the mapping $S : H \to H$ defined by $Sg = g + \gamma(h - Tg)$ is a strict contraction mapping, and hence the solution to $Tg = h$ can be found by iterating S.*

Proof. For g, $g' \in H$, we have

$$Sg - Sg' = g - g' - \gamma(Tg - Tg'),$$

and so

$$\begin{aligned}\|Sg - Sg'\|^2 &= \|g-g'\|^2 - 2\gamma\langle g-g', Tg - Tg'\rangle + \gamma^2\|T(g-g')\|^2 \\ &\leq \|g-g'\|^2(1-2\alpha\gamma+\gamma^2\|T\|^2) = K^2\|g-g'\|^2\end{aligned}$$

where $K^2 = 1-\alpha^2/\|T\|^2$. ■

We can apply the above proposition to $T = T_\lambda$, $H = H^2$, where we use the real inner product $(g, g') \mapsto \mathrm{Re}\langle g, g'\rangle$. Using Theorem 6 and Equation (7), this provides an algorithm for solving Problem 2.

For the practical implementation of the procedure, it is necessary to be able to calculate the effect of the real-linear operator T_λ defined by (5). Observe first that:

$$T_\lambda g = P(g\chi_I) + \frac{\lambda}{2}P(\chi_J[g-\bar{g}])$$

$$= g + \left(\frac{\lambda}{2} - 1\right) P(g\chi_J) - \frac{\lambda}{2}P(\chi_J\bar{g})\,.$$

Introduce the Hankel operator $\mathcal{H} : \bar{H}^2 \to H^2$ defined by

$$\mathcal{H}\, g = P\,\chi_J\, g\,.$$

If we put $\rho = \frac{\lambda}{2} - 1 > -1$, we get

$$T_\lambda g = (I + \rho\,\mathcal{T})g - (\rho+1)\,\mathcal{H}\,\bar{g}\,,$$

which expresses T_λ in terms of Toeplitz and Hankel operators. Hence, equation (7) can be rephrased as

$$(I+\rho\,\mathcal{T})\,g_2 - (\rho+1)\,\mathcal{H}\,\bar{g_2} = P\,(f \vee 2\,i\,(\rho+1)\,\phi)\,, \tag{9}$$

for some $\rho > -1$, compare with (6).

Whenever $I = \{e^{i\theta} : -a \leq \theta \leq a\,,\ 0 < a < \pi\}$, direct computations on the Fourier basis of H^2 show that:

$$\mathcal{T} = (\mathcal{T}_{k,p})_{k,p\geq 0}\ ,\ \mathcal{T}_{k,p} = -\frac{\sin(p-k)\,a}{\pi\,(p-k)} \text{ if } p \neq k\,,\ \mathcal{T}_{k,k} = 1 - \frac{a}{\pi}\,,$$

and

$$\mathcal{H} = (\mathcal{H}_{k,p})_{k,p\geq 0}\ ,\ \mathcal{H}_{k,p} = \frac{\sin(p+k)\,a}{\pi\,(p+k)} \text{ for } p, k > 0\,,\ \mathcal{H}_{0,0} = 1 - \frac{a}{\pi}\,.$$

Theorems 6 and 7 together with (9) can be re-expressed as the following result. For $\rho > -1$, let $\Gamma_\rho : H^2 \to H^2$ denote the linear operator given by

$$\Gamma_\rho\, g = (I + \rho\,\mathcal{T})\,g - (\rho+1)\,\mathcal{H}\,\bar{g}\,,$$

for $g \in H^2$.

Corollary 10 *Let $M > 0$ and suppose that f is not already the trace of an H^2 function g with $\| \operatorname{Im} g - \phi \|_J \le M$. Then the unique solution g_2 to Problem 2 is given by*

$$g_2 = \Gamma_\rho^{-1} \, P \left(f \vee 2 \, i \, (\rho + 1) \, \phi \right) , \tag{10}$$

for the unique $\rho > -1$ such that $\| \operatorname{Im} g_2 - \phi \|_J = M$. The correspondence between ρ and M determines a mapping which is continuous, monotonic decreasing, and bijective onto its range.

5 Algorithm and numerical experiments

One of the difficulties in formula (10) comes from the fact that $\mathcal{H}$ acts on the sequence $(\bar{g}_p)_{p \ge 0}$ of *complex conjugates* of the Fourier coefficients of the solution. This can be handled by splitting the real-linear complex operator Γ_ρ into two real (linear) operators Γ_R and Γ_I:

$$\Gamma_\rho \, g = \Gamma_R \, \operatorname{Re} g + i \, \Gamma_I \, \operatorname{Im} g \, ,$$

where those are defined (as operators on real-valued harmonic $L^2(\mathbb{T})$ functions) by:

$$\Gamma_R = I + \rho \, \mathcal{T} - (\rho + 1) \, \mathcal{H} \ , \ \Gamma_I = I + \rho \, \mathcal{T} + (\rho + 1) \, \mathcal{H} \, . \tag{11}$$

Putting

$$F_\rho = P \left(f \vee 2 \, i \, (\rho + 1) \, \phi \right) ,$$

formula (10) becomes

$$g_2 = \Gamma_R^{-1} \, \operatorname{Re} F_\rho + i \, \Gamma_I^{-1} \, \operatorname{Im} F_\rho \, . \tag{12}$$

This suggests the following algorithm to solve for Problem 2, whose numerical implementation is feasible to programme. Let $N \in \mathbb{N}$ and choose some initial value $\rho_0 > -1$; put $\rho = \rho_0$.

1. Compute N-dimensional approximations of the above Toeplitz and Hankel operators on the Fourier basis:

$$\mathcal{T}_N = (\mathcal{T}_{k,p})_{0 \le k,p \le N-1} \ , \ \mathcal{H}_N = (\mathcal{T}_{k,p})_{0 \le k,p \le N-1} \, .$$

2. Build the associated $N \times N$ matrices $\Gamma_{R,N}$ and $\Gamma_{I,N}$, using (11), and compute the first N Fourier coefficients F_p, $0 \le p \le N-1$, of F_ρ.

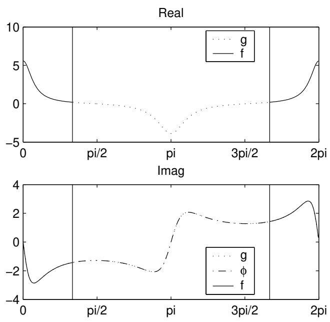
3. Compute $g_\rho = \Gamma_{R,N}^{-1} \, \operatorname{Re} F_p + i \, \Gamma_{I,N}^{-1} \, \operatorname{Im} F_p$ and get $\| \operatorname{Im} g_\rho - \phi \|_J$.

4. If $\| \operatorname{Im} g_\rho - \phi \|_J$ approximately achieves the specified M bound, then $g_2 \approx g_\rho$ and we are done; if not, modify the current value of ρ accordingly to Corollary 10 (dichotomic search) and go back to step 2.

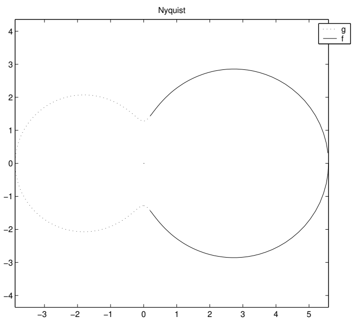
The following simulations of the above algorithm have been performed under `matlab 6` with $N = 512$ and data built from the purely illustrative analytic function:

$$f(z) = \frac{z}{1 - .8\,z} + \frac{z}{1 + .7\,z}$$

on the arc I corresponding to $[-\pi/3, \pi/3]$, with $\phi = \operatorname{Im} f$ on J. Satisfactory reconstruction of f and its real part is obtained for $M = 2 \times 10^{-3}$ and $\rho = -1 + 4 \times 10^{-4}$, the associated value of the criterion on I being 8×10^{-3}. See figures (a) and (b).



(a) Real and Imaginary parts



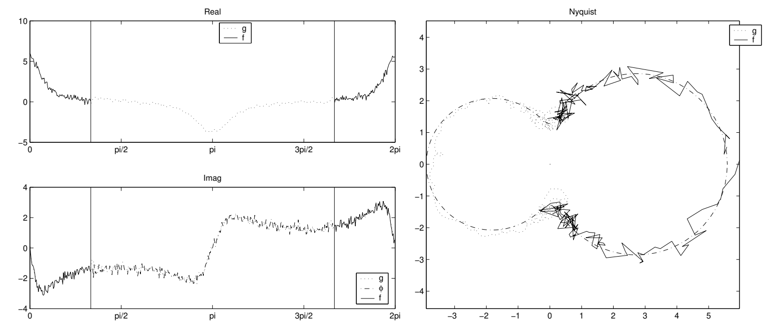
(b) Nyquist plot

The same experiment has been performed with an additive random noise on the data (of standard deviation equal to $.05\,||f||$, which provides a signal / noise ratio equal to 26 dB). We still observe satisfactory results: $\rho = -.997$, $M = .5$, the error on I being equal to .306. See figures (c) and (d).

6 Conclusion

We have presented a solution to an extremal problem of importance in parameter identification, and we have shown how this can be made constructive, illustrating our method with a numerical example.

We also suggest another very natural algorithm, whose interest relies on the fact that a resolution algorithm for Problem 1 is already available (by iterative inversions of the Toeplitz operator). However, it is not known whether this algorithm actually converges.



(c) Real and Imaginary parts (d) Nyquist plot

Let $f \in L^2(I)$, $\phi \in L^2(J)$, $M > 0$, and assume that f is not already the trace of an H^2 function g with $\|\operatorname{Im} g - \phi\|_J \leq M$. Let g_2 be the associated solution to Problem 2. For $n \geq 0$, $g_{1,n} = g_1(f, \psi_n, M)$ stands for the solution to Problem 1 with data $f \in L^2(I)$, $M > 0$, and $\psi_n \in L^2(J)$ defined by:

$$\psi_n = \begin{cases} i\,\phi & \text{for } n = 0\,, \\ i\,\phi + \chi_J \ \operatorname{Re} g_{1,n-1} & \text{for } n > 0\,. \end{cases}$$

Then we conjecture that

$$\lim_{n \to \infty} \|g_{1,n} - g_2\| = 0\,.$$

Further work in this direction is likely to include the use of the L^∞ norm, where at present there seem to be severe technical difficulties to overcome. For instance, the Hilbert transform is not bounded in the L^∞ norm, and thus it is not immediately clear that the analogous problem is well-posed. Problem 1 for the L^∞ norm was solved in [5], and it may again be possible to use this fact in the Solution of Problem 2.

It would also be of interest to study such problems in multiply-connected domains (such as a disc with finitely many holes), and we refer again to [10] for more background and references on this topic.

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