IMA Journal of Applied Mathematics (2009) **74**, 481−506
doi:10.1093/imamat/hxn041
Advance Access publication on December 18, 2008

Robust numerical algorithms based on analytic approximation for the solution of inverse problems in annular domains

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[Received on 20 March 2008; revised on 27 September 2008; accepted on 7 November 2008]

We consider the Cauchy problem of recovering both Neumann and Dirichlet data on the inner part of the boundary of an annular domain from measurements of a harmonic function on some part of the outer boundary. Using tools from complex analysis and best approximation in Hardy classes, we present a family of fast data completion algorithms which are shown to provide constructive and robust identification schemes. These are applied to the computation of an impedance or Robin coefficient and are validated by a thorough numerical study.

Keywords: inverse problems; Cauchy problems; harmonic functions; analytic functions; Hardy spaces; approximation.

1. Introduction

The problem we are dealing with in this contribution is the recovery of both Dirichlet and Neumann data on some part of the inner boundary of an annulus from measurements of a function harmonic in the annulus, taken on some part of the outer boundary. These extended data may be relevant by themselves in some applications or used to compute the electrical impedance (or the Robin coefficient), which is needed in other applications.

Such a problem arises for instance in corrosion detection in tubular domains. Corrosion may occur in many different forms, and several models are encountered in the literature (see Chaabane *et al.*, 2003b; Kaup & Santosa, 1995; Kaup *et al.*, 1996; Santosa *et al.*, 1998). Evaluating the electrical impedance,

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which is actually the Robin coefficient, on the internal wall of a hollow pipe from measurements performed on the external wall turns out to be an appropriate way to locate the corroded parts of the internal wall. Santosa *et al.* (1998) have given a simple linear model proving how corrosion affects the electrical impedance. For this model, there is a significant work due to Fasino & Inglese (1999a,b) for identifying Robin coefficients. This was achieved by means of an imposed flux and measured potential on the accessible part of the boundary of the domain. The numerical scheme adopted was based on a Fourier series expansion and shows instability for thicker domains. Some uniqueness results were given by Chaabane & Jaoua (1999) for simply connected 2D domains for continuous Robin coefficients bounded below: similar properties hold for doubly connected domains, see Leblond *et al.* (2006).

Chaabane *et al.* (2003a) proposed an identification algorithm based on a least squares minimization, an idea attributed to Kohn & Vogelius (1997) and developed in Chaabane & Jaoua (1999); the algorithm consists of comparing solutions corresponding to Robin–Dirichlet and Robin–Neumann boundary conditions, which coincide at the actual solution. This method smooths out possible oscillations in the impedance, which may give information on the regions of corrosion.

Since the Robin coefficient may be recovered from the completed Cauchy data, this problem reduces to solving a Cauchy problem for the Laplace operator. Among recent approaches to the Cauchy problem, we mention Kabanikhin & Karchevsky (1995) who used an optimization (gradient) method in order to minimize the quadratic norm on the accessible part of the boundary. Klibanov & Santosa (1995) used a quasi-reversible method to resolve the problem, combined with Carleman-type estimations. In Kozlov *et al.* (1991), an iterative algorithm is provided, which proceeds by resolving alternatively Neumann and Dirichlet problems; it converges in classes of compatible boundary data, although rather slowly.

The data completion problems that we consider have been widely studied in the case of simply connected domains, which can be conformally mapped on the unit disc, as in Chaabane *et al.* (2003b). The method we wish to generalize here to annular domains is to construct analytic approximations by solving a bounded extremal problem (BEP) there. Such a construction uses an implicit asymptotic expansion of the analytic approximant, and it needs to determine by some appropriate procedure the actual bound of that approximant in order to stabilize the whole algorithm.

The first issue to tackle is thus to obtain asymptotic expansions in annular domains. Provided full data are available on the whole of the outer boundary, such formulae have already been obtained in Smith (2005), and stability estimates for the inverse problem (with suitable norms) have been established as consequences of boundedness properties for functions of weighted Hardy classes in Leblond *et al.* (2006), from which the present work originates. In most practical cases, however, full data cannot be expected. In the present work, implicit formulae of the analytic approximant have therefore been sought and obtained for the incomplete data case. Continuity with respect to the data of the computed approximants has also been proved; this makes it possible to use the formulae as a basic tool in the algorithmic part.

In order to produce an accurate approximant, it has already been noted (see Chaabane *et al.*, 2003b) that the numerical algorithms need sharp information on the actual bound of the data sought. The issues of computing both these data and the bound on them thus need to be dealt with simultaneously. This has been achieved in the present work by characterizing the actual bound as the unique zero of an appropriate function. Robustness properties of the designed procedure are improved by applying it to the nth-order derivatives of the data, instead of the data themselves, working in certain Sobolev classes of smoother functions, provided of course that the prescribed data meet this additional regularity requirements. A whole family of algorithms, more robust as their order increases, is designed this way.

In Section 2 of this paper, we introduce the inverse Robin problem and recall the identifiability and stability results as obtained in Chaabane & Jaoua (1999) and Leblond *et al.* (2006). Section 3 is devoted to deriving the formulae we use to compute the solution in the incomplete external data case and to proving continuity of these solutions with respect to the prescribed data. The identification algorithms are presented and studied in Section 4, and their numerical implementation and results are finally discussed in Section 5.

2. Setting the inverse problem

2.1 *The physical model*

Pipelines are widely used to transport gas and petroleum from their production spots to their processing or consumption places. These pipelines are subject to internal corrosion (caused by hydrogen sulphides and carbon dioxide in the case of gas pipelines and by sulphato-reducing bacteria in the case of oil pipelines). Non-destructive testing techniques are used in order to check whether the pipeline needs to be repaired, before failure occurs. Electrical impedance tomography is one of these techniques. It consists in prescribing a current flux on the external wall, and from the measured voltage potential there, to evaluate the location of corrosion if any, thus helping one to make a decision on whether the pipeline needs repairs (see Kaup *et al.*, 1996; Colton *et al.*, 1990).

Assuming the pipe is infinite in the z-direction and the current circulates in this direction, the electrical potential thus obeys the 2D Laplace equation in the annular (x, y) section G of the pipe:

$$\Delta u = 0 \quad \text{in } G.$$

The boundary conditions are of both Neumann and Dirichlet type on the external part of the boundary where the current flux has been prescribed and the voltage potential measured. As for the boundary condition on the internal wall, several models have been proposed, and particularly a nonlinear one—due to Butler and Volmer:

$$\partial_n u = q(\mathrm{e}^{\alpha u} - \mathrm{e}^{-(1-\alpha)u}). \tag{2.1}$$

This model has been analysed in depth by Jones (1996) and recently discussed by Bryan & Vogelius (2002), Kavian & Vogelius (2003) and Vogelius & Xu (1998). A linearized version of this boundary condition as proposed by Santosa *et al.* (1998) is nothing but the Robin condition

$$\partial_n u = qu.$$

Note that if one regards u as a contrast produced by the effects of corrosion, then it can be regarded as small, and so a linear model is acceptable. Buttazzo & Kohn (1987) also provides an additional justification for working with this linear model, using homogenization techniques. Using this, the corrosion effects, which are actually material damages due to chemical reactions, reduce, as regards the solution of the Laplace equation, to their impact on the impedance q. Recovering the modified impedance (or Robin coefficient) would therefore permit one to locate the corroded zones and evaluate the damage. The inverse problem of corrosion detection becomes a Robin inverse problem, the unknown of which is the impedance on the internal boundary.

2.2 *The Robin inverse problem*

We shall restrict our study to annular domains. There are two reasons for this. The first is that this is usually the shape of a cross-section of a pipeline. The second one is that, up to a conformal mapping,

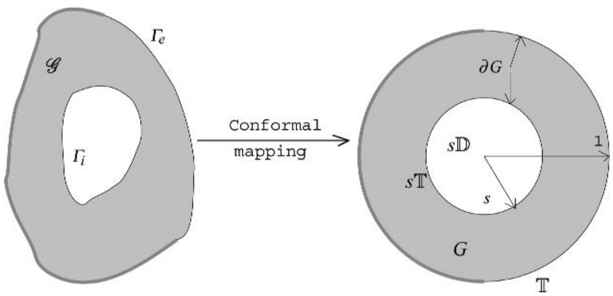

FIG. 1. Annular domain derived by a conformal mapping.

any doubly connected domain with a smooth boundary made of two non-intersecting closed smooth Jordan curves may be seen as an annular domain (see Fig. 1 and Pommerenke, 1992).

Let therefore $\mathbb{D}$ be the unit disc and G be the annulus $G = \mathbb{D} \setminus s\mathbb{D}$ for some fixed s with $0 < s < 1$ and denote $\partial G = \mathbb{T} \cup s\mathbb{T}$, where $\mathbb{T}$ is the unit circle. We provide each circle with normalized Lebesgue measure.

Let I be a non-null measurable subset of $\mathbb{T}$, and let $J = \partial G \setminus I$. We consider the following inverse problem: given functions u_d and ϕ, or a number of pointwise measurements, with $\phi \not\equiv 0$, find a function q such that a solution u to

$$\begin{cases} \Delta u = 0 & \text{in } G \text{ (i)}, \\ u = u_d & \text{on } I \text{ (ii)}, \\ \partial_n u = \phi & \text{on } I \text{ (iii)} \end{cases} \tag{2.2}$$

also satisfies

$$\partial_n u + qu = 0 \quad \text{on } J, \tag{2.3}$$

where ∂_n stands for the partial derivative w.r.t. the outer normal unit vector to $\mathbb{T}$. In the electrical framework, u_d and ϕ correspond to the measured voltage potential and the prescribed current flux on the outer boundary of some plane section of a tube, while q is the electrical impedance to be recovered on the associated inner boundary.

Let $\underline{c}, \overline{c} > 0$ and introduce the following class of 'admissible' electrical impedances:

$$\mathcal{Q}^n = \{q \in \mathcal{C}^n(J); |q^{(k)}(x)| \leqslant \overline{c}, 0 \leqslant k \leqslant n, \text{and } q(x) \geqslant \underline{c}, \quad \forall x \in J\}.$$

For $k \geqslant 1$, let $W^{k,2}(I)$ denote the usual Sobolev space of functions $f \in L^2(I)$, the derivatives of which are also up to the kth derivative denoted by $f^{(k)}$ in $L^2(I)$. For consistency, we shall also denote by $W^{0,2}(I)$ the space $L^2(I)$. The Sobolev spaces $W^{k,2}(G)$ and $W^{k,2}(\partial G)$ are defined analogously.

Chaabane *et al.* (2003b) and Leblond *et al.* (2006) have already discussed the existence and the uniqueness issues for the forward problem in the unit disc and a doubly connected domain, respectively. They have shown that provided $\phi \in W^{n,2}(I)$, $\phi \geqslant 0$, $\phi \not\equiv 0$, $n \geqslant 0$, and $q \in \mathcal{Q}^n$ for some constants $\underline{c}$, $\overline{c} > 0$, then there exists a unique function $u \in W^{n+3/2,2}(G)$ whence $u_{|\partial G} \in W^{n+1,2}(\partial G)$, a solution to a direct problem. Further, there exist constants $m > 0$ and κ (depending on the class $\mathcal{Q}^n$) such that for all $q \in \mathcal{Q}^n$ and $\phi \in W^{n,2}(I)$,

$$u \geqslant m > 0 \;\text{ on } J \quad \text{and} \quad \|u\|_{W^{n+1,2}(\partial G)} \leqslant \kappa.$$

Also, they have examined the questions of the uniqueness of the solution q of the inverse problem and have proved that if $u_{1_{|I}} = u_{2_{|I}}$, then $q_1 = q_2$, where $q_1, q_2 \in \mathcal{Q}^0$ and u_1, u_2 be the associated solutions.

2.3 *From the Robin inverse problem to the Cauchy problem for analytic functions*

We propose here to solve the Robin inverse problem by taking advantage of the analytic extension theory, which provides with explicit or quasi-explicit formulae for the computation of the extended data. Let $\phi \in L^2(I)$ and assume that $q \in \mathcal{Q}^0$. In this case, $u_{|\partial G} \in W^{1,2}(\partial G)$ (Leblond *et al.*, 2006, Theorem 1). Then there exists a function v harmonic in G such that $\partial_\theta v = \partial_n u$ on ∂G, where ∂_θ stands for the tangential partial derivative on ∂G, from the Cauchy–Riemann equations. Hence, v is given on I up to a constant by

$$v_{|I}(\mathrm{e}^{\mathrm{i}\theta}) = \int_{\theta_0}^{\theta} \phi(\mathrm{e}^{\mathrm{i}\tau})\mathrm{d}\tau.$$

Further, from the M. Riesz theorem (Duren, 2000, Theorem 4.1), the harmonic conjugate operator is bounded in $L^2(\partial G)$, whence $v_{|\partial G} \in W^{1,2}(\partial G)$. Thus, $f = u + \mathrm{i}v$ is analytic in G and $f_{|\partial G} \in W^{1,2}(\partial G)$; it is given on I by

$$f(\mathrm{e}^{\mathrm{i}\theta}) = u_d(\mathrm{e}^{\mathrm{i}\theta}) + \mathrm{i}\int_{\theta_0}^{\theta} \phi(\mathrm{e}^{\mathrm{i}\tau})\mathrm{d}\tau.$$

Then on J, we have

$$q = -\frac{\partial_\theta v}{u} = -\frac{\partial_\theta \operatorname{Im} f}{\operatorname{Re} f}, \tag{2.4}$$

which gives the link to be used between q and f, in order to recover q from approximations to f on the subset I of the boundary ∂G.

The annulus is not simply connected, but it is possible to define f globally in G as a single-valued function. Indeed, there is a single-valued analytic function $\tilde{f}$ defined on G such that

$$\tilde{f}(z) = f(z) - \frac{c}{2\pi}\log z; \tag{2.5}$$

hence $u(z) = \operatorname{Re}\tilde{f}(z) + \frac{c}{2\pi}\log|z|$, where

$$c = \int_0^{2\pi} \phi(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta, \tag{2.6}$$

for $I = \mathbb{T}$, and

$$c = \int_I \phi(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta + \int_{\mathbb{T}\setminus I} \partial_\theta v(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta,$$

if $I \subsetneq \mathbb{T}$. In both situations, this allows us to work with single-valued analytic functions, as in Leblond *et al.* (2006).

Let us introduce here the Hardy space $H^2(\mathbb{D})$ of analytic functions in the unit disc $\mathbb{D}$ whose L^2-norms on the unit circle $\mathbb{T}$ are bounded (see Duren, 2000). Let $\bar{H}_0^2(s\mathbb{D})$ be the Hardy space consisting of the analytic functions on the complement of $\overline{s\mathbb{D}}$ that have boundary values in $L^2(\partial G)$ and vanish at infinity.

From the above-mentioned regularity properties, the function f is bounded in $L^2(I)$, and we then find an extension of f in the so-called Hardy space denoted by $H^2(G) = H^2(\mathbb{D}) \oplus \bar{H}_0^2(s\mathbb{D})$ defined in Rudin (1955). It is also possible to define the Hardy spaces $H^2(\partial G)$ as the closure in $L^2(\partial G)$ of the set R_G of rational functions whose poles lie in $\mathbb{C} \setminus \overline{G}$.

The spaces $H^2(G)$ and $H^2(\partial G)$ are then isomorphic in a natural way, and so we identify the two spaces (see Chalendar & Partington, 1999).

So, a function $f \in H^2(\partial G)$ has the following expansion:

$$f(z) = \sum_{n\in\mathbb{Z}} a_n z^n \quad \text{for } z \in G, \quad \text{where } \|f\|^2_{L^2(\partial G)} = \sum_{n\in\mathbb{Z}} (1+s^{2n})|a_n|^2.$$

Recalling that $\partial G = s\mathbb{T} \cup \mathbb{T}$ where each circle has normalized Lebesgue measure, we see that the space $H^2(\partial G)$ has a canonical orthonormal basis consisting of the functions

$$e_n(z) = (z^n/\sqrt{1+s^{2n}})_{n\in\mathbb{Z}}.$$

We write $\chi_I g$ for the function in $L^2(\partial G)$ that coincides with g on I and vanishes on J.

2.4 *Approximation in Hardy classes and BEP problems*

We assume that $I = [-\theta_0, \theta_0] \subset \mathbb{T}$, $0 < \theta_0 < \pi$. We write $L^2(\partial G) = L^2(I) \oplus L^2(J)$. Whenever κ_1 is defined on I and κ_2 on J, we write $\kappa_1 \vee \kappa_2$ for the function equal to κ_1 on I and κ_2 on J.

Suppose that we are given $f \in L^2(I)$ and we wish to approximate f as well as possible by the restriction to I of an $H^2(\partial G)$ function, i.e. $g_{|_I}$ for $g \in H^2(\partial G)$. In view of the results established in Chalendar & Partington (1999), the space $H^2(\partial G)_{|_I}$ is dense in $L^2(I)$. Then there will exist a sequence (g_n) of $H^2(\partial G)$ functions such that $\|g_{n|_I} - f\|_{L^2(I)} \to 0$. However, if $f \neq g_{|_I}$ for any $g \in H^2(\partial G)$, then it will follow that $\|g_{n|_J}\|_{L^2(J)} \to \infty$, which means that the approximation problem is ill-posed.

In our work, we are interested in the determination of an extension on J. To prevent instability from appearing, imposing a bound for the approximation on J may be a solution. This motivates the following BEP, which is a problem of analytic approximation of incomplete data in Hardy classes.

To fix ideas, we consider the following minimization problem:

$$(\text{BEP})\begin{cases}\text{Given } f \in L^2(I) \setminus H^2(\partial G)_{|_I},\ f_1 \in L^2(J) \text{ and } M > 0,\\ \text{find a function } g \in H^2(\partial G) \text{ such that } \|g - f_1\|_{L^2(J)} \leqslant M \text{ and}\\ \|f - g\|_{L^2(I)} = \inf\{\|f - \psi\|_{L^2(I)}\colon \psi \in H^2(\partial G), \|\psi - f_1\|_{L^2(J)} \leqslant M\}.\end{cases}$$

In practice, f corresponds to the data, I is the part where these data can be measured and f_1 is a reference behaviour of the data on the part of the boundary where they are unknown. Such a problem is convex and admits a unique solution which can be obtained by solving a spectral equation for the Toeplitz operator $\mathcal{T}$ with symbol χ_J, the characteristic function of the component J:

$$\mathcal{T}\colon H^2(\partial G) \to H^2(\partial G),$$
$$g \mapsto P_{H^2(\partial G)}\chi_J g,$$

where $P_{H^2(\partial G)}\colon L^2(\partial G) \to H^2(\partial G)$ is the orthogonal projection. More precisely, the unique solution g to the (BEP) problem solves the following.

PROPOSITION 2.1 (Chalendar & Partington, 1999) The unique solution g of the (BEP) problem is given by the formula

$$g = (Id + \lambda \mathcal{T})^{-1} P_{H^2(\partial G)}[f \vee (1+\lambda) f_1], \tag{2.7}$$

for the unique $\lambda > -1$ such that

$$\|g - f_1\|_{L^2(J)} = M. \tag{2.8}$$

REMARK 2.2 Let us note that λ plays the role of a Lagrange multiplier which makes implicit the dependence of the solution on M and which can be adjusted by dichotomy. A consequence of Proposition 2.1 is that the error $e(\lambda) = \|f - g(\lambda)\|_{L^2(I)}$ smoothly decreases to 0 as $\lambda \to -1$ and we refer to Torkhani (1995), that $\lambda \to M(\lambda)$ is C^1, bijective and decreasing on $(-1, +\infty) \to (0, +\infty)$.

When f is the trace on I of some $H^2(\partial G)$ function, the (BEP) problem becomes one of interpolation. In this case, for simplicity, we will continue to denote by f the $H^2(\partial G)$ function defined on the whole of ∂G. The error $e(\lambda)$ decreases strictly to zero as M increases to $\|f - f_1\|_{L^2(J)}$ and vanishes identically for $M \geqslant \|f - f_1\|_{L^2(J)}$.

3. Solutions for the (BEP) problem

Now, using a Fourier series development on the $(e_n)_n$ basis, we are able to propose a quasi-explicit method to solve (2.7).

Let a_n and b_n be, respectively, the Fourier coefficients of $\phi = f \vee (1+\lambda) f_1 \in L^2(\partial G)$ on $\mathbb{T}$ and $s\mathbb{T}$, defined by

$$a_n = \frac{1}{2\pi}\left(\int_{-\theta_0}^{\theta_0} f(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta + (1+\lambda)\int_{\theta_0}^{2\pi-\theta_0} f_1(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta\right)$$

and

$$b_n s^n = \frac{1+\lambda}{2\pi}\int_0^{2\pi} f_1(s\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}n\theta}\,\mathrm{d}\theta.$$

Moreover, let

$$\mathcal{B} = P_{H^2(\partial G)}\phi = P_{H^2(\partial G)}[f \vee (1+\lambda) f_1] = \sum_{n\in\mathbb{Z}} \mathcal{B}_n e_n,$$

where

$$\mathcal{B}_n = \frac{a_n + b_n s^{2n}}{\sqrt{1+s^{2n}}}.$$

3.1 *Constructive formulae*

The following theorem then holds.

THEOREM 3.1 The solution g of the (BEP) problem, viewed as the (infinite) vector as defined by its Fourier coefficients $(g_n)_{n\in\mathbb{Z}}$, solves the following equation:

$$(Id + \lambda\mathcal{T})g = \mathcal{B}, \tag{3.1}$$

where $\mathcal{T}$ is the Toeplitz operator represented in the $\{e_n\}$ basis by the infinite Toeplitz matrix defined by

$$\text{for } n, m \in \mathbb{Z}, \quad \mathcal{T}_{n,m} = \begin{cases} \frac{1}{1+s^{2n}}\left(1 + s^{2n} - \frac{\theta_0}{\pi}\right), & \text{when } n = m, \\ -\frac{1}{\sqrt{(1+s^{2n})(1+s^{2m})}}\frac{\sin(m-n)\theta_0}{\pi(m-n)}, & \text{when } n \neq m. \end{cases} \tag{3.2}$$

REMARK 3.2

1. This result is similar to the one obtained in the unit disc by Jacob *et al.* (2002). Both lead to an infinite linear system, here indexed by $\mathbb{Z}$ whereas it was indexed by $\mathbb{N}$ for the problem in the unit disc. Let us denote by g_N the approximate solution obtained by solving the truncated system in the basis $(e_n)_{-N\leqslant n\leqslant N}$

$$((Id + \lambda\mathcal{T})g_N)_N = \mathcal{B}_N, \tag{3.3}$$

   where $\mathcal{B}_N$ is the truncated Fourier series of $\mathcal{B}$.
   The linear system so obtained has a symmetric positive-definite matrix, which can be factorized using the Cholesky method. Iterating then on λ until (2.8) holds leads to the solution of the (BEP) problem for a given bound M. Further details are given in Section 4.
2. A particular case is that of full external data ($J = s\mathbb{T}$). It has been established in Abrahamse (1974) that the Toeplitz operator is diagonalizable, and an expression of the (BEP) solution has been obtained in Smith (2005):

$$g(z) = \sum_{n\in\mathbb{Z}} \frac{a_n + \alpha b_n s^{2n}}{1 + \alpha s^{2n}} e_n(z),$$

   where $\alpha = 1 + \lambda > 0$ is the unique constant such that

$$\sum_{n\in\mathbb{Z}} \frac{|(a_n - b_n)|^2 s^{2n}}{(1 + \alpha s^{2n})^2} = M^2.$$

The proof of Theorem 3.1 is now a straightforward consequence of the following two lemmas, whose proofs are provided in the appendix.

LEMMA 3.3

$$P_{H^2(\partial G)}\phi(z) = \sum_{n\in\mathbb{Z}} \frac{a_n + b_n s^{2n}}{\sqrt{1 + s^{2n}}} e_n(z),$$

LEMMA 3.4 Let $g \in H^2(\partial G)$ such that $g(z) = \sum_{n\in\mathbb{Z}} g_n e_n(z)$ for $z \in G$ and $\mathcal{T}$ the Toeplitz operator. Then

$$\mathcal{T}g(z) = \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{1+s^{2n}}}\left(\frac{g_n}{\sqrt{1+s^{2n}}}\left(1 + s^{2n} - \frac{\theta_0}{\pi}\right) - \sum_{m\neq n} \frac{g_m}{\sqrt{1+s^{2m}}}\frac{\sin(m-n)\theta_0}{\pi(m-n)}\right) e_n(z).$$

Since we have no prior information on how the data behave on the part J of the boundary, we shall choose from now on $f_1 = 0$, whence $b_n = 0$, $\forall n \in \mathbb{Z}$.

Aiming to make use of these formulae in order to set up robust numerical computation algorithms, a crucial point to investigate is continuity of the so computed solutions with respect to the data. This is the matter of Section 3.2.

3.2 *Continuity of the solutions with respect to the data*

In this section, we shall investigate continuity properties of the solutions of (BEP) problem with respect to the data f and M. Let $\mathbb{R}_+^*$ denote the set of strictly positive real numbers and let g be the mapping defined by

$$\Psi\colon L^2(I) \times \mathbb{R}_+^* \to H^2(\partial G),$$
$$(f, M) \mapsto g(f, M),$$

where $g(f, M)$ solve the (BEP) problem associated to the data f and M.

Let $\mathcal{D} = \{(h, M) \in H^2(\partial G)_{|I} \times \mathbb{R}_+^* | \|h\|_{L^2(J)} < M\}$.

THEOREM 3.5 The mapping Ψ is continuous on $(L^2(I) \times \mathbb{R}_+^*)\backslash\mathcal{D}$, but not on the whole of $L^2(I) \times \mathbb{R}_+^*$. However, if $(f_n, M_n) \to (f, M)$ in $L^2(I) \times \mathbb{R}_+^*$, then $g(f_n, M_n) \rightharpoonup g(f, M)$ weakly in $H^2(\partial G)$, whereas $g(f_n, M_n) \to g(f, M)$ in $L^2(I)$.

Proof. First, consider the mapping e_f defined by

$$e_f\colon \mathbb{R}_+^* \to \mathbb{R}_+,$$
$$M \mapsto \|g(f, M) - f\|_{L^2(I)}.$$

The mapping e_f is convex and decreasing, thus continuous on $\mathbb{R}_+^*$.

Next, let (f_n) be a sequence in $L^2(I)$ such that $\|f_n - f\|_{L^2(I)} \to 0$ and suppose that (M_n) is a sequence in $\mathbb{R}_+^*$ such that $M_n \to M$. We claim that

$$\lim_{n\to\infty} e_{f_n}(M_n) = e_f(M). \tag{3.4}$$

Indeed, let $\delta > 0$ and assume that either $e_{f_n}(M_n) > e_f(M) + \delta$ or $e_{f_n}(M_n) < e_f(M) - \delta$ infinitely often. In the first case, since

$$\|g(f, M_n) - f_n\|_{L^2(I)} \leqslant \|g(f, M_n) - f\|_{L^2(I)} + \|f_n - f\|_{L^2(I)}$$

and because e_f is continuous, we have infinitely often

$$\|g(f, M_n) - f_n\|_{L^2(I)} < e_f(M) + \delta < e_{f_n}(M_n),$$

which contradicts the fact that $g(f_n, M_n)$ is optimal. In the second case,

$$\|g(f_n, M_n) - f\|_{L^2(I)} \leqslant e_{f_n}(M_n) + \|f_n - f\|_{L^2(I)},$$

which implies that we have infinitely often

$$e_f(M_n) < e_f(M) - \frac{\delta}{2}$$

and contradicts the continuity of e_f established above.

Next, the sequence $(g(f_n, M_n))$ is bounded. We show that each of its subsequences admits a further subsequence which converges to $g(f, M)$. We pass to a subsequence that converges weakly to, say, $\tilde{g} \in H^2(\partial G)$. By relabelling, we still call it $(g(f_n, M_n))$. It follows directly from the assumptions and (3.4) that

$$\|\tilde{g} - f\|_{L^2(I)} \leqslant e_f(M), \quad \|\tilde{g}\|_{L^2(J)} \leqslant M.$$

Now, because the solution to (BEP) is unique (by the strict convexity of the norm), we necessarily have that $\tilde{g} = g(f, M)$. This shows the weak convergence in $H^2(\partial G)$.

On the other hand, it holds from (3.4) that $\|g(f_n, M_n) - f_n\|_{L^2(I)} \to \|g(f, M) - f\|_{L^2(I)}$ which implies that strong convergence always holds in $L^2(I)$.

Finally, whenever $(f, M) \notin \mathcal{D}$, then

$$\limsup_{n\to\infty} \|g(f_n, M_n)\|_{L^2(J)} \leqslant \limsup_{n\to\infty} M_n = M = \|g(f, M)\|_{L^2(J)},$$

and since we have also $g(f_n, M_n)$ converging weakly to $g(f, M)$ in $H^2(\partial G)$, then we obtain strong convergence on J. □

In order to achieve convergence of the reconstruction scheme, continuity ensured by Theorem 3.5 is hardly sufficient. Aiming to ensure strong convergence of the extended data, one needs to deal with higher-order methods. These methods consist in solving the (BEP) problem for the data derivatives, instead of the data themselves, provided some additional regularity is available in order to allow that. Let us define to that end the appropriate Hardy–Sobolev spaces. For $n \geqslant 1$, define

$$H^{n,2}(\partial G) = H^2(\partial G) \cap W^{n,2}(\partial G) = \{f \in H^2(\partial G);\ f^{(k)} \in H^2(\partial G), 1 \leqslant k \leqslant n\}.$$

For consistency, we shall also denote by $H^{0,2}(\partial G)$ the space $H^2(G)$.

Let now Ψ_n be the mapping

$$\Psi_n\colon W^{n,2}(I) \times \mathbb{R}_+^* \to H^{n,2}(\partial G) \tag{3.5}$$

defined by

$$[\Psi_n(f, M)]^{(n)} = \Psi(f^{(n)}, M), \quad [\Psi_n(f, M)]^{(k)}(z_0) = f^{(k)}(z_0), \quad 0 \leqslant k \leqslant n-1,$$

for some fixed $z_0 \in I$. Note that $\Psi_0 = \Psi$. An order-n version of the (BEP) problem consists in solving (BEP) with bound M for the nth derivative $f^{(n)}$ of f, and then integrating n times using the initial conditions provided above, in order to get $\Psi_n(f, M)$ as a function of $H^{n,2}(\partial G)$; see also Baratchart & Leblond (1992).

Finally, let us define as above

$$\mathcal{D}_n = \{(h, M) \in H^{n,2}(\partial G)_{|I} \times \mathbb{R}_+^* | (h^{(n)}, M) \in \mathcal{D}\}.$$

Similarly to the previous theorem, the convergence result that holds in $\mathcal{D}_n$ is weaker than the one holding outside $\mathcal{D}_n$.

THEOREM 3.6 The mapping Ψ_n is continuous on $(W^{n,2}(I) \times \mathbb{R}_+^*) \backslash \mathcal{D}_n$, but not on the whole of $W^{n,2}(I) \times \mathbb{R}_+^*$. However, if $(f_k, M_k) \to (f, M)$ in $W^{n,2}(I) \times \mathbb{R}_+^*$, then $g(f_k, M_k) \rightharpoonup g(f, M)$ weakly in $H^{n,2}(\partial G)$, whereas $g(f_k, M_k) \to g(f, M)$ in $W^{n,2}(I)$. Thus, Ψ_n is continuous on $H^{n-1,2}(\partial G)$.

Proof. The first two statements are direct consequences of Theorem 3.5 applied to the first n derivatives of the function g_n. Regarding the third one, this follows since if (f_k) is a sequence in $W^{1,2}(J)$ such that $f_k(z_0) = f(z_0)$, $z_0 \in J$, with derivative f_k' converging weakly to $f' \in L^2(J)$, then $f_k \to f$ pointwise in J and hence, by the Lebesgue dominated convergence theorem, $\|f_k - f\|_{L^2(J)} \to 0$. □

4. Identification algorithms

We present in this section a family of numerical algorithms permitting to compute the Robin inverse problem solution. Still in the electrical framework, once the current flux and the voltage potential have been computed on the inaccessible boundary J, we can evaluate the impedance (or Robin coefficient) q from (2.4):

$$q = -\frac{\partial_\theta \operatorname{Im} g(f, M)}{\operatorname{Re} g(f, M)} \quad \text{on } J,$$

where f is the prescribed data and $g(f, M)$ the extended data computed by solving the (BEP) problem using f and the bound M.

Actually, the data we are dealing with are usually noisy ones

$$f_\varepsilon = f + \varepsilon,$$

where $f \in H^2(\partial G)\,|_I$ and $\varepsilon \in L^2(I)$, but $\varepsilon \notin H^2(\partial G)|_I$. In that case, what can be derived from the above section, namely from Theorem 3.5, is that in order to provide with extended data 'close' to the actual ones, the (BEP) problem needs to use a bound M close enough to the actual one

$$M_0 = \| f \|_{L^2(J)}.$$

Moreover, since the prescribed data do not belong to the Hardy class $H^2(\partial G)$, the computed extension will saturate the prescribed bound whatever its value is, i.e.

$$\|g(f_\varepsilon, M)\|_{L^2(J)} = M.$$

Properly choosing the bound is therefore mandatory to get an accurate approximation on these extended data. The point is that the actual bound M_0 is unknown, since it depends on the unknown part of the data. Any constructive algorithm will thus need to tackle together the tasks of computing both the extended data and the bound on them. To make the paper easier to read, we shall however describe separately in the sequel how to go through each of these tasks.

4.1 *Determination of the actual bound*

In Chaabane *et al.* (2003b), the authors have proposed, in order to determine the bound, a cross-validation procedure using some part of the prescribed data. Though efficient, this method turns out to be costly in the present case, since a smaller amount of data is available, due to the multiply connected geometry. It is thus preferable to devote the whole of the data to the reconstruction task, which requires one to build up an alternative'non-data-consuming' method in order to compute the bound. We shall be presenting that alternative method in the sequel.

Given a positive real number M, $g(f, M)$ denotes as usual the solution of the (BEP) problem with data f and bound M, whereas $g_\varepsilon = g(f_\varepsilon, M)$ solves the same problem with f_ε as a data set and the same bound M. The convergence results of the previous section indicate that $g(f_\varepsilon, M)$ is close to f, which is equal to $g(f, M_0)$, provided that M is close to M_0 and f_ε close to f. Since we do not know M_0, let us try to evaluate the difference $f - g_\varepsilon$. An approximation of this function on I may be given by $f_\varepsilon - g_\varepsilon$, whereas a rough estimate of the bound may be given by

$$e_{f_\varepsilon}(M) = \| f_\varepsilon - g_\varepsilon \|_{L^2(I)}.$$

Let then w_ε solve the (BEP) problem with these data

$$w_\varepsilon = g\left(f_\varepsilon - g_\varepsilon, e_{f_\varepsilon}(M)\right).$$

Therefore, $g_\varepsilon + w_\varepsilon$ is likely to provide a better approximation to f than g_ε. As a matter of fact, let us define

$$\begin{aligned} \tau_\varepsilon\colon\quad & \mathbb{R}_+ \mapsto \mathbb{R}_+, \\ & M \mapsto \left|\|g_\varepsilon\|_{L^2(J)} - \|g_\varepsilon + w_\varepsilon(M)\|_{L^2(J)}\right|. \end{aligned} \tag{4.1}$$

The closer M becomes to the actual bound, the better the approximation becomes, and the closer to zero $\tau_\varepsilon(M)$ becomes. Minimizing $\tau_\varepsilon(M)$ seems thus a reasonable way to find out the actual bound M_0. This is what we are going to prove in Theorem 4.1 for analytic data. First, let us notice that

$$\tau_\varepsilon(M) \leqslant e_{f_\varepsilon}(M), \quad \forall\, M \in \mathbb{R}_+^*. \tag{4.2}$$

Indeed,

$$\tau_\varepsilon(M) = \left|\|g_\varepsilon\|_{L^2(J)} - \|g_\varepsilon + w_\varepsilon(M)\|_{L^2(J)}\right| \leqslant \|w_\varepsilon(M)\|_{L^2(J)} = e_{f_\varepsilon}(M).$$

THEOREM 4.1 (Bound determination for analytic data)
In case the data are analytic (i.e. $\varepsilon = 0$), then M_0 is the smallest positive real number that minimizes the mapping τ_0 and moreover $\tau_0(M_0) = 0$.

Proof. Since $f \in H^2(\partial G)_{|I}$, then for each $M \geqslant M_0$ one has $g(f, M) = f$ on I, therefore $e_f(M) = 0$. Then we have $\tau_0(M) = 0$, $\forall\, M \geqslant M_0$.

On the other hand, suppose $M < M_0$. Since $g(f, M)$ solves the (BEP) problem with respect to (f, M), we have

$$e_f(M) = \|f - g(f, M)\|_{L^2(I)} = \inf\{\|f - g\|_{L^2(I)}\colon g \in H^2(\partial G), \|g\|_{L^2(J)} \leqslant M\} > 0$$

and, since $w_0(M)$ solves the (BEP) problem with respect to $(f - g(f, M), e_f(M))$, we have

$$\|f - g(f, M) - w_0(M)\|_{L^2(I)} = \inf\{\|f - g(f, M) - w\|_{L^2(I)}, w \in H^2(\partial G), \|w\|_{L^2(J)} \leqslant e_f(M)\}.$$

Since the null function $w = 0$ is in $H^2(\partial G)$ (and $\|w\|_{L^2(J)} = 0 < e_f(M)$), then

$$\|f - g(f, M) - w_0(M)\|_{L^2(I)} \leqslant \|f - g(f, M)\|_{L^2(I)}. \tag{4.3}$$

If there exists a real $M < M_0$ such that $\tau_0(M) = 0$, then $\|g(f, M) + w_0(M)\|_{L^2(J)} = M$. Therefore, from (4.3) and uniqueness of the solution of the (BEP) problem, we have $g(f, M) + w_0(M) = g(f, M)$ on G and then $w_0(M) = 0$ on G. This implies that $e_f(M) = \|w_0(M)\|_{L^2(J)} = 0$, which contradicts the fact that $e_f(M) \neq 0$. □

The case of non-analytic data is however the one we are interested in. Figure 2 illustrates the behaviour of the functions $\tau_0(M)$ and $\tau_\varepsilon(M)$ for the rational function

$$f(z) = c + \frac{2(z-1)}{z-a}, \quad a \in s\mathbb{D}, \tag{4.4}$$

with $c = 12$, $a = 0.1$ and $s = 0.6$. For $\varepsilon \neq 0$, the function τ_ε seems to have a minimum, the argument of which is equal to the correct value M_0 of the bound. This is confirmed by numerical computations, although at this stage we only prove a somewhat weaker result.

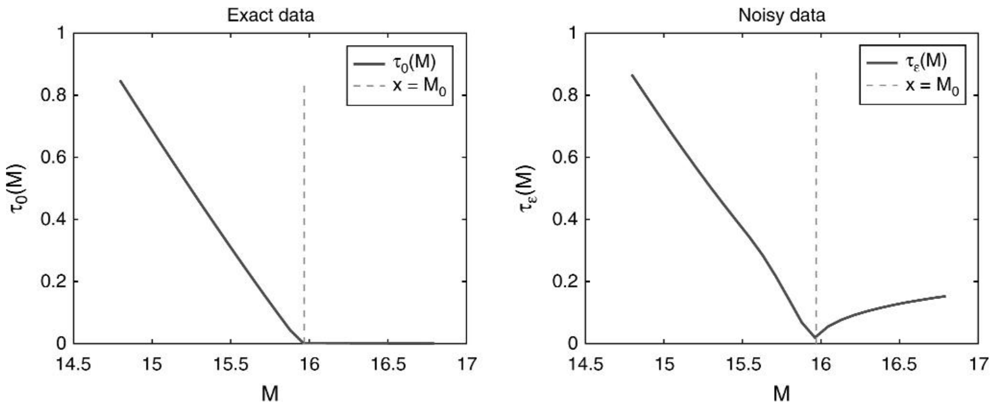

FIG. 2. Plots of $\tau_0(M)$ (left) and $\tau_\varepsilon(M)$, $\varepsilon \neq 0$ (right).

THEOREM 4.2 Let α and β be two positive numbers such that $0 < \alpha < \beta$ and $M_0 \in [\alpha, \beta]$, and let $\varepsilon \in L^2(I)$ be a positive function.

(i) The function τ_ε has at least one minimum M_ε in $[\alpha, \beta]$. Moreover, defining

$$\delta_\varepsilon = \inf_{M \in [\alpha,\beta]} \tau_\varepsilon(M) = \tau_\varepsilon(M_\varepsilon), \qquad \text{we have} \lim_{\|\varepsilon\|_{L^2(I)} \to 0} \delta_\varepsilon = 0.$$

(ii) Let $\mathcal{I}_\varepsilon = \{M_\varepsilon \in [\alpha, \beta]\colon \delta_\varepsilon = \tau_\varepsilon(M_\varepsilon)\}$. Then $\mathcal{I}_\varepsilon$ has a minimum point $\underline{M}_\varepsilon$.

(iii) Any accumulation point $\underline{M}_0$ of the family $(\underline{M}_\varepsilon)_\varepsilon$ is such that $\underline{M}_0 \geqslant M_0$.

(iv) When $\|\varepsilon\|_{L^2(I)} \to 0$, then $g(f_\varepsilon, \underline{M}_\varepsilon) \rightharpoonup f$ weakly in $H^2(\partial G)$, hence also in the weak topology of $L^2(J)$, and $g(f_\varepsilon, \underline{M}_\varepsilon) \to f$ in $L^2(I)$.

Proof.

(i) Since the data f_ε are not analytic, we get from Theorem 3.5

$$\lim_{M_n \to M} \|g(f_\varepsilon, M_n) - g(f_\varepsilon, M)\|_{L^2(J)} = 0,$$

$$\text{and also} \lim_{M_n \to M} \|w_\varepsilon(M_n) - w_\varepsilon(M)\|_{L^2(J)} = 0;$$

τ_ε is thus continuous on the compact set $[\alpha, \beta]$, and there exists some real number $M_\varepsilon \in [\alpha, \beta]$ such that $\tau_\varepsilon(M_\varepsilon) = \delta_\varepsilon$.
Let $(\varepsilon_n)_n$ be a sequence such that $\lim_{n\to\infty} \varepsilon_n = 0$. Since $M_0 \in [\alpha, \beta]$, we have

$$0 \leqslant \delta_{\varepsilon_n} \leqslant \tau_{\varepsilon_n}(M_0).$$

From (4.2), we obtain

$$0 \leqslant \delta_{\varepsilon_n} \leqslant e_{f_{\varepsilon_n}}(M_0); \tag{4.5}$$

we conclude then, from (3.4), that

$$0 \leqslant \lim_{n\to\infty} \delta_{\varepsilon_n} \leqslant e_f(M_0) = 0. \tag{4.6}$$

(ii) Let $\mathcal{I}_\varepsilon = \{M_\varepsilon \in [\alpha, \beta]\colon \delta_\varepsilon = \tau_\varepsilon(M_\varepsilon)\}$. Now $\mathcal{I}_\varepsilon$ is a closed subset of $[\alpha, \beta]$ since $\mathcal{I}_\varepsilon = \tau_\varepsilon^{-1}(\delta_\varepsilon)$, then it is a compact set and therefore $\underline{M}_\varepsilon$ exists.

(iii) Assume there exists a subsequence $\left(\underline{M}_{\varepsilon_n}\right)_n$ of $(\underline{M}_\varepsilon)_\varepsilon$ such that $\lim_{n\to\infty} \underline{M}_{\varepsilon_n} = \underline{M}_0 < M_0$. Introduce the notation $g_\varepsilon(M) = g(f_\varepsilon, M)$, $g_0(M) = g(f, M)$, and similarly for the functions w_ε and w_0. By Theorem 3.5, we have

$$\left\| g_{\varepsilon_n}\left(\underline{M}_{\varepsilon_n}\right)\right\|_{L^2(J)} \to \|g_0(\underline{M}_0)\|_{L^2(J)},$$

and by (3.4),

$$\left\| w_{\varepsilon_n}\left(\underline{M}_{\varepsilon_n}\right)\right\|_{L^2(J)} \to \|w_0(\underline{M}_0)\|_{L^2(J)}.$$

Then

$$0 = \lim_{n\to\infty} \delta_{\varepsilon_n} = \tau_0(\underline{M}_0),$$

and we deduce that

$$\underline{M}_0 = \|g_0(\underline{M}_0) + w_0(\underline{M}_0)\|_{L^2(J)}.$$

Since

$$\| f - g_0(\underline{M}_0) - w_0(\underline{M}_0)\|_{L^2(I)} \leqslant \| f - g_0(\underline{M}_0)\|_{L^2(I)},$$

therefore $g_0(\underline{M}_0) + w_0(\underline{M}_0) = g_0(\underline{M}_0)$, which implies that $w_0(\underline{M}_0) = 0$, and in this case we deduce that $e_f(\underline{M}_0) = 0$; then $\| f - g_0(\underline{M}_0)\|_{L^2(I)} = 0$, i.e. $f = g_0(\underline{M}_0)$ and $M_0 = \underline{M}_0$, which is a contradiction.

(iv) This is a straightforward consequence of Theorem 3.5 and the point (iii) above. Note that weak convergence in $H^2(\partial G)$ implies weak convergence when restricted to $L^2(J)$, since the traces of functions in $H^2(\partial G)$ are dense in $L^2(J)$.

□

REMARK 4.3 Theorem 4.2 does not provide us with the actual bound for non-analytic data as Theorem 4.1 does for analytic data. However, it provides us with a family of bounds permitting one to compute extended data that converge—although weakly—to the required extension. The independent operation of the two tasks (bound determination and data extension) here reaches its limits. In the following subsections, we shall need to combine them again in order to build up robust reconstruction algorithms.

4.2 *The zero-order algorithm* ($\mathbf{A}_0$)

Theorem 4.2 does not ensure the convergence of the bound $\underline{M}_\varepsilon$ to M_0, since only weak convergence of the analytic extensions $g(f_\varepsilon, \underline{M}_\varepsilon)$ to f holds on J. Still, despite its ineffectiveness at least from a theoretical point of view, it is interesting to describe the so-called ‘zero-order’ algorithm that we shall use in the sequel as the basis on which higher-order algorithms will be built up.

The ($\mathbf{A}_0$) algorithm:

1. Given $M > 0$, solve the (BEP) problem with respect to (f_ε, M) and get $g_\varepsilon(M) = g(f_\varepsilon, M)$, and $e_{f_\varepsilon}(M) = \| f_\varepsilon - g_\varepsilon(M)\|_{L^2(I)}$.
2. Solve the (BEP) problem w.r.t. $\left(f_\varepsilon - g_\varepsilon(M)_{|I}, e_{f_\varepsilon}(M)\right)$ and get $w_\varepsilon = g\left(f_\varepsilon - g_\varepsilon(M)_{|I}, e_{f_\varepsilon}(M)\right)$;
3. Compute $\underline{M}_\varepsilon = \text{Argmin}_{M>0}\, \tau_\varepsilon(M)$ by some numerical method such as the golden section search, see Kiefer (1953);
4. Compute
$$q_\varepsilon = -\frac{\partial_\theta \operatorname{Im} g_\varepsilon(\underline{M}_\varepsilon)}{\operatorname{Re} g_\varepsilon(\underline{M}_\varepsilon)} \quad \text{on } J.$$

Note that 'Argmin' represents the value of the argument M at which the functional τ_ε achieves its minimal value.

The numerical implementation of this algorithm has been done using Matlab. The discrete Fourier transform function *fft* and the inverse discrete Fourier transform *ifft* have been used in order to compute the Fourier coefficients, whereas the Toeplitz matrix coefficients have been computed using the function *toeplitz*. The finite differences function *diff* has been used to compute the function derivatives.

4.2.1 *Solving the (BEP) problem for a prescribed multiplier λ.* We are given data (f_ε), a prescribed bound M and the related multiplier λ; we shall describe how to derive λ from the bound M and compute it, in the next sub-subsection. The solution of the (BEP) problem is obtained by solving the infinite linear equation (3.1), (3.2) given by Theorem 3.1. A discretization is needed, using a finite basis of Fourier functions $\{e_n(z), -N \leqslant n \leqslant N\}$. The proper value N to choose has been derived from an error study: given the data (4.4) with $a = 0.1$ and $c = 12$, and the noisy data f_ε derived from it by adding a perturbation of uniform norm varying from 1% to 15%, we have plotted the error $\| f - f_{\varepsilon,N}\|_{L^\infty(I)}$ between the actual data and the truncated noisy data (Fig. 3). It turns out actually that a value $N = 7$ (15 basis functions) is sufficient to bring this error below the noise level, if the level is around 15%. Furthermore, the figure shows that it is not worthwhile to choose more than 35 basis functions ($N = 17$), since the error is stabilized starting from that point. In order to fix our computations, and since these computations are quite cheap, we have, however, chosen $N = 25$ (51 basis functions).

Let us now describe the computations. Having prescribed data f on the part I of the boundary, discretizing equation (3.1) leads to the following:

$$((Id + \lambda \mathcal{T}_J) g_N)_N = P_{H^2}(\chi_J f_\varepsilon)_N, \tag{4.7}$$

which can also be written as follows:

$$[(\mathcal{T}_I + (1+\lambda)\mathcal{T}_J) g_N]_N = P_{H^2}(\chi_J f_\varepsilon)_N, \tag{4.8}$$

where $\mathcal{T}_I = P_{H^2}\chi_I$ is the Toeplitz operator associated to the characteristic function related to the part I of the boundary (in our implementation, $I = (\mathrm{e}^{-\mathrm{i}\theta_0}, \mathrm{e}^{\mathrm{i}\theta_0})$, $\theta_0 \in [0, \pi]$).

The Toeplitz matrices of operators $\mathcal{T}_J$ (and $\mathcal{T}_I = Id - \mathcal{T}_J$) with respect to the basis $\{e_n(z), n = -N, \ldots, N\}$ are obtained by truncating the infinite matrix $(\mathcal{T}_{n,m})_{(n,m)\in\mathbb{Z}\times\mathbb{Z}}$ given in (3.2) for

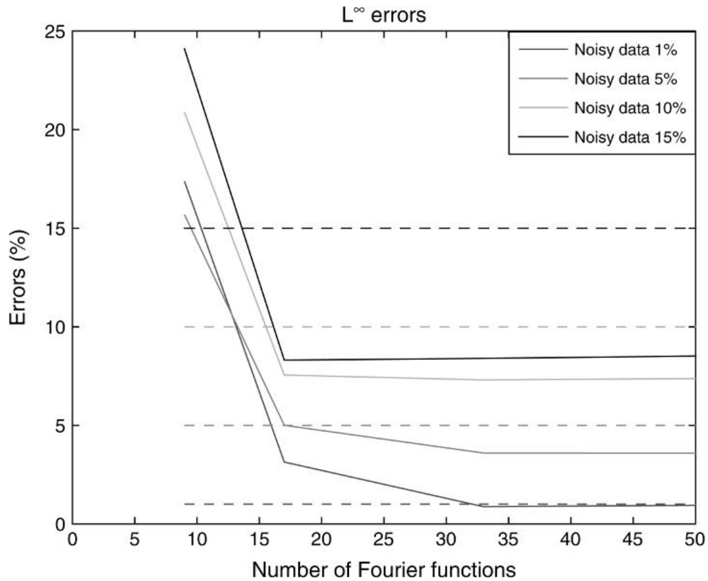

FIG. 3. Error approximations w.r.t. the number of Fourier functions used.

$-N \leqslant n, m \leqslant N$. The linear system (4.8) is finally solved using the Matlab. Matlab's Cholesky algorithm called *pinv*.

So far we have described how to go through items (1)–(2) of the above ($\mathbf{A}_0$) algorithm, provided the multiplier λ is known. Let us now describe how to derive it from the bound M.

4.2.2 *Determining the multiplier λ associated to the bound M.* It has already been mentioned in Proposition 2.1 and Remark 2.2 that the mapping $\lambda \to M(\lambda)$ is C^1, bijective and decreasing from $(-1, +\infty)$ to $(0, +\infty)$. Using the change of variables

$$\lambda = \frac{r}{1-r} - 1$$

gives us a function $M(\lambda(r))$ decreasing on $[0, 1)$. Furthermore, we know that the right value of λ is that ensuring that the computed (BEP) solution w.r.t. λ saturates the bound M, which actually means

$$V(\lambda) = \|g(f, \lambda)\|_{L^2(J)} = M.$$

Having prescribed some threshold, a bisection method has been used to find r, increasing r if $V(\lambda(r)) > M$ and decreasing it otherwise.

4.2.3 *Computing the right bound M.* Let us now describe how to compute the right bound M, which is Step 3 of the ($\mathbf{A}_0$) algorithm. Given a bound M, one needs to solve two (BEP) problems—as described by items (1)–(2) of the ($\mathbf{A}_0$) algorithm—in order to compute $\tau_\varepsilon(M)$:

1. Solve the (BEP) problem w.r.t. (f_ε, M) and get $g_\varepsilon(M) = g(f_\varepsilon, M)$.
2. Solve the (BEP) problem w.r.t. $\left(f_\varepsilon - g_\varepsilon(M)_{|I}, \|f_\varepsilon - g_\varepsilon(M)\|_{L^2(I)}\right)$ and get $w_\varepsilon(M)$.

Having done so, we have, as defined above in (4.1),

$$\tau_\varepsilon(M) = |\|g_\varepsilon\|_{L^2(J)} - \|g_\varepsilon + w_\varepsilon(M)\|_{L^2(J)}|.$$

Minimizing τ_ε w.r.t. M has been done using the golden section search method for $M \in [A, B]$.

In case the bound provided by the algorithm is equal to A (respectively B), one needs to run it once again, after enlarging the interval $[A, B]$ on the left-hand side (respectively on the right-hand side).

Finally, we compute the (BEP) extension g_ε associated with the bound M_ε so obtained, in order to compute the Robin coefficient, which requires us first to differentiate its imaginary part using finite differences.

4.3 *The higher-order algorithms* ($\mathbf{A}_n$)

The fourth statement in Theorem 4.2 can be seen as a weak robustness result for the ($\mathbf{A}_0$) algorithm. This is not strong enough even for the data extension, and it is definitely less than our needs for the impedance computation. This is the reason why, in search of better robustness properties, we shall now investigate higher-order algorithms based on the same tools.

The basic idea is actually to apply the above-described zero-order algorithm to the nth derivatives of the prescribed data and then to integrate n times the so extended derivatives.

Let $f_\varepsilon = f + \varepsilon$, where ε is a non-analytic, but still smooth, perturbation ($\varepsilon \in W^{n,2}(I) \setminus H^{n,2}(\partial G)_{|I}$), and assume $f \in H^{n,2}(\partial G)$. The ($\mathbf{A}_n$) algorithm is thus expressed as follows:

The ($\mathbf{A}_n$) algorithm:

1. Compute the nth derivative $f_\varepsilon^{(n)}$ of f_ε on I;
2. Apply the zero order method to the data $f_\varepsilon^{(n)}$, and get $g_\varepsilon^{(n)}$;
3. Integrate n times $g_\varepsilon^{(n)}$ and get $g_{n,\varepsilon}$;
4. Compute

$$q_{n,\varepsilon} = -\frac{\partial_\theta \operatorname{Im} g_{n,\varepsilon}}{\operatorname{Re} g_{n,\varepsilon}} \quad \text{on } J.$$

Because of the continuity properties of Section 3.2, these algorithms have much better robustness properties than the zero-order one. This is the content of the next theorem.

THEOREM 4.4 (Robustness of the nth-order method)
Suppose $\phi \in W^{n,2}(I)$, $q \in \mathcal{Q}^n$, $n \geqslant 1$. Let then $f_\varepsilon = u_d + \mathrm{i}\int \phi \, \mathrm{d}\theta + \varepsilon \in W^{n,2}(I)$ and $g_{n,\varepsilon}$ as above. As $\|\varepsilon\|_{W^{n,2}(I)} \to 0$, it holds that

$$\operatorname{Re} g_{n,\varepsilon} \to u \ \text{ in } W^{n,2}(\partial G), \quad \partial_\theta \operatorname{Im} g_{n,\varepsilon} \to \partial_n u \ \text{ in } W^{n-1,2}(\partial G).$$

Also

$$q_{n,\varepsilon} \to q \ \text{ in } W^{n-1,2}(J).$$

Proof. This is an immediate consequence of Theorem 3.6. □

REMARK 4.5 This 'robustness' result is obtained for smooth noise ($\varepsilon \in W^{n,2}(I)$), a feature that noise is actually not expected to have. Suppose now $\varepsilon \in L^{\infty}(I)$ with $|\varepsilon(x)| \leqslant \epsilon$ for x a.e. in I. Let us denote by $\tilde{f}^{\varepsilon}$ the smoothed function obtained by using cubic B-splines with a path length h. It has been proved in Chaabane *et al.* (2008) that we then have the following estimates:

$$\|\tilde{f}^{\varepsilon} - f\|_{L^{\infty}(I)} \leqslant c(\epsilon + h^2), \quad \|(\tilde{f}^{\varepsilon} - f)'\|_{L^{\infty}(I)} \leqslant c\left(\frac{\epsilon}{h} + h\right).$$

Choosing now $h = O(\sqrt{\epsilon})$, we get a $\sqrt{\epsilon}$ error on f', which means $(\tilde{f}^{\varepsilon})'$ can be seen as noisy data w.r.t. f', with a noise level $\sqrt{\epsilon}$. By 'bootstrapping' with the B-spline approximation, we can thus get an estimate of order $\epsilon^{\frac{1}{2p}}$ on the pth derivative of f.

This means that the smoothing of noisy data by using proper B-splines provides us with 'smoothed noisy data' that meet the assumptions of the above theorem. Actually, this is the way numerical results are usually run: data are smoothed prior to being processed. Observe that in this situation, a computational algorithm for a Hardy–Sobolev approximant can be directly used, as in Baratchart & Leblond (1992) in the framework of the unit disc.

In Section 5, we are going to confirm these robustness properties by a thorough numerical study which shows the efficiency of the higher-order methods in the tasks of both extending the data and recovering the electrical impedance coefficients.

5. Numerical validation

In the numerical results we are presenting in this section, we have considered the cases of both full prescribed data (i.e. data prescribed on the whole of the outer boundary) and incomplete data (i.e. data prescribed on some part of the outer boundary). The latter case is actually the most realistic one, particularly concerning non-destructive control applications. The impact on the outcome of several parameters has been studied:

- regularity of the data to be reconstructed,
- amount of prescribed data,
- noise level.

The non-singular data we have considered are those resulting from

$$f(z) = \exp(z), \tag{5.1}$$

whereas data with a singularity a in $s\mathbb{D}$ have been generated by (4.4) with $c = 12$. The closer a becomes to the circle $s\mathbb{T}$, with $s = 0.6$, the more 'singular' the data to reconstruct become.

5.1 *Case of full external data*

Extension formulae at order zero are provided by Chalendar *et al.* (2001), Chalendar & Partington (1999) and Smith (2005), and formulae for the $(\mathbf{A}_n)$ algorithms ($n = 0, 1, 2$) have been straightforwardly derived from them.

First, the non-singular data to be reconstructed are those resulting from the function (5.1). Figure 4 shows that the three methods (zero, first and second order) provide very accurate results, as regards the analytic extension as well as the electrical impedance computed from it.

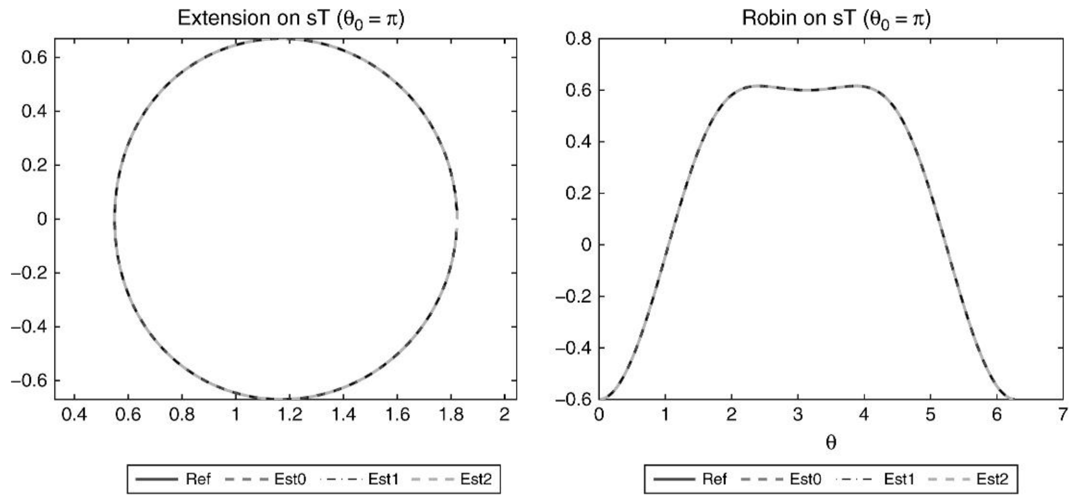

FIG. 4. Nyquist plot of the extended data ((Re $g_{n,\varepsilon}(z)$, Im $g_{n,\varepsilon}(z)$) as z varies in $s\mathbb{T}$) (left) and plot of the electrical impedance (right) obtained from full external smooth data and algorithms of order 0, 1, 2.

Things change however when it comes to noisy data, as can be noticed in Fig. 5. Although the extended data using the zero- and first-order methods remain acceptable up to a 10% level noise, the accuracy of the reconstructed electrical impedance drops dramatically when the noise level increases. Actually, the zero-order method turns out to be definitely unsuitable for the electrical impedance recovery task, whereas the first- and second-order ones behave quite well in that respect.

These conclusions were predictable from the theoretical results on robustness proved in Section 4. The zero-order method possesses only weak robustness properties, regarding the extended data but not the electrical impedance. From Theorem 4.4, we derive that the first-order method is the lowest possible ensuring L^2 convergence for the electrical impedance.

The sensitivity of the reconstruction method with respect to a, which parameterizes the singularity of the data generated by the function (4.4), is summarized in Fig. 6. As expected, the accuracy on the electrical impedance computed drops when a gets too close to the internal boundary. The first- and second-order methods do not show qualitative differences, though the second-order one is more accurate, even for data singular near $s\mathbb{T}$. The next part of the study will thus focus on the second-order algorithm.

5.2 *Case of incomplete external data*

In this section, we are concerned with the behaviour of the algorithm when data are lacking on some part of the outer boundary. This situation is likely to happen quite often in practice, and this is the reason why we have run quite an extensive numerical study, investigating the impact of the following parameters on the result:

- amount of prescribed data, as measured by the ratio $\rho = \frac{|I|}{2\pi}$, where $|I|$ is the Lebesgue measure of the prescription area I on the outer boundary, whose length is 2π;

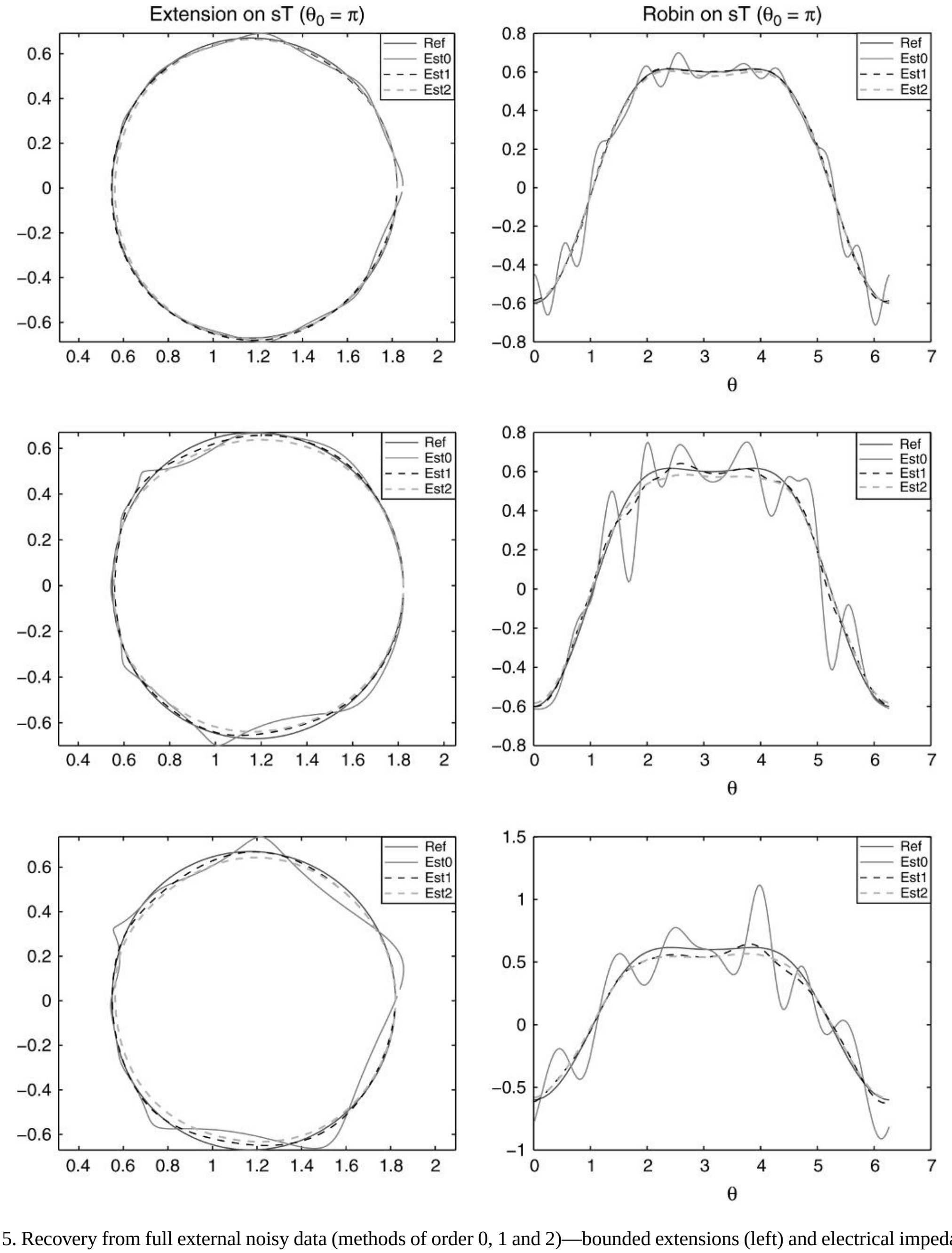

FIG. 5. Recovery from full external noisy data (methods of order 0, 1 and 2)—bounded extensions (left) and electrical impedances (right) for 1%, 5% and 10% noise.

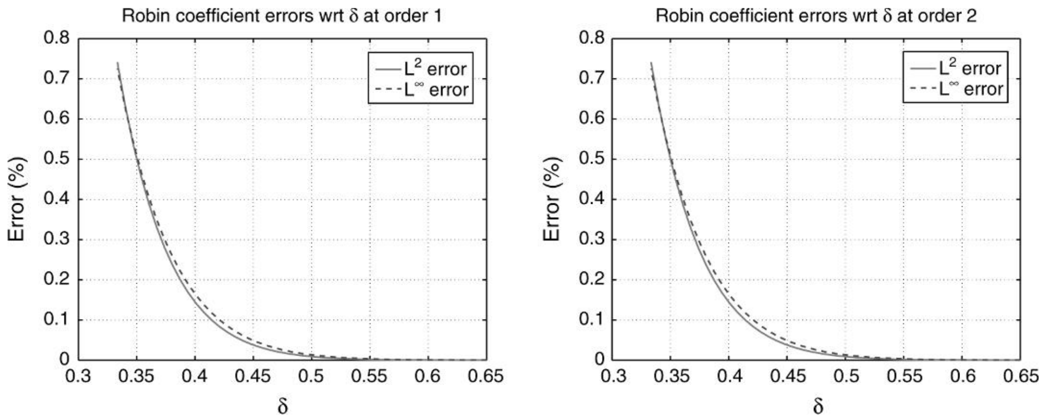

FIG. 6. Plot of L^{∞}- and L^2-errors on q w.r.t. the distance of the singularity to the inner boundary: first-order (left) and second-order method (right).

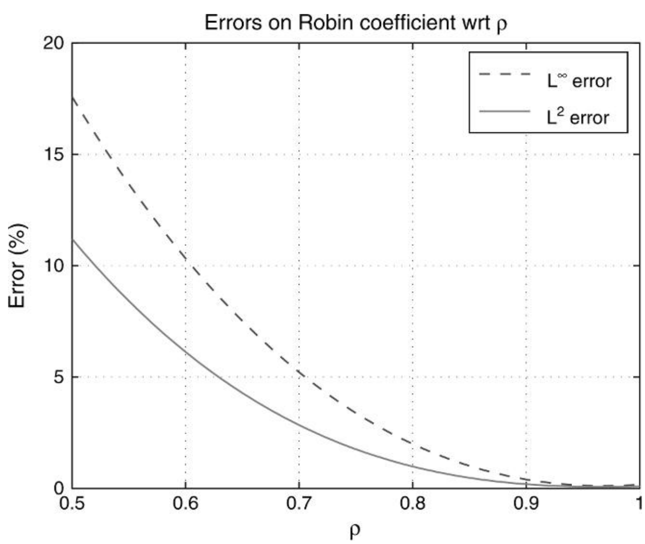

FIG. 7. Plot of L^2- and L^{∞}-errors w.r.t. the amount of prescribed data on the outer boundary (second-order algorithm).

- singularity of the data, as parameterized by $\delta = \frac{1}{s} d(a, s\mathbb{T}) = 1 - |a|/s$, where a is the complex number defined in the previous section (location of a singularity inside the inner disc);
- noise level.

In the case of non-noisy data (5.1), Fig. 7 shows that the error on the Robin coefficient remains acceptable for as small a quantity of data as that prescribed on half the outer boundary, and the error decreases quite fast with respect to the amount of prescribed data.

Let us now study the sensitivity of the reconstruction method to the data regularity. By making a closer to the circle $s\mathbb{T}$, the behaviour of the function (4.4) gets harsher, though remaining smooth as stated in Theorem 4.4. Unsurprisingly, the left-hand plot in Fig. 8 shows that the harsher the data, the lower the accuracy on the computed electrical impedance. However, the plots in the right-hand side of Fig. 8 also indicate how to make up for the lack of regularity by increasing the amount of prescribed data. Highly singular functions need an almost complete set of external data in order to compute the electrical impedance with an acceptable accuracy.

The noise effects are somewhat similar. The right-hand plot of Fig. 9 displays curves relating the noise level to the amount of prescribed data for different targeted error levels. Once again, we observe that to some extent, one can make up for the noise effects by increasing the amount of available data.

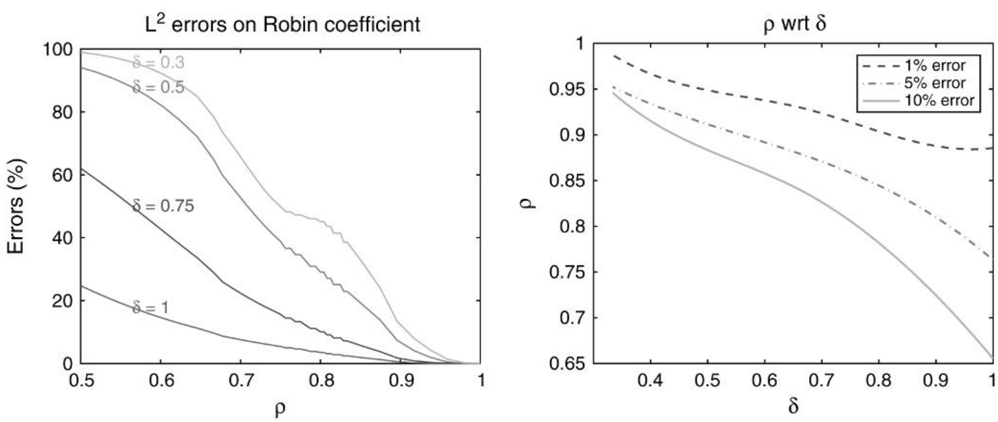

FIG. 8. Plots of errors w.r.t. amount of data (left) and of ρ versus δ for a 1%, 5% and 10% error level (right).

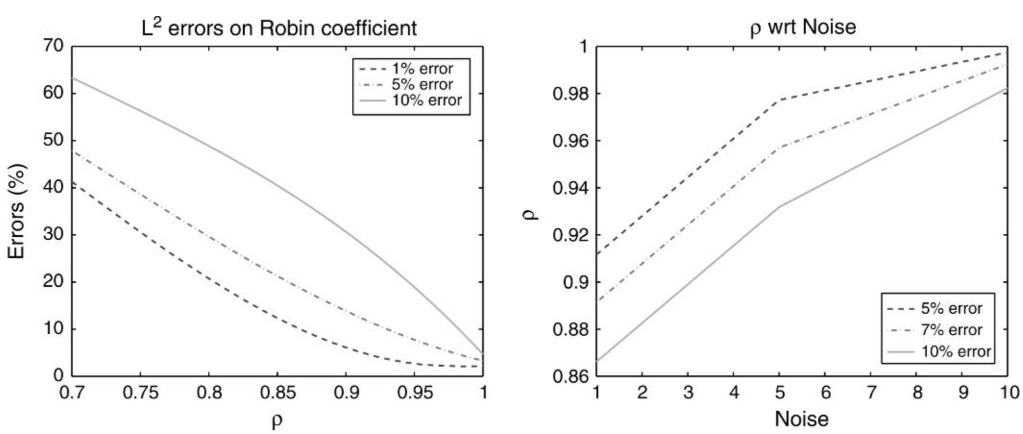

FIG. 9. Plots of errors w.r.t. ρ in the case of noisy data (left) and of ρ versus noise at various error levels.

6. Conclusion

The methods we have been presenting in this work constitute a family of fast data completion algorithms solving the Cauchy problem for the Laplace equation in an annular domain, and up to a conformal mapping, in similar domains in the plane. The main goal was to compute accurately from these data the electrical impedance on the inaccessible inner part of the boundary from the extended data.

To that end, we have derived new explicit formulae in the case when the set of available data on the outer boundary is not complete. These formulae have been implemented in order to build up algorithms using the BEPs and needing the actual bound on the unknown data to be computed at the same time as the data are extended. These algorithms use a new stabilization technique that proves to be fast and efficient. Beside their efficiency, the so designed algorithms have been proved to be robust with respect to noise, and a thorough numerical study has been run that widely confirms these theoretical predictions.

Despite their valuable qualities (accuracy and robustness at a low computational cost), these algorithms have two limitations. The first is related to the exclusive focus on the Laplace equation. Though not restrictive for corrosion detection, this limitation would need to be lifted, since extensions to other operators such as Maxwell's equations electroencephalography (EEG) and magnetoencephalography (MEG) would be highly appreciated. This is not, however, a straightforward extension of the present work. On the other hand, lifting the limitation to 2D problems is the crucial issue to investigate in order to deal with 'real-life problems'. But before tackling these two challenging developments, the next step is to study 'real 2D problems', i.e. 2D problems in other domains than the annulus, in order to obtain a clearer idea of how the conformal mapping affects the numerical results.

Extensions of these methods to 3D domains are a subject of current investigation, see Atfeh *et al.* (2008).

Acknowledgement

The authors thank the referees for their constructive comments.

Funding

AIRE *développement*, using resources allocated by the French Ministry of Foreign Affairs; the cooperative research program (Direction Générale de la Recherche Scientifique et Technique, Tunisia) - INRIA, Institut Francais de Cooperation and the Tunisian Ministry for Higher Education, Scientific Research and Technology (to the LAMSIN researchers' work); the British Engineering and Physical Sciences Research Council.

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Appendix

Proof of Lemma 3.3.
From Smith (2005), we can write

$$P_{H^2(\partial G)}\phi(z) = \sum_{n\in\mathbb{Z}} \frac{a_n + b_n s^{2n}}{1+s^{2n}} z^n,$$

where

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \phi(e^{i\theta})e^{-in\theta}d\theta = \frac{1}{2\pi}\int_{-\theta_0}^{\theta_0} \phi(e^{i\theta})e^{-in\theta}\,d\theta + \frac{1+\lambda}{2\pi}\int_{\theta_0}^{2\pi-\theta_0} \phi(e^{i\theta})e^{-in\theta}\,d\theta$$

$$= \frac{1}{2\pi}\int_{-\theta_0}^{\theta_0} f(e^{i\theta})e^{-in\theta}\,d\theta + \frac{1+\lambda}{2\pi}\int_{\theta_0}^{2\pi-\theta_0} f_1(e^{i\theta})e^{-in\theta}\,d\theta$$

and

$$b_n s^n = \frac{1}{2\pi}\int_0^{2\pi} \phi(s\,e^{i\theta})e^{-in\theta}\,d\theta = \frac{1+\lambda}{2\pi}\int_0^{2\pi} f_1(s\,e^{i\theta})e^{-in\theta}\,d\theta.$$

□

Proof of Lemma 3.4.
Let $\phi = \chi_J g$. Then we have from Lemma 3.3

$$P_{H^2(\partial G)}\phi(z) = \sum_{n\in\mathbb{Z}} \frac{c_n + d_n s^{2n}}{1+s^{2n}} z^n,$$

where

$$c_n = \frac{1}{2\pi}\int_{\theta_0}^{2\pi-\theta_0} g(e^{i\theta})e^{-in\theta}\,d\theta$$

and

$$d_n s^n = \frac{1}{2\pi}\int_0^{2\pi} g(s\,e^{i\theta})e^{-in\theta}\,d\theta.$$

Let $g(z) = \sum_{n\in\mathbb{Z}} g_n z^n$ for $z \in G$, then

$$c_n = \frac{1}{2\pi}\int_{\theta_0}^{2\pi-\theta_0} \sum_{m\in\mathbb{Z}} g_m\, e^{im\theta}\, e^{-in\theta}\,d\theta$$

$$= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} g_m \int_{\theta_0}^{2\pi-\theta_0} e^{i(m-n)\theta}\,d\theta$$

$$= g_n\left(1 - \frac{\theta_0}{\pi}\right) - \sum_{m\neq n} g_m \frac{\sin(m-n)\theta_0}{\pi(m-n)}$$

and

$$d_n s^n = \frac{1}{2\pi}\int_0^{2\pi}\sum_{m\in\mathbb{Z}} g_m s^m\, \mathrm{e}^{\mathrm{i}m\theta}\, \mathrm{e}^{-\mathrm{i}n\theta}\, \mathrm{d}\theta = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} g_m s^m \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(m-n)\theta}\, \mathrm{d}\theta = g_n s^n.$$

Then we deduce that

$$P_{H^2(\partial G)}\chi_J g(z) = \sum_{n\in\mathbb{Z}} \frac{1}{1+s^{2n}}\left(g_n\left(1-\frac{\theta_0}{\pi}\right) + g_n s^{2n} - \sum_{m\neq n} g_m \frac{\sin{(m-n)\theta_0}}{\pi\,(m-n)}\right) z^n;$$

therefore,

$$\mathcal{T}g(z) = \sum_{n\in\mathbb{Z}} \frac{1}{1+s^{2n}}\left(g_n\left(1+s^{2n}-\frac{\theta_0}{\pi}\right) - \sum_{m\neq n} g_m \frac{\sin{(m-n)\theta_0}}{\pi\,(m-n)}\right) z^n. \qquad \square$$