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Source localization in ellipsoids by the best meromorphic approximation in planar sections

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Abstract

We consider the inverse problem of localizing dipolar sources in an ellipsoid from boundary data, which we approach and constructively solve with techniques from harmonic and complex analysis. We use ellipsoidal harmonics to isolate the singular part of the solution, which we consider on a family of two-dimensional sections of the domain. We then use approximation theory to locate its singularities, and provide an algorithm which allows us to recover the sources from these singularities. We provide numerical illustrations related to the localization of pointwise dipolar sources in the human brain from electroencephalography (EEG).

(Some figures in this article are in colour only in the electronic version)

1. Introduction

A classical inverse problem for the Laplace operator will be considered in this paper. It consists of the recovery of finitely many dipolar sources located in a 3D domain from measurements of their potential and current flux (normal derivative) on the boundary, see [20, 28, 33, 40].

For instance, in medical engineering, the inverse EEG (electroencephalography) problem involves detecting pointwise dipolar current sources, modeling epileptic foci located in the brain, from measurements of the electrical potential on the scalp, see [15, 21, 22].

The problem covers geophysics interests too, more precisely the determination of the mass–density distribution of the earth, which can be approximated by an ellipsoid, or that of equipotential surfaces (geoids), from measurements of the gravitational potential at the earth's surface (see, for example [24–26]).

Concerning the EEG issue, the so-called spherical model, where the 3D domain is a ball, has been handled in [5] where it appears that this recovery issue reduces to a sequence of 2D inverse problems. All 2D issues concern the recovery of the branchpoints and singularities of some holomorphic function in a disk. These problems may then be solved using the best rational or meromorphic approximation algorithms, see, for instance [1]. Finally, those 2D singularities are strongly linked with the original dipolar sources, which they allow us to approximately locate.

Though various possible approaches to this inverse problem exist, they mainly deal with the iterative resolution of the associated direct problem, see, e.g. [28]. We will describe here an efficient algorithm based on meromorphic approximation in families of planar cross-sections of Ω. Those are costless since they run with boundary data only.

We will investigate in this paper the case where the 3D domain is an ellipsoid. Concerning the EEG inverse problems, this is a first step toward a more realistic geometry for the brain, see [29] for the computation of solutions to the direct problem, [38] for basic potential problems in electrostatics and [24–26] for geodetic applications.

We will show that if the 3D domain is an ellipsoid, then the issue can still be approached by a sequence of 2D inverse problems in a family of ellipses (cross-sections), hence in a family of disks as in the spherical case, using the fact that the boundary of an ellipse is the image by a rational function of the unit circle. We will then face once again the issue of recovering the singularities that a holomorphic function has in a disk.

Let us first briefly recall how the problem is solved in the case where $\Omega = \mathbb{B}$ is the unit ball and $\partial\Omega = \mathbb{S}$ is the unit sphere. Let $v(X)$ be the difference of potential with respect to $X \in \bar{\Omega}$. The following equation with mixed boundary conditions arises when studying the EEG inverse problem, and assuming the conductivity of the head to be constant:

$$\begin{cases} \Delta v = F & \text{in} \quad \mathbb{R}^3 \\ v = g & \text{on} \quad \partial\Omega \\ \dfrac{\partial v}{\partial \vec{n}} = \phi & \text{on} \quad \partial\Omega, \end{cases}$$

where $g \in L^2(\partial\Omega)$ and $\phi \in L^2(\partial\Omega)$ are the given measurements of the difference of the potential and current flux on (the scalp) $\partial\Omega$, with

$$\int_{\partial\Omega} \phi \, \mathrm{d}s = 0,$$

and

$$F = \sum_{k=1}^{m} \langle p_k, \nabla\delta_{c_k} \rangle$$

is the distribution corresponding to the pointwise dipolar sources located at $c_k \in \Omega$ with moments $p_k \in \mathbb{R}^3$.

Since

$$E(X) = \frac{1}{4\pi \|X\|}$$

is a fundamental solution of the Laplacian in $\mathbb{R}^3$, we get that

$$v(X) = h(X) + \sum_{k=1}^{m} \frac{\langle p_k, X - c_k \rangle}{4\pi \|X - c_k\|^3}, \qquad X \in \Omega, \tag{1}$$

where h is a harmonic function in $\mathbb{R}^3$.

The decomposition of a harmonic function in a neighborhood of the unit sphere in terms of spherical harmonics using the knowledge of the function and normal derivative gives an explicit decomposition of this function as the sum of two functions, one harmonic in a neighborhood of the unit ball, and the other one harmonic outside a compact set included in the unit ball (cf [2, theorem 9.6]). This explicit decomposition gives exactly the singular term

$$u(X) = \sum_{k=1}^{m} \frac{\langle p_k, X - c_k \rangle}{4\pi \|X - c_k\|^3}.$$

Then we aim at recovering the singularities of $u^2(x, y, z_p)$ for a family of cross-sections $\{z = z_p\}$ of the sphere Ω from its values on the 2D boundaries (circles). This is done by means of the best meromorphic approximation of the function $[u_{|\{z=z_p\}}]^2$ on the corresponding circle. The poles of such an approximant allow us to approximately locate the m singularities of $[u_{|\{z=z_p\}}]^2$ in $\mathbb{D}$ which finally enable us to recover the location of the sources c_k [5].

If Ω is an ellipsoid, the above decomposition $v = h + u$ can still be explicitly computed with an expansion into ellipsoidal harmonics. The next step will also consist in recovering the singularities of $u^2(x, y, z_p)$ for a family of cross-sections $\{z = z_p\}$ of Ω from its values on the 2D boundaries (ellipses).

This is done by using a rational correspondence φ_p between these ellipses $\{z = z_p\} \cap \partial\Omega$ and the unit circle $\mathbb{T}$, and then, as in the spherical situation, by means of the best meromorphic approximation of the function $f_p = [u_{|\{z=z_p\}} \circ \varphi_p]^2$ on $\mathbb{T}$. Again, the poles of the approximant allow us to approximately locate the $2m$ singularities of f_p in $\mathbb{D}$ and, finally, the sources c_k.

Let us specify that the singularities of $u^2(x, y, z_p)$ induced by the dipolar sources will appear also as polar singularities of order 3, whence in the expression of $u(x, y, z_p)$, they appear only as ramified singularities of order 3/2. This is a framework in which the above rational approximation algorithms are more efficient.

The main steps of the recovery algorithm that we consider in this work are thus summarized as follows:

Step 1: get the (trace on the boundary of the) singular part $u_{|\partial\Omega}$ from the data $v_{|\partial\Omega} = g$, $(\partial v/\partial \vec{n})_{|\partial\Omega} = \phi$, using ellipsoidal harmonics;
Step 2: find (approximately) the singularities (branchpoints) $\zeta^{k,p}$ of $f_p = [u_{|\{z=z_p\}} \circ \varphi_p]^2$ using
(2.i) a rational correspondence φ_p between the ellipse $\{z = z_p\} \cap \partial\Omega$ and the circle $\mathbb{T}$,
(2.ii) the best approximation on $\mathbb{T}$ by meromorphic functions in the disk;
Step 3: recover c_k from the estimated $\zeta^{k,p}$.

The ellipsoidal harmonics (required for step 1) will be briefly explained in section 2, which accounts for a synthesis of available results in the literature [14, 23, 41].

In section 3, we briefly recall some results about the best meromorphic approximation and potential theory in the complex plane, see [5] and references therein, that are to the effect that the poles of the approximants converge (in a sense to be made precise) to the singularities as the degree increases; this is the basis of step (2.ii).

We will then be back to the inverse problem in section 4, and explain how to localize the sources inside the ellipsoid from related 2D singularities in disks, which is step 3, using the fact that the boundaries of ellipses are the images under rational mappings of the unit circle, for step (2.i).

We end up in section 5 with a numerical illustration of the above scheme and some concluding comments in section 6.

2. Ellipsoidal harmonics

In this section, we will give a precise definition of ellipsoidal harmonics. The reader may be referred to [14, 16, 17, 23, 41] for a more complete treatment and [30, so.2] for a self-contained exposition of what we need precisely in this paper.

Now let $\partial\Omega = \mathcal{E}$ be the ellipsoid whose Cartesian equation is

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1, \tag{2}$$

where $a_1 > a_2 > a_3 > 0$. Let $h_1 > 0$, $h_2 > 0$ and $h_3 > 0$ be the semifocal distances defined by

$$h_1^2 = a_2^2 - a_3^2, \qquad h_2^2 = a_1^2 - a_3^2, \qquad h_3^2 = a_1^2 - a_2^2.$$

For $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ such that $0 < \lambda_3^2 < h_3^2 < \lambda_2^2 < h_2^2 < \lambda_1^2$, we consider the following three surfaces (ellipsoid, hyperboloïd of one sheet and hyperboloïd of two sheets) whose equations are

$$\begin{aligned}
&(\mathcal{E}_{\lambda_1}) : \frac{x_1^2}{\lambda_1^2} + \frac{x_2^2}{\lambda_1^2 - h_3^2} + \frac{x_3^2}{\lambda_1^2 - h_2^2} = 1,\\
&\frac{x_1^2}{\lambda_2^2} + \frac{x_2^2}{\lambda_2^2 - h_3^2} - \frac{x_3^2}{h_2^2 - \lambda_2^2} = 1,\\
&\frac{x_1^2}{\lambda_3^2} - \frac{x_2^2}{h_3^2 - \lambda_3^2} - \frac{x_3^2}{h_2^2 - \lambda_3^2} = 1.
\end{aligned} \tag{3}$$

Denoting $k_1^2 = 0$, $k_2^2 = h_3^2$ and $k_3^2 = h_2^2$, one can write these three equations under the form

$$\sum_{j=1}^{3} \frac{x_j^2}{\lambda_i^2 - k_j^2} = 1, \qquad i = 1, 2, 3, \tag{4}$$

and we find that for $i = 1, 2, 3$

$$x_i^2 = \frac{\prod_{j=1}^{3}\left(\lambda_j^2 - k_i^2\right)}{\prod_{\substack{j=1,2,3\\ j\neq i}}\left(k_j^2 - k_i^2\right)}. \tag{5}$$

We then say that $(\lambda_1, \lambda_2, \lambda_3)$ are the ellipsoidal coordinates of the point (x_1, x_2, x_3).

In this ellipsoidal coordinates system, the equation $\Delta V = 0$ can be rewritten as

$$\begin{aligned}
\Delta'_\lambda(V) := {}& \left(\lambda_2^2 - \lambda_3^2\right)\left[\left(\lambda_1^2 - h_2^2\right)\left(\lambda_1^2 - h_3^2\right)\frac{\partial^2 V}{\partial \lambda_1^2} + \lambda_1\left(2\lambda_1^2 - h_2^2 - h_3^2\right)\frac{\partial V}{\partial \lambda_1}\right]\\
&+ \left(\lambda_3^2 - \lambda_1^2\right)\left[\left(\lambda_2^2 - h_2^2\right)\left(\lambda_2^2 - h_3^2\right)\frac{\partial^2 V}{\partial \lambda_2^2} + \lambda_2\left(2\lambda_2^2 - h_2^2 - h_3^2\right)\frac{\partial V}{\partial \lambda_2}\right]\\
&+ \left(\lambda_1^2 - \lambda_2^2\right)\left[\left(\lambda_3^2 - h_2^2\right)\left(\lambda_3^2 - h_3^2\right)\frac{\partial^2 V}{\partial \lambda_3^2} + \lambda_3\left(2\lambda_3^2 - h_2^2 - h_3^2\right)\frac{\partial V}{\partial \lambda_3}\right] = 0.
\end{aligned} \tag{6}$$

Applying the method of separation of variables, we obtain that, for every natural integer n, there are $2n + 1$ linearly independent functions $E_n^{p_k^n}$ for $k = 1, \ldots, 2n + 1$ which are polynomials of degree n in λ, $\sqrt{\lambda^2 - h_2^2}$ and $\sqrt{\lambda^2 - h_3^2}$ such that

$$E_n^{p_k^n}(\lambda_1) E_n^{p_k^n}(\lambda_2) E_n^{p_k^n}(\lambda_3)$$

are solutions to (6).

These functions are called the ellipsoidal harmonics of the first kind.

For a given n, the $2n + 1$ functions

$$E_n^{p_k^n}(\lambda_1) E_n^{p_k^n}(\lambda_2) E_n^{p_k^n}(\lambda_3), \qquad k = 1, \ldots, 2n + 1,$$

are polynomials of degree n with respect to the Cartesian coordinates.

With respect to ellipsoidal coordinates, the normal derivative of a function f at some point $(x_1, x_2, x_3) \in \mathcal{E}_{\lambda_1}$ is

$$\left(\sqrt{\frac{\left(\lambda_1^2 - h_2^2\right)\left(\lambda_1^2 - h_3^2\right)}{\left(\lambda_1^2 - \lambda_2^2\right)\left(\lambda_1^2 - \lambda_3^2\right)}}\right) \frac{\partial f}{\partial \lambda_1}(\lambda_1, \lambda_2, \lambda_3). \tag{7}$$

The following propositions and theorem close this section. Let dS be the Lebesgue surface measure on $\mathcal{E}_{\lambda_1}$.

Proposition 2.1. *Let $\lambda_1 \in]h_2, +\infty[$. For two real-valued functions f and g defined and continuous on the ellipsoid $\mathcal{E}_{\lambda_1}$, we define*

$$\langle f, g\rangle_{\lambda_1} = \int_{\mathcal{E}_{\lambda_1}} fg \, \mathrm{d}S.$$

Then for every $\lambda_1 \in]h_2, +\infty[$, the family of polynomials

$$\left\{E_n^{p_k^n}(\lambda_1) E_n^{p_k^n}(\lambda_2) E_n^{p_k^n}(\lambda_3)\right\}_{n,k} = \left\{V_n^{p_k^n}(x_1, x_2, x_3)\right\}_{n, p_k^n}$$

is orthogonal with respect to $\langle\cdot,\cdot\rangle_{\lambda_1}$.

Proposition 2.2. *Let $\lambda_1 > h_2$. Then every function $f \in L^2\left(\mathcal{E}_{\lambda_1}, \mathrm{d}S\right)$ can be written as*

$$f(\lambda_2, \lambda_3) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \alpha_n^k E_n^{p_k^n}(\lambda_2) E_n^{p_k^n}(\lambda_3),$$

where

$$\alpha_n^k = \frac{\left\langle f, E_n^{p_k^n}(\lambda_2) E_n^{p_k^n}(\lambda_3)\right\rangle_{\lambda_1}}{\left\langle E_n^{p_k^n}(\lambda_2) E_n^{p_k^n}(\lambda_3), E_n^{p_k^n}(\lambda_2) E_n^{p_k^n}(\lambda_3)\right\rangle_{\lambda_1}}.$$

Theorem 2.1. *Every function f harmonic in a neighborhood of the ellipsoid $\mathcal{E} = \mathcal{E}_{a_1}$ can be written in ellipsoidal coordinates in the form*

$$f(\lambda_1, \lambda_2, \lambda_3) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \alpha_n^k E_n^{p_k^n}(\lambda_1) E_n^{p_k^n}(\lambda_2) E_n^{p_k^n}(\lambda_3) + \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \beta_n^k F_n^{p_k^n}(\lambda_1) E_n^{p_k^n}(\lambda_2) E_n^{p_k^n}(\lambda_3),$$

where

$$F_n^p(\lambda_1) = (2n+1) E_n^p(\lambda_1) \int_{\lambda_1}^{\infty} \frac{\mathrm{d}\xi}{\sqrt{\left(\xi^2 - h_2^2\right)\left(\xi^2 - h_3^2\right)\left[E_n^p(\xi)\right]^2}}. \tag{8}$$

Moreover, there exists r_1 and r_2 such that $r_1 < 1 < r_2$ and such that the first sum converges uniformly to a harmonic function inside the solid ellipsoid

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} < r_2, \tag{9}$$

and such that the second sum converges uniformly to a harmonic function outside the solid ellipsoid

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} < r_1. \tag{10}$$

The functions $F_n^{p_k^n}(\lambda_1) E_n^{p_k^n}(\lambda_2) E_n^{p_k^n}(\lambda_3)$ are called the ellipsoidal harmonics of second kind. Observe that, in the above results, the quantities involving p_k^n are explicitly available only for small values of n, while they must be numerically computed for larger n. This is feasible, however, since the p_k^n are the roots of explicit polynomials and because $E_n^{p_k^n}$ satisfy (also explicit) recurrence formulae, see [16, 30] and related comments in section 6.

3. About meromorphic approximation in $\mathbb{D}$

3.1. The best meromorphic approximation

Let H^2 and H^∞ be the familiar Hardy spaces of the disk:

$$H^2 = \left\{ f \text{ analytic in } \mathbb{D}, \sup_{r<1} \int_0^{2\pi} |f(r\,\mathrm{e}^{\mathrm{i}\theta})|^2 \,\mathrm{d}\theta < +\infty \right\}$$

$$= \left\{ f \in L^2(\mathbb{T}), f(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{n=0}^{\infty} a_n \,\mathrm{e}^{\mathrm{i}n\theta}, (a_n) \in \ell^2 \right\}$$

and

$$H^\infty = \left\{ f \text{ analytic in } \mathbb{D}, \sup_{r\,\mathrm{e}^{\mathrm{i}\theta}\in\mathbb{D}} |f(r\,\mathrm{e}^{\mathrm{i}\theta})| < +\infty \right\}.$$

We denote by $\bar{H}_0^2$ the orthogonal subspace of H^2 in $L^2(\mathbb{T})$. Let $\mathcal{R}_n = \mathcal{R}_n(\mathbb{D})$ be the set of rational fractions having at most n poles in $\mathbb{D}$. Introduce the set H_n^q of meromorphic function with at most n poles in $\mathbb{D}$ by setting, for $q = 2, \infty$,

$$H_n^q = H^q + \mathcal{R}_n.$$

Let now $f \in L^q(\mathbb{T})$, for $q = 2, \infty$. A best meromorphic $L^q(\mathbb{T})$-approximant to f with at most n poles is a function $R_n \in H_n^q$ such that

$$\|f - R_n\|_{L^q(\mathbb{T})} = \inf_{R\in H_n^q} \|f - R\|_{L^q(\mathbb{T})}. \tag{11}$$

For $p = \infty$, by the Adamyan–Arov–Krein theory [1, 35, 36, 42], a best meromorphic approximant with at most n poles uniquely exists provided that $f \in H^\infty + C(\mathbb{T})$. Moreover, it can be computed from the singular value decomposition of the Hankel operator with the symbol f. More precisely, let P_+ (resp. P_-) be the orthogonal (analytic) projection from $L^2(\mathbb{T})$ into H^2 (resp. $\bar{H}_0^2$), and Γ_f be the Hankel operator defined by

$$\Gamma_f : H^2 \to \bar{H}_0^2, \qquad g \mapsto P_-(fg).$$

It follows from Hartman's theorem [35, theorem 2.2.5] that Γ_f is compact if, and only if, $f \in H^\infty + C(\mathbb{T})$, and from Kronecker's theorem [35, theorem 2.4.4] that Γ_f is of finite rank if, and only if, $f \in H^\infty + \mathcal{R}_N$, for some $N \geqslant 0$.

Assume that $f \in H^\infty + C(\mathbb{T})$. If we take the sequence of singular values σ_m of Γ_f arranged decreasingly, and the associated singular vectors v_m satisfying the following:

$$\Gamma_f^* \Gamma_f(v_m) = \sigma_m^2 v_m$$

((σ_m, v_m) is the so-called nth Schmidt pair of Γ_f), we get that the solution R_n to (11) (with $q = \infty$) is given by

$$R_n = f - \frac{\Gamma_f v_n}{v_n} = \frac{P_+(f v_n)}{v_n}.$$

Further, $|f - R_n| = \sigma_n$ a.e. on $\mathbb{T}$ (circular error). Hence, R_n can be computed using the singular value decomposition of the Hankel operator Γ_f.

Additional results concerning the convergence properties of R_n and the continuity properties of the best approximation operator may be found in [36].

For $q = 2$ and $f \in L^2(\mathbb{T})$, the existence and uniqueness properties of solutions to (11) are discussed in [3, 6, 7, 10]. Constructive algorithms to generate local minima can be obtained using Schur parametrization [31, 32].

Note that such approximants are more generally studied in [9] for every $q \geqslant 2$.

3.2. Behavior of poles

Let $\mathcal{A}_\varepsilon \subset \mathbb{D}$ be the annulus,

$$\mathcal{A}_\varepsilon = \{z \in \mathbb{D}, 1 - \varepsilon < |z|\},$$

and $\mathcal{BP}$ be the class of functions that are continuous in $\bar{\mathcal{A}}_\varepsilon$ and holomorphic in $\mathcal{A}_\varepsilon$, for some e > 0, and that can be analytically extended to $\mathbb{D}$ except for finitely many poles and branchpoints.

Theorem 3.1 [7, 37]. *If $f \in \mathcal{BP}$, there exists a unique connected open $V_f \subset \bar{\mathbb{D}}$ with $\bar{\mathcal{A}}_\varepsilon \subset V_f$ such that f extends holomorphically to V_f which has the properties that $\mathbb{D}\backslash V_f$ is of minimal Green capacity among such sets and that V_f contains every such sets. Furthermore, $\mathbb{D}\backslash V_f$ consists of the poles and branchpoints of f and finitely many analytic cuts (between branchpoints, with no loops).*

Recall that the Green capacity $C_K \geqslant 0$ of a compact set $K \subset \mathbb{D}$ is defined by

$$\frac{1}{C_K} = \inf_{\mu\in\mathcal{P}_K} \int\!\!\int \log\left|\frac{1-\bar{t}z}{z-t}\right| \mathrm{d}\mu(t)\,\mathrm{d}\mu(z),$$

where $\mathcal{P}_K$ is the set of all probability measures supported on K. If $C_k > 0$, there exists a unique measure in $\mathcal{P}_K$ which achieves the above infimum: it is the Green equilibrium measure of K.

The set V_f is the so-called extremal domain associated with f.

Denote by $(s_{j,n})_{j=1,\ldots,n}$ the poles in $\mathbb{D}$, counted with their multiplicities, of the solution $R_n = R_n(f)$ to (11). Define the sequence of their counting probability measure by

$$\mu_n = \mu_n(f) = \frac{1}{n}\sum_{j=1}^{n} \delta_{s_{j,n}}.$$

Theorem 3.2 [8]. *If $f \in \mathcal{BP}$ is not single valued, then the measure μ_n converges weak-* to the Green equilibrium distribution of $\mathbb{D}\backslash V_f$ as $n \to \infty$.*

Note that analogous results have been recently obtained in [11, 12] for classes of functions that are proper subsets of $\mathcal{BP}$, namely sums of Cauchy-type integrals and rational functions; they still concern the behavior of poles, but also convergence properties of the approximants themselves.

We use theorem 3.2 to approach the inverse source problem, which consists in recovering singularities that appear both as branchpoints and poles of f. Indeed, the counting measure μ_n will asymptotically charge the endpoints of $\mathbb{D}\backslash V_f$, because the equilibrium distribution is infinite there. By theorem 3.1, $\mathbb{D}\backslash V_f$ includes the poles and branchpoints of f, whence computing μ_n for increasing values of n will allow us to approximately locate them. Of course, one has to keep in mind that computations are in practice restricted to limited values of n, thus the quality of the recovery scheme will strongly depend on how fast the poles converge.

4. Localization of the sources

Recall from section 1 that we aim at recovering the sources $c_k \in \Omega$ being given the values of the function v of the form (1) and that of its normal derivative on the ellipsoid $\mathcal{E} = \partial\Omega$. We now explain how to handle this issue in several steps, using the results of sections 2 and 3.

4.1. The use of ellipsoidal harmonics for computing the singular part: step 1

Thanks to theorem 2.1, we can compute the singular part u of v on $\partial\Omega$. Indeed, using the ellipsoidal harmonics and ellipsoidal coordinates λ_i, we get that

$$v(\lambda_1, \lambda_2, \lambda_3) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \alpha_n^k E_n^{p_k^n}(\lambda_1) E_n^{p_k^n}(\lambda_2) E_n^{p_k^n}(\lambda_3) + \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \beta_n^k F_n^{p_k^n}(\lambda_1) E_n^{p_k^n}(\lambda_2) E_n^{p_k^n}(\lambda_3).$$

Also, the given data g and ϕ on $\partial\Omega$ allow us to compute the following expressions (thanks to proposition 2.2):

$$g(\lambda_2, \lambda_3) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \gamma_n^k E_n^{p_k^n}(\lambda_2) E_n^{p_k^n}(\lambda_3)$$

and

$$\sqrt{\frac{\left(a_1^2 - \lambda_2^2\right)\left(a_1^2 - \lambda_3^2\right)}{\left(a_1^2 - h_2^2\right)\left(a_1^2 - h_3^2\right)}}\,\phi(\lambda_2, \lambda_3) = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \phi_n^k E_n^{p_k^n}(\lambda_2) E_n^{p_k^n}(\lambda_3).$$

Identifying $v(a_1, \lambda_2, \lambda_3)$ to $g(\lambda_2, \lambda_3)$, we first get

$$\alpha_n^k E_n^{p_k^n}(a_1) + \beta_n^k F_n^{p_k^n}(a_1) = \gamma_n^k.$$

Next, appealing to (7) and identifying the normal derivative of v to ϕ on $\partial\Omega$, we obtain

$$\alpha_n^k E_n^{p_k^n\prime}(a_1) + \beta_n^k F_n^{p_k^n\prime}(a_1) = \phi_n^k.$$

We thus get the following system:

$$\begin{cases} \alpha_n^k E_n^{p_k^n}(a_1) + \beta_n^k F_n^{p_k^n}(a_1) = \gamma_n^k \\ \alpha_n^k E_n^{p_k^n\prime}(a_1) + \beta_n^k F_n^{p_k^n\prime}(a_1) = \phi_n^k. \end{cases}$$

Observe that its determinant (see theorem 2.1)

$$\begin{aligned} E_n^{p_k^n}(a_1) F_n^{p_k^n\prime}(a_1) - F_n^{p_k^n}(a_1) E_n^{p_k^n\prime}(a_1) &= \left(E_n^{p_k^n}(a_1)\right)^2 \left(\frac{F_n^{p_k^n}}{E_n^{p_k^n}}\right)'(a_1) \\ &= -\frac{2n+1}{\sqrt{\left(a_1^2 - h_2^2\right)\left(a_1^2 - h_3^2\right)}} \neq 0, \end{aligned}$$

thus one can compute the coefficients β_n^k in the above expression of v and obtain its singular part:

$$u(X) = \sum_{k=1}^{m} \frac{\langle p_k, X - c_k \rangle}{4\pi \, \|X - c_k\|^3} = \sum_{n=0}^{\infty} \sum_{k=1}^{2n+1} \beta_n^k F_n^{p_k^n}(a_1) E_n^{p_k^n}(\lambda_2) E_n^{p_k^n}(\lambda_3), \tag{12}$$

where $X \in \partial\Omega = \mathcal{E}$ has the ellipsoidal coordinates $(a_1, \lambda_2, \lambda_3)$.

4.2. Expressions on the boundary of the slices $z = z_p$: from ellipses to circles: step (2.i)

The next step will consist in recovering the singularities of $u^2(x, y, z_p)$ for a family of cross-sections $\{z = z_p\}$ of Ω from its values on the 2D boundaries (ellipses). This is done by using a rational correspondence φ_p between these ellipses $\{z = z_p\} \cap \partial\Omega$ and the unit circle $\mathbb{T}$, and then, as in the spherical situation, by means of the best meromorphic approximation of the function $f_p = [u_{|\{z=z_p\}} \circ \varphi_p]^2$ on $\mathbb{T}$. Again, the poles of the approximant allow us to approximately locate the $2m$ singularities of f_p in $\mathbb{D}$ and, finally, the sources c_k.

We hence have

$$u^2(X) = \frac{\Phi(X)}{\prod_{k=1}^{m} \|X - c_k\|^6} = \frac{\Phi(X)}{\mathcal{Q}(X)},$$

where

$$\Phi(X) = \left[\frac{1}{4\pi} \sum_{k=1}^{m} \left(\langle p_k, X - c_k \rangle \prod_{\substack{i=1 \\ i \neq k}}^{m} \| X - c_i \|^3 \right) \right]^2 \tag{13}$$

is continuous on $\overline{\Omega}$. In every horizontal slice $\{z = z_p\}$, we have

$$\mathcal{Q}(X) = \prod_{k=1}^{m} \mathcal{Q}_k(X),$$

where

$$\mathcal{Q}_k(X) = \left[(x - x_k)^2 + (y - y_k)^2 + (z_p - z_k)^2\right]^3, \qquad c_k = (x_k, y_k, z_k).$$

Denoting

$$h_{k,p} = z_p - z_k, \qquad w = x + \mathrm{i}y, \qquad w_k = x_k + \mathrm{i}y_k,$$

we then get

$$\mathcal{Q}_k(X) = \left[|w - w_k|^2 + h_{k,p}^2\right]^3.$$

The intersection of the ellipsoid $\partial\Omega$, whose equation is given by (2), and the plane $\{z = z_p\}$, is the ellipse whose equation is

$$\frac{x^2}{a_{1,p}^2} + \frac{y^2}{a_{2,p}^2} = 1,$$

with

$$a_{1,p} = a_1 \sqrt{1 - \frac{z_p^2}{a_3^2}} \qquad \text{and} \qquad a_{2,p} = a_2 \sqrt{1 - \frac{z_p^2}{a_3^2}}.$$

On this ellipse, we have

$$w = a_{1,p} x + \mathrm{i} a_{2,p} y = \underbrace{a_{1,p} \frac{\zeta + \frac{1}{\zeta}}{2} + a_{2,p} \frac{\zeta - \frac{1}{\zeta}}{2}}_{=\varphi_p(\zeta)},$$

with $\zeta \in \mathbb{T}$. We have

$$\varphi_p(\zeta) = \frac{\alpha_p \zeta^2 + \beta_p}{\zeta},$$

where

$$\alpha_p = \tfrac{1}{2}(a_{1,p} + a_{2,p}) \qquad \text{and} \qquad \beta_p = \tfrac{1}{2}(a_{1,p} - a_{2,p}).$$

Observe that $\beta_p > 0$ since, by assumption, $a_1 > a_2$. If we denote by

$$Q_{k,p}(\zeta) = \zeta^2 [\mathcal{Q}_k(\varphi_p(\zeta), z_p)]^{\frac{1}{3}},$$

then

$$\begin{aligned} Q_{k,p}(\zeta) &= \zeta^2 \left[(\varphi_p(\zeta) - w_k)(\overline{\varphi_p(\zeta)} - \overline{w}_k) + h_{k,p}^2\right] \\ &= \zeta^2 \left[\left(\frac{\alpha_p \zeta^2 + \beta_p}{\zeta} - w_k \right) \left(\frac{\alpha_p \bar{\zeta}^2 + \beta_p}{\bar{\zeta}} - \overline{w}_k \right) + h_{k,p}^2 \right]. \end{aligned}$$

Using the fact that, for $\zeta \in \mathbb{T}$, we have

$$\bar{\zeta} = \frac{1}{\zeta},$$

we get

$$
\begin{aligned}
Q_{k,p}(\zeta) &= (\alpha_p\zeta^2 + \beta_p - w_k\zeta)(\alpha_p + \beta_p\zeta^2 - \bar{w}_k\zeta) + h_{k,p}^2\zeta^2 \\
&= \alpha_p\beta_p\zeta^4 - (\alpha_p\bar{w}_k + \beta_p w_k)\zeta^3 + (\alpha_p^2 + \beta_p^2 + |w_k|^2 + h_{k,p}^2)\zeta^2 \\
&\quad - (\alpha_p w_k + \beta_p\bar{w}_k)\zeta + \alpha_p\beta_p \\
&= \alpha_p\beta_p\zeta^4 + \varpi_3^{k,p}\zeta^3 + \varpi_2^{k,p}\zeta^2 + \overline{\varpi_3^{k,p}}\zeta + \alpha_p\beta_p,
\end{aligned} \tag{14}
$$

with

$$
\varpi_2^{k,p} = \alpha_p^2 + \beta_p^2 + |w_k|^2 + h_{k,p}^2, \qquad \varpi_3^{k,p} = -(\alpha_p\bar{w}_k + \beta_p w_k).
$$

Thus, on $\mathbb{T}$, and $\forall p$,

$$
u^2(\varphi_p(\zeta), z_p) = \frac{\zeta^{6m}\Phi(\varphi_p(\zeta), z_p)}{\prod_{k=1}^m Q_{k,p}^3(\zeta)}. \tag{15}
$$

Put $p_k = (p_{kx}, p_{ky}, p_{kz})$ for the moments, and $\pi_k = p_{kx} + \mathrm{i}p_{ky}$ for their affix in the plane $z = z_p$. In the situation where $m = 1$, we get from (13) that

$$
\Phi(X) = \left[\frac{1}{4\pi}\langle p_1, X - c_1\rangle\right]^2,
$$

whence

$$
(4\pi)^2\Phi(\varphi_p(\zeta), z_p) = [\mathrm{Re}(\bar{\pi}_1(\varphi_p(\zeta) - w_1)) + p_{1z}h_{1,p}]^2.
$$

Observe now that on $\mathbb{T}$, if we denote by $\Psi_{1,p}^2(\zeta)$ the function

$$
4\zeta^6(4\pi)^2\Phi(\varphi_p(\zeta), z_p)
$$

then we get (using the same computational trick as before, namely that on $\mathbb{T}$, $\bar{\zeta} = 1/\zeta$):

$$
\Psi_{1,p}(\zeta) = \zeta^2\left((\pi_1\beta_p + \bar{\pi}_1\alpha_p)\zeta^2 + 2(h_{1,p}p_{1,z} - \mathrm{Re}(\pi_1\bar{w}_1))\zeta + \pi_1\alpha_p + \bar{\pi}_1\beta_p\right).
$$

In this single source situation, $u^2(\varphi_p(\zeta), z_p)$ thus coincides on $\mathbb{T}$ with a rational function whose numerator is equal to $\Psi_{1,p}^2(\zeta)$, the polynomial of degree 8, and whose denominator is $Q_{1,p}^3(\zeta)$, of degree 12 (up to multiplication by a constant).

Whenever $m > 1$, we get from (13) that

$$
\begin{aligned}
(4\pi)^2\Phi(\varphi_p(\zeta), z_p) &= \left[\sum_{k=1}^m [\mathrm{Re}(\bar{\pi}_k(\varphi_p(\zeta) - w_k)) + p_{kz}h_{k,p}]\prod_{\substack{i=1\\ i\neq k}}^m \left[|\varphi_p(\zeta) - w_i|^2 + h_{i,p}^2\right]^{3/2}\right]^2 \\
&= \frac{1}{4\zeta^{6m-4}}\left[\sum_{k=1}^m \Psi_{k,p}(\zeta)\prod_{\substack{i=1\\ i\neq k}}^m Q_i(\zeta)^{3/2}\right]^2 \quad \text{on } \mathbb{T}.
\end{aligned}
$$

Let now

$$
\Psi_{k,p}(\zeta) = \zeta^2\left((\pi_k\beta_p + \bar{\pi}_k\alpha_p)\zeta^2 + 2(h_{k,p}p_{k,z} - \mathrm{Re}(\pi_k\bar{w}_k))\zeta + \pi_k\alpha_p + \bar{\pi}_k\beta_p\right),
$$

and

$$
\Psi_p^2(\zeta) = 4(4\pi)^2\Phi(\varphi_p(\zeta), z_p)\zeta^{6m} = \zeta^4\left[\sum_{k=1}^m \Psi_{k,p}(\zeta)\prod_{\substack{i=1\\ i\neq k}}^m Q_i(\zeta)^{3/2}\right]^2.
$$

Finally, we obtain from (15) that $u^2(\varphi_p(\zeta), z_p)$ coincides on $\mathbb{T}$ with a function whose numerator has both (triple) poles and branched singularities at the roots of $\prod_{k=1}^m Q_k$:

$$
u^2(\varphi_p(\zeta), z_p) = \frac{1}{4(4\pi)^2}\frac{\Psi_p^2(\zeta)}{\prod_{k=1}^m Q_{k,p}^3(\zeta)} \quad \text{on } \mathbb{T}. \tag{16}
$$

4.3. About the zeros of $Q_{k,p}$ and w_k, z_k: step (2.i) (ctn) and step 3

Proposition 4.1. *At a fixed p, the polynomial $Q_{k,p}$ has two roots $\zeta_1^{k,p}$ and $\zeta_2^{k,p}$ inside $\mathbb{D}$ such that $\arg\zeta_1^{k,p} = -\arg\zeta_2^{k,p}$, and two roots in $\mathbb{C}\backslash\bar{\mathbb{D}}$ given by $1\big/\overline{\zeta_1^{k,p}}$ and $1\big/\overline{\zeta_2^{k,p}}$.*

Proof. Let p and k be fixed at the moment.

- First of all, we get that the polynomial $Q_{k,p}$ is autoreciprocal, whence satisfies
$$Q_{k,p}(\zeta) = \zeta^4\overline{Q_{k,p}\left(\frac{1}{\bar{\zeta}}\right)}.$$
- Hence, if ζ is a root of $Q_{k,p}$, then $1/\bar{\zeta}$ is a root too.
- From the fact that there are no sources on $\partial\Omega$, we get that $Q_{k,p}$ has no root of modulus 1. In particular, $Q_{k,p}$ has two roots $\zeta_1^{k,p}$ and $\zeta_2^{k,p}$ inside the unit disk, whereas the other two roots $1\big/\overline{\zeta_1^{k,p}}$ and $1\big/\overline{\zeta_2^{k,p}}$ are outside of the unit disk and have the same argument as $\zeta_1^{k,p}$ and $\zeta_2^{k,p}$, respectively.
- The product of the roots of $Q_{k,p}$ is
$$\zeta_1^{k,p}\zeta_2^{k,p}\frac{1}{\overline{\zeta_1^{k,p}}}\frac{1}{\overline{\zeta_2^{k,p}}} = 1,$$
hence $\zeta_1^{k,p}\zeta_2^{k,p}$ is real, which implies that the arguments of $\zeta_1^{k,p}$ and $\zeta_2^{k,p}$ are opposed or complementary.
- We then have two cases:
$$\zeta_1^{k,p} = r_1^{k,p}\,\mathrm{e}^{\mathrm{i}\theta_{k,p}}, \quad \zeta_2^{k,p} = r_2^{k,p}\,\mathrm{e}^{-\mathrm{i}\theta_{k,p}} \quad \text{or} \quad \zeta_1^{k,p} = r_1^{k,p}\,\mathrm{e}^{\mathrm{i}\theta_{k,p}}, \quad \zeta_2^{k,p} = -r_2^{k,p}\,\mathrm{e}^{-\mathrm{i}\theta_{k,p}}.$$
In the second case, since the sum of the products two by two of the roots of the polynomial $Q_{k,p}$ is equal on the one hand to
$$-\left(r_1^{k,p}r_2^{k,p} + \frac{r_1^{k,p}}{r_2^{k,p}} + \frac{r_2^{k,p}}{r_1^{k,p}} + \frac{1}{r_1^{k,p}r_2^{k,p}}\right) + 2\cos 2\theta_{k,p} \leqslant 0,$$
and on the other hand, from (14), to
$$\frac{\varpi_2^{k,p}}{\alpha_p\beta_p} = \frac{\alpha_p^2 + \beta_p^2 + |w_k|^2 + h_{k,p}^2}{\alpha_p\beta_p} > 0,$$
we get a contradiction. Hence the first case holds which means that the roots of $Q_{k,p}(\zeta)$ in the unit disk have opposed arguments. □

Remark 4.1. Because $\zeta_1^{k,p} = r_1^{k,p}\,\mathrm{e}^{\mathrm{i}\theta_{k,p}}$, $\zeta_2^{k,p} = r_2^{k,p}\,\mathrm{e}^{-\mathrm{i}\theta_{k,p}}$, the coefficient $\varpi_3^{k,p}$ of ζ^3 in $Q_{k,p}(\zeta)$ is given by
$$\varpi_3^{k,p} = \left(r_1^{k,p} + \frac{1}{r_1^{k,p}}\right)\mathrm{e}^{\mathrm{i}\theta_{k,p}} + \left(r_2^{k,p} + \frac{1}{r_2^{k,p}}\right)\mathrm{e}^{-\mathrm{i}\theta_{k,p}}.$$
If we note $w_k = \rho_k\,\mathrm{e}^{\mathrm{i}\vartheta_k}$, we also get from (14) that
$$\varpi_3^{k,p} = (\alpha_p\rho_k + \beta_p\rho_k)\cos\vartheta_k + \mathrm{i}(\alpha_p\rho_k - \beta_p\rho_k)\sin\vartheta_k.$$
This gives us
$$\vartheta_k = \arctan\left(\frac{\alpha_p + \beta_p}{\alpha_p - \beta_p}\tan\arg\varpi_3^{k,p}\right),$$

while

$$\rho_k = \frac{\varpi_3^{k,p}}{(\alpha_p + \beta_p)\cos\vartheta_k + \mathrm{i}(\alpha_p - \beta_p)\sin\vartheta_k}.$$

In particular, since ϑ_k is independent of p, we deduce that the quantity

$$\frac{\alpha_p + \beta_p}{\alpha_p - \beta_p} \arctan\left(\frac{r_1^{k,p} r_2^{k,p} - 1}{r_1^{k,p} r_2^{k,p} + 1} \frac{r_1^{k,p} - r_2^{k,p}}{r_1^{k,p} + r_2^{k,p}} \tan\theta_{k,p}\right)$$

does not depend on p. This allows us, at least in principle, to compute w_k and then z_k, from the knowledge of $\zeta_1^{k,p}, \zeta_2^{k,p}$ at some p. But we may also choose a variational approach, using $\zeta_1^{k,p}, \zeta_2^{k,p}$ for all p, which will be more robust in numerical computations, see section 4.5.

4.4. How to get $(\zeta^{k,p})_{k,p}$ using AAK: step (2.ii)?

Let

$$f_p(\zeta) = [u(\varphi_p(\zeta), z_p)]^2, \text{ for each } p. \tag{17}$$

We now look for the best meromorphic approximant $R_{p,N}$ of degree N to f_p, solution to (11) on the circle $\mathbb{T}$ which is sent by φ_p onto the ellipse $\{z = z_p\} \cap \mathcal{E}$.

We get from (16) and the computations of the previous section that, for all p, $f_p \in \mathcal{BP} \subset C(\mathbb{T})$ and possesses $2m$ triple poles in $\mathbb{D}$ that coincide with branched singularities, $\left(\zeta_1^{k,p}, \zeta_2^{k,p}\right)_{k=1,\ldots,m}$:

$$f_p(\zeta) = \frac{H_{m,p}(\zeta)}{\prod_{k=1}^m \left(\zeta - \zeta_1^{k,p}\right)^3 \left(\zeta - \zeta_2^{k,p}\right)^3} \in C(\mathbb{T}),$$

where

$$H_{m,p}(\zeta) = \frac{1}{4(4\pi)^2} \frac{\Psi_p^2(\zeta)}{\prod_{k=1}^m \left(\zeta - 1/\overline{\zeta_1^{k,p}}\right)^3 \left(\zeta - 1/\overline{\zeta_2^{k,p}}\right)^3} \in C(\mathbb{T}).$$

It thus follows from the results of section 3 that, for each p, the N poles, say s_j, of the best rational (meromorphic) approximant $R_{p,N}$ will asymptotically locate at $\widetilde{\zeta}_i^{k,p}$, $i = 1, 2, k = 1, \ldots, m$, near the singularities $\zeta_i^{k,p}$, $i = 1, 2, k = 1, \ldots, m$: this allows us to approximately locate them.

Let us discuss a bit more about the above scheme, in the two particular situations $m = 0$ (no source) and $m = 1$ (a single source). When $m = 0$, we clearly obtain that $u = 0$ because v is then harmonic inside Ω.

More interesting is the case where $m = 1$, because then f_p is a rational function of degree 12, which in fact belongs to $\mathcal{R}_6$, as follows from the above formula and from the computations of section 4.2. We then get that the infimum in (11) is equal to 0 for $n = 6$, and thus that $R_6 = f_p$: in this situation, the poles of the best approximants of degree 6 precisely coincide with those of f_p, thus allowing us to exactly recover the source.

Of course, this is no longer true in the general situation where $m \geqslant 2$ for which we can only expect an approximate recovery, since the function f_p to be approximated does no longer belong to the class $\mathcal{R}_N$ of rational functions.

4.5. How to get c_k from $(\widetilde{\zeta}^{k,p})_p$: step 3 (ctn)?

Let us now assume that the singularities $\zeta_i^{k,p}$, $i = 1, 2, k = 1, \ldots, m$, inside the unit disk of the above function are (approximately) known from the above step, and that we are given $\widetilde{\zeta}^{k,p}$, $i = 1, 2, k = 1, \ldots, m$.

Figure 1. Source c_1 at $(0.2, 0.5, -0.3)$.

For each p, we then have $2m$ triple poles inside the unit disk. We first class these $2m$ poles by pairs for each p, using the remarks of section 4.3: we take a pair $\left(\widetilde{\zeta}_1^{1,p}, \widetilde{\zeta}_2^{1,p}\right)$ such that their arguments are opposed. They correspond to a first source whose projection on the plane $z = z_p$ has an affix w_1.

We do the same thing for the other pairs: for $k \in \{2, \ldots, m\}$, to the pair $\left(\widetilde{\zeta}_1^{k,p}, \widetilde{\zeta}_2^{k,p}\right)$ such that the arguments are opposed, we know that there exists a kth source. The affix of the projection of this source on the plane $z = z_p$ is w_k.

Doing this for every p, and using the above invariants, see remark 8, we can sort these different pairs along the horizontal sections $z = z_p$, since the pairs $\left(\widetilde{\zeta}_1^{k,p}, \widetilde{\zeta}_2^{k,p}\right)$ correspond to the same (still unknown) source w_k, for all p. We thus get the family $\left(\widetilde{\zeta}_1^{k,p}, \widetilde{\zeta}_2^{k,p}\right)_p$ for each k.

Hence, for each p and k, we can form the polynomials

$$\begin{aligned} q_{k,p}(\zeta) &= \alpha_p\beta_p\left(\zeta - \widetilde{\zeta}_1^{k,p}\right)\left(\zeta - \widetilde{\zeta}_2^{k,p}\right)\left(\zeta - 1/\overline{\widetilde{\zeta}_1^{k,p}}\right)\left(\zeta - 1/\overline{\widetilde{\zeta}_2^{k,p}}\right) \\ &= \alpha_p\beta_p\zeta^4 + \widetilde{\varpi}_3^{k,p}\zeta^3 + \widetilde{\varpi}_2^{k,p}\zeta^2 + \overline{\widetilde{\varpi}_3^{k,p}}\zeta + \alpha_p\beta_p, \end{aligned}$$

where we use $\widetilde{\varpi}_2^{k,p}$ and $\widetilde{\varpi}_3^{k,p}$ to denote the coefficients of the polynomials $q_{k,p}$. Because

$$q_{k,p}(\zeta) \approx Q_{k,p}(\zeta),$$

we get

$$\widetilde{\varpi}_2^{k,p} \approx \varpi_2^{k,p} = \alpha_p^2 + \beta_p^2 + |w_k|^2 + h_{k,p}^2,$$

hence, for fixed k, the expression

$$\varpi_2^{k,p} - \left(\alpha_p^2 + \beta_p^2\right) \simeq \widetilde{\varpi}_2^{k,p} - \left(\alpha_p^2 + \beta_p^2\right) \tag{18}$$

is minimal among the indices p if and only if the slice $\{z = z_p\}$ contains the source w_k. Because also $\widetilde{\varpi}_3^{k,p} \approx \varpi_3^{k,p}$, the discussion of remark 8 allows us to reconstruct w_k.

5. Numerical illustration

In order to provide a preliminary numerical illustration of the above scheme (implemented with Matlab7), let us work with an ellipsoid $\mathcal{E}$ with average semi-axes equal to $a_1 = 3$, $a_2 = 2$ and $a_3 = 1$ in (2). We will consider $z_p = -1, \ldots, 1$ with an increment of 0.1.

Assume further that we are in a single source situation ($m = 1$) where $c_1 = (0.2, 0.5, -0.3)$, $p_1 = (0.7, 0, -0.7)$. In this case, we computed the singular part u given by (12) of the associated solution to the EEG inverse problem. Figures 1–4 are related to this case.

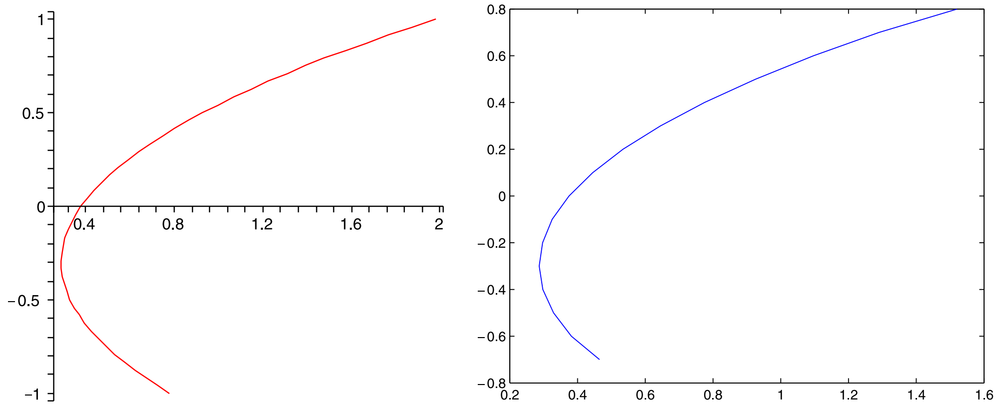

Figure 2. The functions $\varpi_2^{1,p} - (\alpha_p^2 + \beta_p^2)$ and $\widetilde{\varpi}_2^{1,p} - (\alpha_p^2 + \beta_p^2)$ (x-axis) w.r.t. the height z_p of the slice (y-axis).

Table 1. $2m$ estimated singularities in disks ($m = 1$ source).

z_p	$\zeta_1^{1,p}$	$\zeta_1^{2,p}$
	$\widetilde{\zeta}_1^{1,p}$	$\widetilde{\zeta}_1^{2,p}$
−0.3	0.05 + 0.56i	0.03 − 0.35i
	0.05 + 0.56i	0.03 − 0.35i
−0.6	0.06 + 0.62i	0.04 − 0.37i
	0.06 + 0.58i	0.04 − 0.33i
0.8	0.06 + 0.46i	0.03 − 0.27i
	0.06 + 0.46i	0.03 − 0.27i

Figure 1 shows the location of c_1 in the interior of $\mathcal{E}$. Figure 2 shows (in abscissa) the formal (left-hand side plot) and computed (right-hand side one) behavior of criterion (18), which takes its minimal value at the slice $z_p = -0.3$ containing the source c_1. Figures 3 and 4, and table 1, are related to the three values of p corresponding to the slices $z_p = -0.6$, $z_p = -0.3$ and $z_p = 0.8$. The singularities $\zeta_1^{1,p}, \zeta_2^{1,p}$ of f_p are the *, while the six poles of its best rational approximant $R_{p,6}$ are the black points $\cdot$ (there are three poles around or below each *, and this accounts for the fact that those are the triple poles of f_p). We can thus estimate the positions $\widetilde{\zeta}_1^{1,p}, \widetilde{\zeta}_2^{1,p}$ by computing the barycenters of those two groups of poles, plotted as '$\diamond$' in figures 3 and 4; in this situation, the poles are so close to each other and to $\zeta_i^{1,p}$ that it is pretty hard to distinguish between them all, except on the zoom around $\zeta_2^{1,p}$ in figure 3.

This allows us to recover $w_1 \simeq \varphi_p\big(\widetilde{\zeta}_1^{2,p}\big) = 0.2 + 0.5\mathrm{i}$ whence c_1, up to a relative error smaller than 10^{-3}.

Considering a situation with two sources ($m = 2$) at $c_1 = (-0.5, 0.2, -0.5)$ and $c_2 = (0.5, -0.5, 0.3)$, and associated moments $p_1 = (10, 0, 0)$, $p_2 = (0, 1, 0)$, see figure 5, we show the behavior of criterion (18), in figures 6 for $k = 1$ and 7 for $k = 2$. The left-hand side plots are derived from formal computations, while the right-hand side ones come from numerical simulations; one can see if the minimum in figure 6 is achieved for $z_p = -0.5$ in both cases, this is not the case in figure 7. Indeed, the computed minimum of (18) related to the source c_2 is achieved for $z_p = 0.2$ (close but not equal to its true value 0.3).

We also computed the singularities $\zeta_1^{k,p}, \zeta_2^{k,p}$ of the function f_p defined by (17) for the three values of p corresponding to the slices $z_p = -0.5$, $z_p = 0$, $z_p = 0.2$ and $z_p = 0.7$

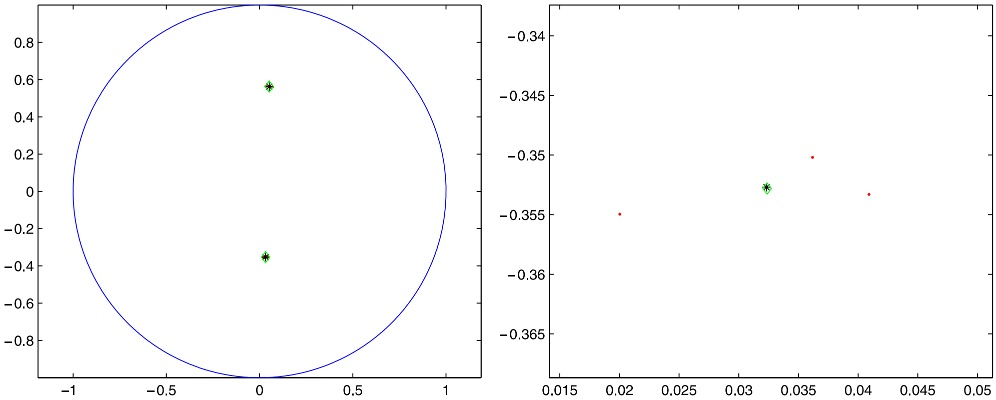

Figure 3. Slice $z_p = -0.3$, zoom around $\zeta_2^{1,p}$.

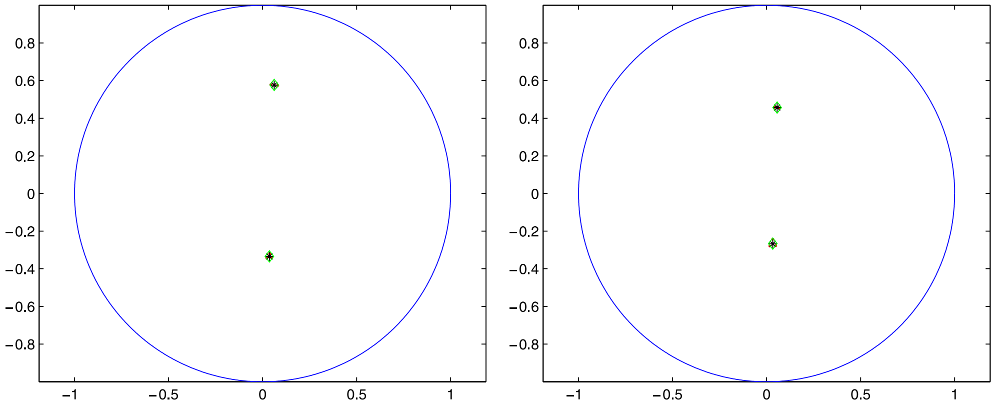

Figure 4. Slices $z_p = -0.6$ and $z_p = 0.8$.

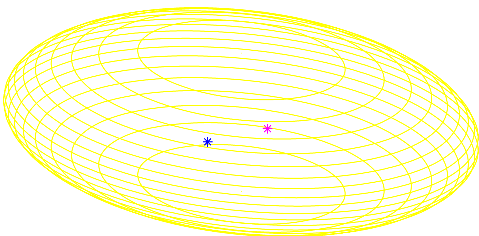

Figure 5. Two sources $c_1 = (-0.5, 0.2, -0.5)$ and $c_2 = (0.5, -0.5, 0.3)$.

(see table 2). For $z_p = -0.5, 0, 0.7$, they are plotted as $*$ in figures 8 and 9, where the poles of $R_{p,N}$ are the black points $\cdot$ and their barycenters (which provide us with the estimates $\widetilde{\zeta}_i^{k,p}$) are the $\diamond$.

In all cases, one can check that $\zeta_1^{k,p}$ and $\zeta_2^{k,p}$ have opposed arguments (although the arguments of $\widetilde{\zeta}_1^{1,p}, \widetilde{\zeta}_2^{2,p}$ are in the present situation more accurately recovered than the others).

At $z_p = -0.5$, we thus get $\varphi_p\left(\widetilde{\zeta}_1^{1,p}\right) \simeq w_1$ and allows us to locate c_1 with an error less than 10^{-3}.

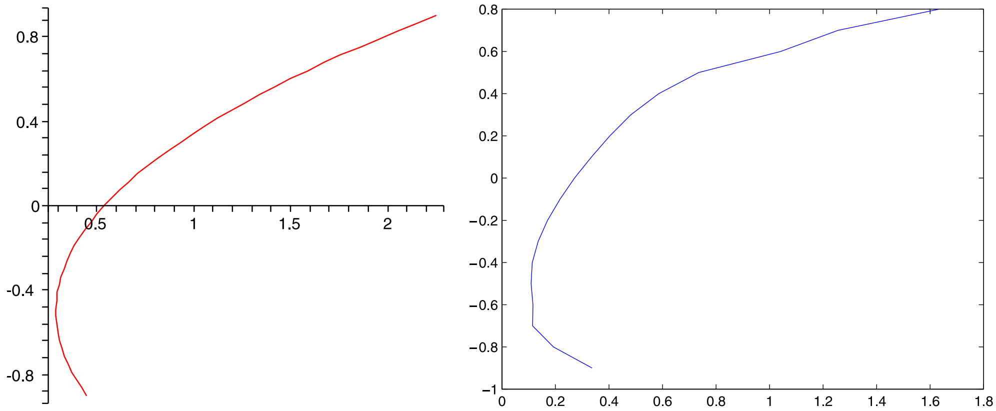

Figure 6. The functions $\varpi_2^{1,p} - (\alpha_p^2 + \beta_p^2)$ and $\widetilde{\varpi}_2^{1,p} - (\alpha_p^2 + \beta_p^2)$ (x-axis) w.r.t. the height z_p of the slice (y-axis).

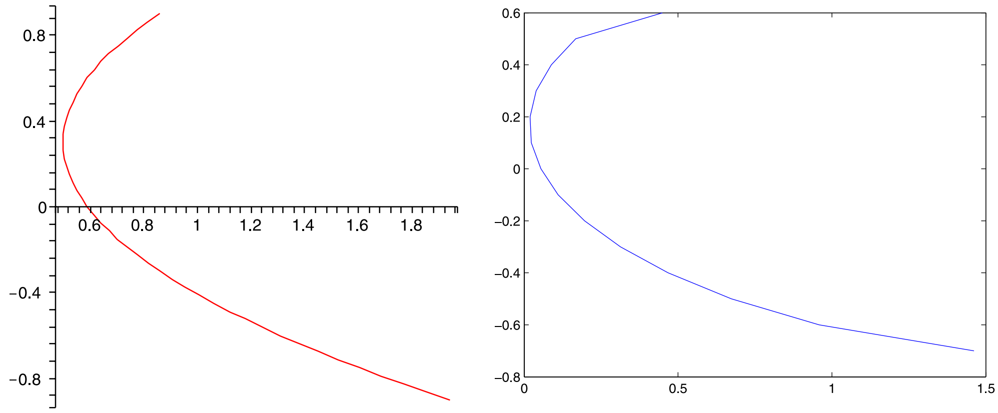

Figure 7. The functions $\varpi_2^{2,p} - (\alpha_p^2 + \beta_p^2)$ and $\widetilde{\varpi}_2^{2,p} - (\alpha_p^2 + \beta_p^2)$ (x-axis) w.r.t. the height z_p of the slice (y-axis).

Table 2. $2m$ estimated singularities in disks ($m = 2$ sources).

z_p	$\zeta_1^{1,p}$	$\zeta_1^{2,p}$	$\zeta_2^{1,p}$	$\zeta_2^{2,p}$
	$\widetilde{\zeta}_1^{1,p}$	$\widetilde{\zeta}_1^{2,p}$	$\widetilde{\zeta}_2^{1,p}$	$\widetilde{\zeta}_2^{2,p}$
−0.5	−0.13 + 0.48i	−0.1 − 0.39i	0.08 + 0.32i	0.13 − 0.5i
	−0.13 + 0.48i	−0.12 − 0.4i	0.1 + 0.33i	0.13 − 0.5i
0	−0.11 + 0.47i	−0.09 − 0.39i	0.08 + 0.35i	0.12 − 0.54i
	−0.1 + 0.47i	−0.12 − 0.4i	0.1 + 0.36i	0.13 − 0.56i
0.7	−0.11 + 0.39i	−0.09 − 0.32i	0.09 + 0.3i	0.17 − 0.56i
	−0.11 + 0.39i	−0.11 − 0.35i	0.12 + 0.31i	0.17 − 0.56i
0.2	−0.1 + 0.46i	−0.09 − 0.38i	0.08 + 0.35i	0.13 − 0.55i
	−0.1 + 0.46i	−0.12 − 0.4i	0.11 + 0.36i	0.14 − 0.57i

At $z_p = 0.2$, we get $\varphi_p\left(\widetilde{\zeta}_2^{2,p}\right) = 0.52 - 0.5\mathrm{i} \simeq w_2$, whence we get $(0.52, -0.5, 0.2) \simeq c_2$ up to 10^{-1}. Observe, however, that the efficiency of the algorithm should be improved by

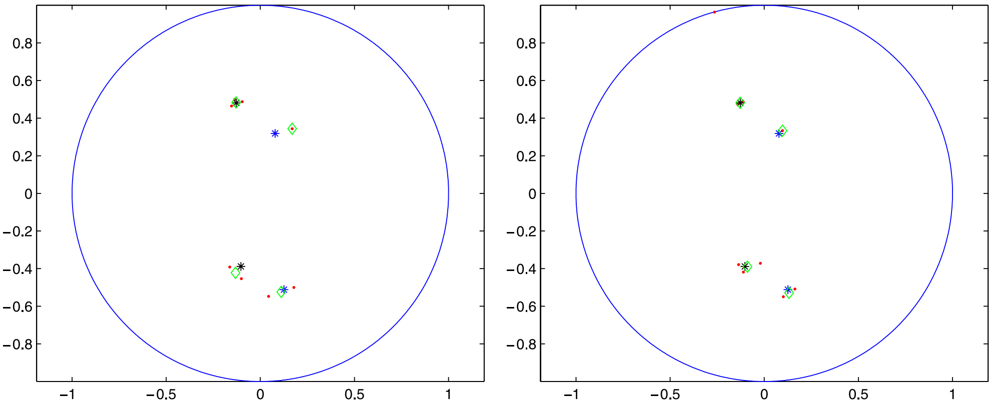

Figure 8. Singularities $\zeta_1^{k,p}$, $\zeta_2^{k,p}$ for $k = 1, 2$ at $z_p = -0.5$; poles of $R_{p,8}$ and $R_{p,10}$.

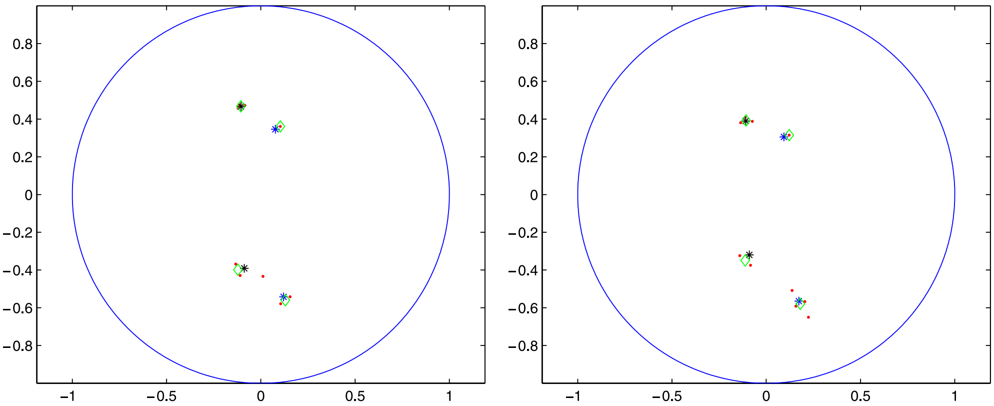

Figure 9. $z_p = 0.7$, poles of $R_{p,10}$; $z_p = 0$, poles of $R_{p,9}$.

refining the slicing along z_p and by looking to more accurate approximants, namely rational functions constrained to have triple poles, or more prospectively to functions with branched singularities of order $3/2$.

6. Conclusion

Numerous algorithmical and computational aspects of the present study remain to be approached. Amongst them, we can list the computation of the moments p_k, but also questions related to the selection of the estimated singularities $\widetilde{\zeta}_1^{k,p}$, $\widetilde{\zeta}_2^{k,p}$ for fixed k, as p varies. From this numerical point of view, in order to be more precise, we have to reduce the increment between two consecutive slices, to run the algorithm along slices which are parallel to other directions, and then to cluster the estimated singularities.

The present computation requires to compute rational approximants of huge degree. This is a numerical limitation which comes from the fact that the algorithm is mostly dedicated to the recovery of simple poles. This point is reinforced by the fact that we have $2m$ singularities to be recovered on each slice p, which is already twice the complexity of the spherical situation where there were only m singularities in each slice [5]. Furthermore, convergence properties only ensure the asymptotic approximation of the singularities. More efficient would be a scheme constrained to find triple poles, as well as the study of approximation by algebraic functions, instead of rational ones.

Despite these numerical limitations, the extension of the present work to more general geometries should in principle be feasible, at least for revolution domains whose plane sections by the family $\{z = z_p\}$ are so-called quadrature domains, as described in [19]. These are 2D simply connected domains whose boundaries can be described by a rational function defined on the unit circle (which generalizes the rational map φ_p). Observe that the number of singularities in each slice will then be equal to m times the degree of this rational function.

Also, for the EEG application, the preliminary cortical mapping step should be considered in this ellipsoidal setup, using decompositions on ellipsoidal harmonics. This step consists in solving a Cauchy-type issue for the potential v, which is practically given by pointwise measurements (electrodes) on the scalp, together with a vanishing current flux there, and has to be transmitted as a harmonic function up to the boundary of the brain, where the sources are seeked. A classical model is to consider the head as made up of a number of layers of constant conductivity (brain, skull, scalp).

Note also that computational difficulties will arise from the fact that the present scheme may require large-order expansions of the solution in order to solve the cortical mapping step with real data, as indicated in [27]. As already observed in section 2, the quantities involving p_k^n are explicitly available only for small values of n, and should be numerically computed for large orders, see [16, 30]. These computations may be carried out numerically in a forthcoming work [13], where more realistic situations with respect to the EEG application will be considered, as the case of sources closer to the boundary than in the above numerical examples; multipolar sources should also be considered, see [34], together with a fair comparison of available resolution schemes.

A feasible way to handle this recovery problem directly in 3D, and no longer using this slicing process and 2D approximation schemes, might be the use of quaternionic analysis [39], although a number of definitions of analyticity are available in this framework and an appropriate one needs to be chosen. These issues will also be the topics of a forthcoming work.

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References

[1] Adamyan V M, Arov D Z and Krein M G 1971 Analytic properties of Schmidt pairs, Hankel operators, and the generalized Schur–Takagi problem *Mat. Sb.* **128** 34–75

Adamyan V M, Arov D Z and Krein M G 1971 Analytic properties of Schmidt pairs, Hankel operators, and the generalized Schur–Takagi problem *Math. USRR Sb.* **15** (Engl. transl.)

[2] Axler S, Bourdon P and Ramey W 2001 *Harmonic Function Theory 2nd edn GTM* (Berlin: Springer)

[3] Baratchart L 1986 Existence and generic properties for L^2-approximants of linear systems *IMA J. Math. Control Inform.* **3** 89–101

[4] Baratchart L, Ben Abda A, Ben Hassen F and Leblond J 2005 Recovery of pointwise sources or small inclusions in 2D domains and rational approximation *Inverse Problems* **21** 51–74

[5] Baratchart L, Leblond J and Marmorat J-P 2006 Inverse source problem in a 3D ball from best meromorphic approximation on 2D slices *Electron. Trans. Numer. Anal.* **25** 41–53
[6] Baratchart L, Olivi M and Wielonsky F 1992 On a rational approximation problem in the real Hardy space H^2 *Theor. Comput. Sci.* **94** 175–97
[7] Baratchart L, Saff E B and Wielonsky F 1996 A criterion for uniqueness of a critical point in H^2 rational approximation *J. Anal. Math.* **70** 225–66
[8] Baratchart L and Stahl H Asymptotic spectrum of Hankel operator with branched symbol (in preparation)
[9] Baratchart L and Seyfert F 2002 An L^p analog to AAK theory for $p \geqslant 2$ *J. Funct. Anal.* **191** 52–122
[10] Baratchart L, Stahl H and Wielonsky F 2001 Asymptotic uniqueness of best rational approximants of given degree to Markov functions in L^2 of the circle *Constr. Approx.* **17** 103–38
[11] Baratchart L and Yattselev M Meromorphic and multipoint Padé approximants for complex Cauchy transforms with polar singularities *Preprint*
[12] Baratchart L and Yattselev M Strong asymptotics for non-Hermitian orthogonal polynomials and Blaschke products on a segment (submitted)
[13] Ben Hassen F, Leblond J, Rigat S and Zghal M EEG inverse problems in ellipsoid by data approximation on the boundary (in preparation)
[14] Byerly E 1893 *Fourier's Series* (New York: Dover)
Byerly E 1959 *Fourier's Series* (New York: Dover (second reprint))
[15] Chafik M, El Badia A and Ha-Duong T 2000 On some inverse EEG problems *Inverse Problems in Engineering Mechanics* vol II, pp 537–44
[16] Dassios G and Kariotou F 2003 Magnetoencephalography in ellipsoidal geometry *J. Math. Phys.* **44** 220–41
[17] Dassios G and Kleinman R 2000 Low frequency scattering *Oxford Science Publications: The Clarendon Press* (New York: Oxford University Press) 297 pp
[18] Dautray R and Lions J-L 1987 Analyse Mathématique et Calcul Numérique vol 2 (Paris: Masson)
[19] Ebenfelt P, Khavinson D and Shapiro H S 2001 An inverse problem for the double layer potential *Comput. Methods Funct. Theory* **1** 387–402
[20] El Badia A and Ha-Duong T 2000 An inverse source problem in potential analysis *Inverse Problems* **16** 651–63
[21] Faugeras O *et al* 1999 The inverse EEG and MEG problems: the adjoint state approach: I. The continuous case INRIA Research Report no. 3673
[22] Hämäläinen M, Hari R, Ilmoniemi J, Knuutila J and Lounasmaa O V 1993 Magnetoencephalography theory, instrumentation and applications to noninvasive studies of the working human brain *Rev. Mod. Phys.* **65** 413–97
[23] Hobson E W 1931 *Spherical and Ellipsoidal Harmonics* (Cambridge: Cambridge University Press)
Hobson E W 1965 *Spherical and Ellipsoidal Harmonics* (Cambridge: Cambridge University Press (second reprint))
[24] Hofmann-Wellenhof B and Moritz H 2005 *Physical Geodesy* (New York: Springer)
[25] Holota P 2003 On the computation of ellipsoidal harmonics in potential problems of geodetic relevance *Geophys. Res. Abstr.* **5**
[26] Holota P 2004 An approximation of ellipsoidal harmonics in refined studies on earth gravitational potential EGU, 1st General Assembly, Nice, France, 25–30 April 2004
[27] Irimia A 2005 Electric field and potential calculation for a bioelectric current dipole in an ellipsoid *J. Phys. A: Math. Gen.* **38** 8123–38
[28] Isakov V 1990 Inverse source problems *Mathematical Surveys and Monographs vol 34* (Providence, RI: American Mathematical Society)
[29] Kariotou F 2004 Electroencephalography in ellipsoidal geometry *J. Math. Anal. Appl.* **290** 324–42
[30] Leblond J, Paduret C, Rigat S and Zghal M 2008 Source localization in ellipsoids by best meromorphic approximation in planar sections (unpublished) http://www-sop.inria.fr/apics/personnel/Juliette.Leblond/publications.html
[31] Marmorat J-P and Olivi M 2007 Nudelman interpolation, parametrization of lossless functions and balanced realizations *Automatica* **43** 1329–38
[32] Marmorat J-P, Olivi M, Hanzon B and Peeters R L M 2002 Matrix rational H^2 approximation: a state-space approach using Schur parameters Proc. CDC-IEEE'02, Las-Vegas
[33] Nara T and Ando S 2003 A projective method for an inverse source problem of the Poisson equation *Inverse Problems* **19** 355–69
[34] Nara T 2008 An algebraic method for identification of dipoles and quadrupoles *Inverse Problems* **24** at press
[35] Nikolski N K 2002 Operator, functions, and systems: an easy reading *Hardy, Hankel, and Toeplitz vol 1* (*Mathematical Surveys and Monographs*) p 92 (Providence, RI: American Mathematical Society)
[36] Peller V V 2003 *Hankel Operators and their Applications* (Berlin: Springer)

[37] Stahl H 1985 The structure of extremal domains associated with an analytic function *Complex Var. Theory Appl.* **4** 339–54
[38] Sten J C-E 2006 Ellipsoidal harmonics and their applications in electrostatics *J. Electrost.* **64** 647–54
[39] Sudbery A 1979 Quaternionic analysis *Math. Proc. Camb. Phil. Soc.* **85** 199–225
[40] Vessella S 1992 Locations and strengths of point sources: stability estimates *Inverse Problems* **8** 911–7
[41] Whittaker T and Watson G N 1963 *A Course of Modern Analysis* (Cambridge: Cambridge University Press)
[42] Young N J 1998 *Introduction to Hilbert Space* (Cambridge: Cambridge University Press)