INSTITUTE OF PHYSICS PUBLISHING INVERSE PROBLEMS

Inverse Problems **20** (2004) 47–59 PII: S0266-5611(04)62059-4

Logarithmic stability estimates for a Robin coefficient in two-dimensional Laplace inverse problems

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Received 10 April 2003, in final form 1 October 2003
Published 14 November 2003
Online at stacks.iop.org/IP/20/47 (DOI: 10.1088/0266-5611/20/1/003)

Abstract

We establish some global stability results together with logarithmic estimates in Sobolev norms for the inverse problem of recovering a Robin coefficient on part of the boundary of a smooth 2D domain from overdetermined measurements on the complementary part of a solution to the Laplace equation in the domain, using tools from analytic function theory.

1. Introduction

We are concerned here with stability properties and estimates for the inverse problem (PR) of identifying a Robin coefficient on some nonaccessible part of the boundary of a smooth 2D domain from available overdetermined data on the other part of the boundary corresponding to solutions to the Laplace equation. We establish some global isotropic stability properties, of logarithmic type, by using tools from complex analysis, analytic functions theory, and Hardy spaces. Isotropic stability means that the stability coefficient does not depend on the perturbation direction, unlike the local Lipschitz stability results established in [14]. In return, the stability obtained here is much weaker, namely of logarithmic type. In this framework, the present study can be seen as a sequel of [16], where a constructive procedure is provided in order to solve for (PR), that relies on bounded extremal problems and best approximation in Hardy spaces.

Such an issue arises for example in corrosion detection by electrical impedance tomography, which can be modelled by an effective nonlinear boundary condition for the Laplace equation. In the simplest linear case, the knowledge of the corrosion effects can be reduced to that of a function defined on the corroded boundary part, which is usually called the Robin coefficient [22, 25, 26].

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The stability of the inverse problem of identifying Robin's coefficient by means of boundary measurements has already been the topic of numerous works, among them the following. A local and directional result of Lipschitz type stability is established in [20] for the 2D case. Both local and 'monotone' Lipschitz type stability results have been proved in [14] for 2D or 3D domains with smooth boundaries; identifiability (uniqueness) results are also available there. Recently, a local isotropic stability result has been established in [17, 18], together with logarithmic estimates.

The stability results of the present work are also isotropic and logarithmic but *global*. Actually, for smooth 2D domains and a number of Sobolev norms, if the Robin coefficients to be recovered are *a priori* known to be smooth and bounded (which is a feasible physical assumption), these results mean that the *discrepancy* between them is controlled by the uniform or quadratic norm of the discrepancy between the measurements. The regularity of the boundary, together with that of the Robin problem (1), provide us with that crucial boundedness property. Assuming smoothness on the original boundary is restrictive but reasonable, since one should not expect to get stability in the most general case. The Robin function itself must be sufficiently smooth. In fact, this function will be assumed to belong to an admissible set of bounded and smooth functions, where the bounds are physically linked with the corrosion properties of the material. Recently, while this paper was in the submission process, similar uniform stability estimates were published in [2].

Concerning inverse problems of geometrical type, when an unknown part of the boundary itself is to be recovered, analogous logarithmic stability results were established for subsets of the boundary in [1] (where it is shown that estimates cannot be better than log type ones) and in [11] (where logarithmic estimates are provided, under *a priori* assumptions on the unknown part of the boundary).

The outline of this work is as follows. First, we define some notation. In section 2, we introduce the inverse Robin problem, give some preliminary results, and then state the stability theorems. Section 3 contains some basic facts about harmonic conjugation and Hardy classes, together with boundary norm estimates which are the main tools for the proofs of the stability results, given in section 4. We draw conclusions in section 5.

1.1. Notation

Let D be a simply connected bounded domain of $\mathbb{R}^2$ with boundary T, a $\mathcal{C}^{1,\beta}$ Jordan closed curve, for $\beta \in (0, 1)$ (T is said to be $\mathcal{C}^{n,\beta}$ smooth if it admits a $\mathcal{C}^{n,\beta}$ parametrization [23]).

The Lebesgue measure on T will be denoted μ; however, for $T = \mathbb{T}$, the unit circle, we shall write $\mathrm{d}\theta$ for the Lebesgue measure on $[0, 2\pi)$. For $n \geqslant 0$ and $0 \leqslant \beta \leqslant 1$, we note $\mathcal{C}^{n,\beta}(\overline{D})$ for the space of functions f on $\overline{D}$ whose derivatives $f^{(k)}$ are of Hölder class with order β for $0 \leqslant k \leqslant n$. We put $\mathcal{C}^{n,0} = \mathcal{C}^n$.

For any connected open subset $E \subset T$, let χ_E be the characteristic function of E; traces on E of both functions and spaces will be indicated by $_{|E}$. The Hilbert space $L^2(E)$ of square summable functions with respect to μ on E is equipped with the classical norm and inner product, which we simply write as $\| \ \|_{L^2(E)}$ and $(\ , \ )_{L^2(E)}$, respectively. We assume them to be normalized on T: $\|1\|_{L^2(T)} = 1$.

For $n \in \mathbb{N}$, the norm on $W^{n,2}(E)$ is the usual one:

$$\|f\|^2_{W^{n,2}(E)} = \sum_{k=0}^{n} \|f^{(k)}\|^2_{L^2(E)},$$

where $f^{(k)}$ is the kth derivative of f with respect to arclength on E. Note, as usual, that $W^{0,2}(E) = L^2(E)$. For $s > 0$, the Sobolev Hilbert space $W^{s,2}(D)$ and its norm are classically defined. We put $\mathcal{C}_0^n(E)$ for the subset of $\mathcal{C}^n(E)$ consisting of functions f that vanish at ∂E together with their derivatives $f^{(k)}$, $k = 1, \ldots, n$, if $n \geqslant 1$. The set $W_0^{n,2}(E)$ stands for the $W^{n,2}(E)$ closure of $\mathcal{C}_0^n(E)$. The Sobolev Banach space $W^{n,\infty}(E)$ consists of functions belonging to $L^\infty(E)$ together with their derivatives up to order n.

2. Statement of the problem, stability results

Let γ, K be two nonempty disjoint open subsets of T, satisfying $T = \overline{\gamma} \cup \overline{K}$.

2.1. A 2D Robin inverse problem, identifiability, smoothness

We are concerned with the following inverse problem:

(PR) *Given a prescribed flux ϕ together with measurements u_m on K, recover the function q on γ such that the solution u to*

$$\begin{aligned} \Delta u &= 0 && \text{in } D, \\ \frac{\partial u}{\partial n} &= \phi && \text{on } K, \\ \frac{\partial u}{\partial n} + qu &= 0 && \text{on } \gamma, \end{aligned} \tag{1}$$

satisfies $u_{|_K} = u_m$.

In what follows, we assume that *both the measurement part $K \subset T$ and the corroded part $\gamma = T \setminus K$ have positive Lebesgue measure and finitely many connected components, the simplest case being the one where K is an arc of T.*

In this case, additional measurements are available on a set K of positive measure, and the solution to the above inverse problem is unique, provided that ϕ is sufficiently bounded, as we shall recall in theorem 1.

We address here stability issues that can be expressed as continuity properties in suitable spaces and norms of the map $u_m \mapsto q$ or, more generally, of $(u_m, \phi) \mapsto (u, \partial u/\partial n)_{|_\gamma}$.

Note that (PR) could be considered in slightly more general forms; one may for instance take K to be a disjoint union of, say, K_1 and K_2, put the above Neumann boundary condition on K_1 only while adding a Dirichlet one on K_2, see [2, 16]. One can also assume that additional measurements u_m are available on K_1 only. The results of the present work would remain valid. We stick, however, to the above model (PR), for simplicity.

We now state and discuss a number of *prior assumptions*.

The boundary T. We have already assumed T to be a $\mathcal{C}^{1,\beta}$ Jordan closed curve, and K and γ to have positive measure and finitely many connected components. A number of results below will extend to order n, $n \geqslant 2$, if T is $\mathcal{C}^{n,\beta}$ smooth. As already discussed in the introduction, this assumption is not so severe a restriction, as it mainly means that the initially noncorroded domain D should be smooth.

This first set of assumptions thus concerns T (whence K and γ) and is related to *the values of $n \geqslant 1$ and $\beta \in (0, 1)$*, as well as to *the Lebesgue measure of K and γ, since* $0 < \mu(K)/\mu(T) < 1$. The set $\mathcal{B} = (T, K, \gamma; n, \beta)$ will be used to denote these quantities related to the boundary.

The flux ϕ. We should take care here that *the flux ϕ should not change the sign of K* and we will assume that $\phi \geqslant 0$ there (with $\phi \not\equiv 0$). Although this corresponds to an actual physical restriction, this can be imposed on K where ϕ is chosen. This condition, needed in order to get a non-vanishing solution u, see lemma 1, is weakened in [2] where the flux ϕ is allowed to have variable sign on K while its oscillating character is then assumed to be limited; this allows control of the possible vanishing rate of u. The flux ϕ is another item of the prior data; it will also be subject to smoothness assumptions still described by *the value of the above parameter* n.

The Robin coefficient q. Let $c, c' > 0$ and $\mathcal{K}$ be a non-empty connected subset of γ, the boundary of which does not intersect that of γ. Define *classes* Q^n_{ad} *of admissible smooth Robin coefficients*, for $n \geqslant 0$:

$$Q^n_{\mathrm{ad}} = Q^n_{\mathrm{ad}}(n; \gamma; c, c', \mathcal{K}) = \{q \in \mathcal{C}^n_0(\overline{\gamma}) / |q^{(k)}| \leqslant c', 0 \leqslant k \leqslant n, \text{ and } q \geqslant c\chi_{\mathcal{K}}\}.$$

The restrictions then brought to the sought impedances q by *a priori* assuming that they belong to admissible Q^n_{ad} classes are of two kinds.

- *Boundedness.* The lower bound expresses that the corrosion, on the part of the boundary where it is likely to occur, does not turn the boundary condition to perfectly insulating. In fact, corrosion makes the material less conductive, and this assumption can thus be seen as a limitation of the corrosion effects level we are seeking to recover. As for the upper bound, it is derived from the physical knowledge of the interaction between the two materials present.
- *Smoothness.* This assumption is indeed restrictive. In fact, corrosion might break the original impedance smoothness in the same time as it alters its value. However, avoiding this assumption in our proof scheme seems impossible so far.

We then have the following identifiability result.

Theorem 1 ([14, theorem 1]). *Let* $\phi \in L^2(K)$ *with non-negative values a.e. and for* $i = 1, 2$, *let* $q_i \in Q^0_{\mathrm{ad}}$. *Let* $u_i \in \mathcal{C}^0(\overline{D})$ *be the unique solution of (1) associated with* $q = q_i$. *It holds that, if* $u_{1|_K} = u_{2|_K}$, *then* $q_1 = q_2$ *on* $\overline{\gamma}$.

The solution u to the Neumann–Robin problem (1) carries additional smoothness properties provided the coefficient q possesses sufficient regularity itself. This is established in [16, theorem 2], and recalled below, as a consequence of the following lemma.

Lemma 1 ([14, lemma 1]). *Let* $\phi \in L^2(K)$ *with non-negative values a.e. and* u_q *be the solution of problem (1) associated with* q. *Therefore,*

(i) $\forall q \in Q^0_{\mathrm{ad}}$, *there exists a constant* $\varrho = \varrho(\mathcal{B}, \phi, Q^0_{\mathrm{ad}}) > 0$ *such that* $u_q > \varrho$ *in* $\overline{D}$.
(ii) Let $q_1, q_2 \in Q^0_{\mathrm{ad}}$, *such that* $q_1 \geqslant q_2$ *in* $\overline{\gamma}$. *Then* $u_{q_1} \leqslant u_{q_2}$ *in* $\overline{D}$.
(iii) Let us denote by u_c *the solution of (1) for* $q = c$ *on* $\mathcal{K}$ *and* $q = 0$ *on* $\gamma \setminus \mathcal{K}$ *and* $u_{c'}$ *the solution for* $q = c'$ *on* γ. *Then,* $\forall\, q \in Q^0_{\mathrm{ad}}, 0 < u_{c'} \leqslant u_q \leqslant u_c$.

The smoothness result is then as follows.

Theorem 2 ([16, theorem 2]). *If* $\phi \in W^{1,2}_0(K)$, $\phi \geqslant 0$, *and* $q \in Q^1_{\mathrm{ad}}$, *then the solution* u_q *to (1) belongs to* $W^{\frac{5}{2},2}(D)$; *moreover, its trace* $u_{q|_T}$ *belongs to* $W^{2,2}(T)$ *and there exists a constant* $\kappa = \kappa(\mathcal{B}, \phi, Q^1_{\mathrm{ad}}) > 0$ *(which does not depend on* q*) such that* $\|u_q\|_{W^{2,2}(T)} \leqslant \kappa$. *Further, if* $\phi \in W^{2,2}_0(K)$ *and* $q \in Q^2_{\mathrm{ad}}$, *then* $u_{q|_T} \in W^{3,2}(T)$.

Note that [16, theorem 2] more generally asserts that, if T is $\mathcal{C}^{n,\beta}$ smooth for $n \geqslant 1$, $\phi \in W^{n,2}_0(K)$, $\phi \not\equiv 0$, $\phi \geqslant 0$, and $q \in Q^n_{\mathrm{ad}}$, then $u_q \in C^{n,1/2}(\overline{D})$ and $u_{q|_T} \in W^{n+1,2}(T)$, its norm there being bounded by some constant $\kappa = \kappa(\mathcal{B}, \phi, Q^n_{\mathrm{ad}}) > 0$.

2.2. Stability estimates

We are now in a position to state the following stability property and logarithmic estimates for (PR), whose proofs (given in section 4) strongly rely on Hardy spaces and the functional analysis tools of section 3.3.

Theorem 3. *Let $\phi \in W_0^{1,2}(K)$, $\phi \not\equiv 0$ with non-negative values, and assume that $q_1, q_2 \in \mathcal{Q}_{\mathrm{ad}}^1$; let u_1, u_2 be the associated solutions to problem (1). There exist constants $\rho = \rho(\mathcal{B}) \in (0, 1)$, $C, \tau > 0$ depending on $(\mathcal{B}, \phi, \mathcal{Q}_{\mathrm{ad}}^n)$, τ being small enough, and an increasing function $\varepsilon = \varepsilon_\rho : \mathbb{R}_+ \to \mathbb{R}_+$ that goes to 0 at 0 and satisfies*

$$\varepsilon(x) \leqslant \frac{2 + \log(\rho|\log x|)}{\rho|\log x|}, \qquad \text{for } x \leqslant \mathrm{e}^{-1/\rho},$$

such that

$$\|q_1 - q_2\|_{L^2(\gamma)} \leqslant C\varepsilon(\|u_1 - u_2\|_{L^2(K)}), \tag{2}$$

provided that $\|u_1 - u_2\|_{W^{1,2}(K)} \leqslant \tau$.

In fact, the above constants ρ, C and τ also depend on the choice of some conformal mapping $\mathcal{C}$ from D into the unit disc $\mathbb{D}$, see the proof in section 4.

Some uniform bounds can also be obtained.

Corollary 1. *Assume T to be $\mathcal{C}^{2,\beta}$ smooth, $\phi \in W_0^{2,2}(K)$, $\phi \not\equiv 0$, $\phi \geqslant 0$, and $q_1, q_2 \in \mathcal{Q}_{\mathrm{ad}}^2$. Then, there exist constants ρ, τ, a function $\varepsilon = \varepsilon_\rho$ as in theorem 3 and, for every $a \in (0, 1/2)$, a constant $C_a > 0$ depending on $(a, \mathcal{B}, \phi, \mathcal{Q}_{\mathrm{ad}}^2)$ and such that*

$$\|q_1 - q_2\|_{L^\infty(\gamma)} \leqslant C_a[\varepsilon(\|u_1 - u_2\|_{L^\infty(K)})]^a, \tag{3}$$

provided that $\|u_1 - u_2\|_{W^{1,\infty}(K)} \leqslant \tau$.

Remark 1. The above bound on ε near 0 may not be sharp, as we must have that

$$\varepsilon(x) \leqslant \frac{c}{|\log x|}, \qquad \text{for small } x,$$

and some $c = c(T, K) > 0$, see [8]. This statement is strengthened by the recent paper [2] where similar estimates have been obtained for weaker smoothness assumptions. It seems, however, that higher order stability estimates, such as those of corollary 2, would require more regularity.

Remark 2. The above results are stability properties, not robustness ones. Robustness would require dealing with noisy data, which definitely cannot be assumed to be smooth. Moreover, this issue cannot be tackled regardless of the recovery algorithm [15, 16]. On the other hand, stability involves comparison of close solutions to the Robin problem, whose smoothness proceeds from that of the boundary and of the data. The following counter-example shows how the stability results may fail if smoothness is lacking.

Remark 3. An explicit counter-example for which the above conclusion fails can be set as follows. Consider the (non-smooth) square domain $D =]-1, 0[\times]0, 1[$ with $K = [-1, 0] \times \{0\} \cup [-1, 0] \times \{1\}$ and $\gamma = \{0\} \times [0, 1] \cup \{-1\} \times [0, 1]$. Take the flux ϕ to be

$$\phi(x, y) = \begin{cases} 0 & \text{on } [-1, 0] \times \{0\} \\ 2 & \text{on } [-1, 0] \times \{1\}. \end{cases}$$

Let q be defined on γ by

$$q(x, y) = \frac{1}{1+y^2}.$$

The corresponding solution of the direct problem is then given by $u(x, y) = y^2 - x^2 - x + 1$ and

$$u_{|K}(x, y) = \begin{cases} -x^2 - x + 1 & \text{on } [-1, 0] \times \{0\} \\ -x^2 - x + 2 & \text{on } [-1, 0] \times \{1\}. \end{cases}$$

Consider now

$$u_n(x, y) = u(x, y) - \frac{1}{n\pi} \mathrm{e}^{n\pi x} \cos(n\pi y); \qquad n \in \mathbb{N}.$$

Such a u_n is the solution associated with the above ϕ and to q_n given on γ by

$$q_n(x, y) = \begin{cases} \dfrac{1+\cos(n\pi y)}{y^2 + 1 - (\cos(n\pi y)/n\pi)} & \text{on } \{0\} \times [0, 1] \\ \dfrac{1 - (\cos(n\pi y)/\mathrm{e}^{n\pi})}{y^2 + 1 - (\cos(n\pi y)/n\pi \mathrm{e}^{n\pi})} & \text{on } \{-1\} \times [0, 1]. \end{cases}$$

One can check that $\|u - u_n\|_{L^\infty(K)} \to 0$ though $q - q_n$ has no uniform limit on γ.

The above stability estimates are more generally valid at 'order n'.

Corollary 2. *Assume T to be $\mathcal{C}^{n,\beta}$ smooth for $n \geqslant 2$, $\phi \in W_0^{n,2}(K)$, $\phi \not\equiv 0$, $\phi \geqslant 0$, and $q_1, q_2 \in Q_{\mathrm{ad}}^n$. Then, there exist constants ρ, τ, C_n depending on $(\mathcal{B}, \phi, Q_{\mathrm{ad}}^n)$ and $\forall a \in (0, 1/2)$, $C_{n,a} > 0$ depending on $(a, \mathcal{B}, \phi, Q_{\mathrm{ad}}^n)$ together with a function ε as in theorem 3, such that*

$$\|q_1 - q_2\|_{W^{n-1,2}(\gamma)} \leqslant C_n \varepsilon(\|u_1 - u_2\|_{L^2(K)}), \tag{4}$$

provided that $\|u_1 - u_2\|_{L^2(K)} \leqslant \tau$, and $\forall a \in (0, 1/2)$;

$$\|q_1 - q_2\|_{W^{n-2,\infty}(\gamma)} \leqslant C_{n,a} [\varepsilon(\|u_1 - u_2\|_{L^\infty(K)})]^a, \tag{5}$$

provided that $\|u_1 - u_2\|_{L^\infty(K)} \leqslant \tau$.

Remark 4. Theorem 3 and corollaries 1 and 2 are still valid if u_1, u_2 are respectively associated with two different fluxes ϕ_1, ϕ_2. In particular, corollary 1 then ensures that

$$\|q_1 - q_2\|_{L^\infty(\gamma)} \leqslant C_a [\varepsilon(\max(\|u_1 - u_2\|_{L^\infty(K)}, \|\phi_1 - \phi_2\|_{L^\infty(K)}))]^a,$$

provided that $\max(\|u_1 - u_2\|_{W^{1,\infty}(K)}, \|\phi_1 - \phi_2\|_{L^\infty(K)}) \leqslant \tau$.

3. Hardy classes

3.1. Conformal mapping, from D to $\mathbb{D}$

A preliminary technical step of the proof is to express problem (PR) in the unit disc $\mathbb{D}$, in order to work within a classical framework for Hardy spaces.

Whenever D possesses a $C^{n,\beta}$ boundary T for some $n \geqslant 0$, $\beta \in (0, 1)$, the Kellogg–Warschawski theorem [23, theorem 3.6] means that there exists a conformal mapping from the unit disc $\mathbb{D}$ into D having a $C^{n,\beta}$ extension to $\overline{\mathbb{D}}$. In the present case, this guarantees that the conformal mapping $D \to \mathbb{D}$ admits a $C^{1,\beta}$ extension $\overline{D} \to \overline{\mathbb{D}}$ and allows one to express (PR) as a Robin problem in $\mathbb{D}$, see [13].

In the next two sections, we thus assume that $D = \mathbb{D}$ and $T = \mathbb{T} = \overline{\gamma} \cup \overline{K}$.

The inverse problem (PR) can now be approached using the classical Hardy spaces H^p of $\mathbb{D}$, among which is the Hilbert space H^2, that we introduce now.

3.2. More notation and properties

In the following, in order to deal with analytic functions, we isometrically identify $\mathbb{R}^2 \simeq \mathbb{C}$.

For Hardy spaces of the unit disc $\mathbb{D} \subset \mathbb{C}$, we refer to [19, 24] for their definitions and properties. In the Hilbertian framework, $H^2 = H^2(\mathbb{D})$ can be viewed as the space of functions analytic in $\mathbb{D}$ that are square-summable on circles of radius less than 1 centred at 0. It is a consequence of this definition that traces on the unit circle $\mathbb{T}$ of H^2 functions belong to $L^2(\mathbb{T})$, whence H^2 inherits the normalized $L^2(\mathbb{T})$ inner product. Thus, $H^2_{|\mathbb{T}}$ can also be described as the subspace of $L^2(\mathbb{T})$ consisting of functions whose Fourier coefficients of negative order vanish. Hence, from Parseval's equality and because $\sup_{r<1} \sum_{p\geqslant 0} r^p |g_p|^2 = \sum_{p\geqslant 0} |g_p|^2$,

$$H^2 = \left\{ g(z) = \sum_{p\geqslant 0} g_p z^p, \sum_{p\geqslant 0} |g_p|^2 < \infty \right\} \text{ with } \|g\|_{H^2} = \|g_{|\mathbb{T}}\|_{L^2(\mathbb{T})} = \left(\sum_{p\geqslant 0} |g_p|^2 \right)^{1/2}.$$

A further equivalent definition of H^2 asserts that it is the space of complex valued functions whose real and imaginary parts are both harmonic in $\mathbb{D}$ and such that their L^2 norm on circles of radius $r < 1$ remains bounded as $r \to 1$.

The Banach space $H^\infty = H^\infty(\mathbb{D})$ is defined to be the space of functions analytic in $\mathbb{D}$ that are essentially bounded on circles of radius less than 1 centred at 0:

$$H^\infty = \left\{ g(z) = \sum_{p\geqslant 0} g_p z^p, \sup_{r<1} \sup_{\theta\in[0,2\pi)} |g(r\mathrm{e}^{\mathrm{i}\theta})| < \infty \right\}.$$

In fact,

$$\|g\|_{H^\infty} = \sup_{r<1} \sup_{\theta\in[0,2\pi)} |g(r\mathrm{e}^{\mathrm{i}\theta})| = \sup_{\theta\in[0,2\pi)} |g_{|\mathbb{T}}(\mathrm{e}^{\mathrm{i}\theta})| = \|g_{|\mathbb{T}}\|_{L^\infty(\mathbb{T})},$$

and $H^\infty_{|\mathbb{T}} \subset L^\infty(\mathbb{T})$ can equivalently be described as the subspace of $L^\infty(\mathbb{T})$ consisting of functions whose Fourier coefficients of negative order vanish. Note that $H^\infty \subset H^2$.

For $p = 2, \infty$, we finally introduce the Hardy–Sobolev space $\mathcal{H}^{n,p}$:

$$\mathcal{H}^{n,p} = \{g \in H^p \text{ such that } g^{(k)} \in H^p, 0 \leqslant k \leqslant n\},$$

(here, $g^{(k)}$ is the kth derivative of g with respect to the variable z in $\mathbb{D}$) equipped with the norm $\| \|_{W^{n,p}(\mathbb{T})}$; of course $\mathcal{H}^{0,p} = H^p$.

Lemma 2. *For $n \geqslant 0$,*

(i) $\mathcal{H}^{n,2}_{|\mathbb{T}} = H^2_{|\mathbb{T}} \cap W^{n,2}(\mathbb{T})$*;*
(ii) if u is harmonic in $\mathbb{D}$ and $u_{|\mathbb{T}} \in W^{n,2}(\mathbb{T})$, then $u = \mathrm{Re}\, g$ for functions $g \in \mathcal{H}^{n,2}$ defined by (6) and (7).

Proof. For $n = 0$, (ii) is essentially M Riesz' theorem that guarantees in particular the L^2 boundedness of the conjugation operator [19, theorem 4.1]. For $n = 1$, this is [19, theorem 3.11] while (i) is [5, lemma 1]. Both (i) and (ii) remain true for $n \geqslant 1$, by iteration. □

The following basic uniqueness result in Hardy spaces will also be of interest here.

Proposition 1 ([19, 24]). *Let K be an nonempty subset of $\mathbb{T}$ such that $\mu(K) > 0$ and let $g \in H^2$ verifying $g_{|K} = 0$; then $g \equiv 0$ on the whole unit disc $\mathbb{D}$.*

Finally, we state here a Gagliardo–Nirenberg interpolation inequality which we make crucial use of in the last steps of the proofs of our results from section 2.2.

Proposition 2 ([12, chapter VIII]). *Let $K \subset \mathbb{T}$ such that $\gamma = \mathbb{T} \setminus K$ has non-empty interior. For $p = 2, \infty$, we have*

$$\|u'\|_{L^p(K)} \leqslant C_{p,K} \|u\|^{1/2}_{W^{2,p}(K)} \|u\|^{1/2}_{L^p(K)}, \qquad \forall u \in W^{2,p}(K).$$

3.3. From harmonic functions to Hardy classes

Returning to (1), assume now that ϕ and q satisfy the assumptions of theorem 3: $\phi \in W_0^{1,2}(K)$ and $q \in Q^1_{\mathrm{ad}} \subset \mathcal{C}^1(\overline{\gamma})$. It then follows from theorem 2 that $u_{|\mathbb{T}} \in W^{2,2}(\mathbb{T}) \subset \mathcal{C}^{1,1/2}(\mathbb{T})$.

From the knowledge of $\phi \in W_0^{1,2}(K) \subset L^2(K)$ and $u_m \in W^{2,2}(K) \subset \mathcal{C}^{1,1/2}(K)$ in problem (PR), we can build the trace on $K \subset \mathbb{T}$ of a function which is analytic in $\mathbb{D}$. This holds because u is harmonic in $\mathbb{D}$ and Cauchy–Riemann equations ensure that, if ω is a harmonic function in $\mathbb{D}$ satisfying

$$\frac{\partial \omega}{\partial \theta} = \frac{\partial u}{\partial n} \qquad \text{on } \mathbb{T}, \tag{6}$$

namely if ω is the harmonic conjugate function of u, then

$$g_q = u + \mathrm{i}\omega \tag{7}$$

is an analytic function in $\mathbb{D}$, see [3]; thus, $u = \mathrm{Re}\, g_q$ in $\overline{\mathbb{D}}$. Such an analytic function is unique up to an additional imaginary constant.

Equation (6) together with the boundary conditions in (1) mean that, on K,

$$\omega_{|K} = \int \phi \, \mathrm{d}\theta$$

if we denote by $\int \phi \, \mathrm{d}\theta$ some primitive of ϕ on K. The trace of g_q on K is therefore given by

$$g_{q|K} = u_m + \mathrm{i} \int \phi \, \mathrm{d}\theta, \tag{8}$$

and is, at least in principle, completely determined (up to an additional imaginary constant) by the *available boundary data* on K, in problem (PR). Next, it follows from (6), (7) that

$$q = -\frac{1}{\mathrm{Re}\, g_q} \frac{\partial\, \mathrm{Im}\, g_q}{\partial \theta} \qquad \text{on } \gamma, \tag{9}$$

where the above equality should be properly understood (non-tangential limits of the right-hand side). Now, q is the expected solution to (PR) on γ. Thus, recovering $g_{q|\gamma}$ from the knowledge of $g_{q|K}$ would solve for (PR).

This is the basis of the recovery procedure which is tackled in [16] for identification purposes. Indeed, constructive approximation procedures in Hardy spaces make it possible to reconstruct the function g_q from the knowledge of its trace on K given by (8), see [5, 6].

Further, this allows us to establish the stability properties of the map $u_m \mapsto q$ we are looking for, by using some norm estimates from [9]. This will be the topic of the next sections, after stating two additional results:

From lemma 2 and because $u \in W^{2,2}(\mathbb{T})$, then g_q actually belongs to $\mathcal{H}^{2,2} \subset H^2$ and is thus uniquely determined by its trace on K, in view of proposition 1.

3.4. Convergence estimates in Hardy–Sobolev classes

The proof of the stability theorem 3 strongly relies on the following estimate.

Theorem 4 ([9, lemma 4.2]). *Assume that* $\mu(K) = 2\pi\rho$, $0 < \rho < 1$. *Any* $g \in \mathcal{H}^{1,2}$ *such that* $\|g\|_{W^{1,2}(\mathbb{T})} \leqslant 1$ *and* $\|g\|_{L^2(K)} \leqslant \mathrm{e}^{-1/\rho}$ *also satisfies*

$$\|g\|_{L^2(\mathbb{T})} \leqslant \varepsilon(\|g\|_{L^2(K)}), \tag{10}$$

for some increasing function $\varepsilon = \varepsilon_{T,K} : \mathbb{R}_+ \to \mathbb{R}_+$ *that goes to 0 at 0 such that*

$$\varepsilon(x) \leqslant \frac{2 + \log(\rho|\log x|)}{\rho|\log x|}, \qquad \text{for } x \leqslant \mathrm{e}^{-1/\rho}.$$

Observe that the above estimate concerning ε may not be sharp, see remark 1.

Note also that the above conclusion is false in the general case where $g \in H^2$ only. Indeed, a bounded sequence of H^2 functions (g_n) such that $\|g_n\|_{L^2(K)} \to 0$ will necessarily verify $\|g_n\|_{L^2(\mathbb{T})} \to 0$ but this can generally happen arbitrarily slowly on $\mathbb{T} \setminus K$ even if the decay rate is prescribed on K.

The next corollary will be the main ingredient for our proof of theorem 3.

Corollary 3. *For $n \geqslant 1$, and any $g \in \mathcal{H}^{n+1,2}$ such that $\|g\|_{W^{n+1,2}(\mathbb{T})} \leqslant \kappa$ for some $\kappa > 0$ and $\|g\|_{W^{n,2}(K)} \leqslant \kappa e^{-1/\rho}$, there exist constants $\alpha_n, \alpha_n > 0$ depending on $(n; \kappa)$ and $\tau = \tau(n; K; \kappa, \rho) > 0$ such that*

$$\|g\|_{W^{n,2}(\mathbb{T})} \leqslant \alpha_n \varepsilon(\|g\|_{W^{n,2}(K)}). \tag{11}$$

If, further, $\|g\|_{L^2(K)} \leqslant \tau$, then

$$\|g\|_{W^{n,2}(\mathbb{T})} \leqslant \alpha_n \varepsilon(\|g\|_{L^2(K)}). \tag{12}$$

Proof. Let $n = 1$. If $\kappa > 1$, we have by hypothesis that g/κ and g'/κ satisfy the assumptions of theorem 4. It follows from lemma 2 and theorem 4 that

$$\left\|\frac{g}{\kappa}\right\|^2_{W^{1,2}(\mathbb{T})} = \left\|\frac{g}{\kappa}\right\|^2_{L^2(\mathbb{T})} + \left\|\frac{g'}{\kappa}\right\|^2_{L^2(\mathbb{T})} \leqslant \varepsilon^2\left(\left\|\frac{g}{\kappa}\right\|_{L^2(K)}\right) + \varepsilon^2\left(\left\|\frac{g'}{\kappa}\right\|_{L^2(K)}\right) \leqslant 2\varepsilon^2(\|g\|_{W^{1,2}(K)}),$$

the function ε being a non-decreasing function of $x \leqslant e^{-1/\rho}$. If $\kappa \leqslant 1$, the result follows even more directly, by applying the above argument to g itself, leading to $\|g\|^2_{W^{1,2}(\mathbb{T})} \leqslant 2\varepsilon^2(\|g\|_{W^{1,2}(K)})$. This proves the inequality (11) for $n = 1$ with $\alpha_1 = \sqrt{2}\max(\kappa, 1)$.

Concerning (12), we get from proposition 2 with $p = 2$, putting $C_K = C_{2,K}$, that

$$\|g\|^2_{W^{1,2}(K)} \leqslant \|g\|_{L^2(K)}(\|g\|_{L^2(K)} + C_K\|g\|_{W^{2,2}(K)}) \leqslant \kappa(1 + C_K)\|g\|_{L^2(K)}. \tag{13}$$

Choose $\tau = e^{-2/\rho}/\kappa(1 + C_K)$. By hypothesis, $\|g\|_{L^2(K)} \leqslant \tau$ whence

$$[\kappa(1 + C_K)\|g\|_{L^2(K)}]^{1/2} \leqslant e^{-1/\rho}$$

and, from theorem 4, ε can be chosen there such that

$$\varepsilon\left(\kappa(1 + C_K)\|g\|^{1/2}_{L^2(K)}\right) \leqslant c(K; \kappa)\varepsilon(\|g\|_{L^2(K)}),$$

for some constant $c(K; \kappa)$ depending on C_K and κ. Finally, (12) follows from (11), (13) with $\alpha_1 = \alpha_1 \max(1, c(K; \kappa))$.

The cases $n \geqslant 2$ follow by iteration, with some care in adjusting the various constants. □

The following corollary is the basis of corollary 1.

Corollary 4. *Let $n \geqslant 0$. Let $g \in \mathcal{H}^{n+1,\infty}$ such that $\|g\|_{W^{n+1,\infty}(\mathbb{T})} \leqslant \kappa$ and $\|g\|_{W^{n,\infty}(K)} \leqslant \kappa e^{-1/\rho}$ for some $\kappa > 0$ and $\mu(K) = 2\pi\rho$, $0 < \rho < 1$. Then, there exist a function ε as in theorem 4 and, $\forall a \in (0, 1/2)$, a constant $\alpha_a > 0$ depending on $(a; n; \kappa)$ such that*

$$\|g\|_{W^{n,\infty}(\mathbb{T})} \leqslant \alpha_a[\varepsilon(\|g\|_{W^{n,\infty}(K)})]^a. \tag{14}$$

If, further, $\|g\|_{L^\infty(K)} \leqslant \tau$, for some $\tau = \tau(n; K; \kappa) > 0$ then

$$\|g\|_{W^{n,\infty}(\mathbb{T})} \leqslant \alpha_a[\varepsilon(\|g\|_{L^\infty(K)})]^a. \tag{15}$$

Proof. Let first $n = 0$. Since $H^\infty \subset H^2$, it holds from lemma 2 that $g \in \mathcal{H}^{1,2}$. Also

$$\|g\|_{W^{1,2}(\mathbb{T})} \leqslant \|g\|_{W^{1,\infty}(\mathbb{T})} \leqslant \kappa, \qquad \|g\|_{L^2(K)} \leqslant \|g\|_{L^\infty(K)} \leqslant \kappa e^{-1/\rho},$$

which implies, in view of theorem 4 (see also the proof of corollary 3),

$$\|g\|_{L^2(\mathbb{T})} \leqslant \max(\kappa, 1)\varepsilon(\|g\|_{L^2(K)}). \tag{16}$$

The second step uses the subspace H^2_β of H^2 defined for $\beta \geqslant 0$ by

$$H^2_\beta = \left\{ g \in H^2, g(z) = \sum_{k=0}^{\infty} a_k z^k \text{ such that } \sum_{k=0}^{\infty} |a_k|^2 (1+k^2)^\beta < \infty \right\},$$

which is a Banach space, endowed with the norm

$$\|f\|_\beta = \left(\sum_{k=0}^{\infty} |a_k|^2 (1+k^2)^\beta \right)^{1/2}.$$

If we pick $\beta = 1 - a > 1/2$, we get for $g \in H^2_{1-a}$ that

$$\|g\|_{1-a} \leqslant \|g\|^a_{L^2(\mathbb{T})} \|g\|^{1-a}_{W^{1,2}(\mathbb{T})}. \tag{17}$$

Indeed,

$$\|g\|^2_{1-a} = \sum_{k=0}^{\infty} |a_k|^2 (1+k^2)^{1-a} = \sum_{k=0}^{\infty} |a_k|^{2a} |a_k|^{2(1-a)} (1+k^2)^{1-a}.$$

Applying then the Hölder inequality to the above series with exponents $1/a$ and $1/1-a$ leads to

$$\|g\|^2_{1-a} \leqslant \left(\sum_{k=0}^{\infty} |a_k|^2 \right)^a \left(\sum_{k=0}^{\infty} |a_k|^2 (1+k^2) \right)^{1-a},$$

which is (17). Now, because $\|g\|_{W^{1,2}(\mathbb{T})} \leqslant \kappa$ and from (16), (17),

$$\|g\|_{1-a} \leqslant \kappa^{1-a} [\max(\kappa, 1)]^a (\varepsilon(\|g\|_{L^2(K)}))^a \leqslant c_a (\varepsilon(\|g\|_{L^\infty(K)}))^a, \tag{18}$$

since ε is non-decreasing and putting $c_a = \kappa^{1-a}[\max(\kappa, 1)]^a$. Further, there exists a positive constant K_a such that

$$\|g\|_{L^\infty(\mathbb{T})} \leqslant K_a \|g\|_{1-a}, \qquad \forall g \in H^2_{1-a}. \tag{19}$$

It is given by

$$K_a^2 = \sum_{k \geqslant 0} \frac{1}{(1+k^2)^{1-a}}.$$

Finally, in view of (18) and (19) we conclude that

$$\|g\|_{L^\infty(\mathbb{T})} \leqslant c_a K_a [\varepsilon(\|g\|_{L^\infty(K)})]^a,$$

which is (14) or (15) for $n = 0$ with $\alpha_a = c_a K_a$; see also [9, corollary 4.4] and [21] for more details.

Take now $n = 1$. Both g and g' satisfy the assumptions of the present corollary with $n = 0$, whence

$$\|g^{(i)}\|_{L^\infty(\mathbb{T})} \leqslant \alpha_a [\varepsilon(\|g^{(i)}\|_{L^\infty(K)})]^a, \qquad i = 0, 1,$$

and

$$\|g\|_{W^{1,\infty}(\mathbb{T})} = \max_{i=0,1} \|g^{(i)}\|_{L^\infty(\mathbb{T})} \leqslant \alpha_a [\varepsilon(\|g\|_{W^{1,\infty}(K)})]^a.$$

Next, proposition 2 with $p = \infty$ and $C_K = C_\infty K$ means that

$$\|g'\|^2_{L^\infty(K)} \leqslant \kappa C_K \|g\|_{L^\infty(K)},$$

and if we choose $\tau = \mathrm{e}^{-2/\rho} / \kappa C_K$, we obtain, as in the proof of the previous corollary, that

$$\varepsilon(\|g'\|_{L^\infty(K)}) \leqslant c(K;\kappa) \varepsilon(\|g\|_{L^\infty(K)}),$$

for some constant $c(K;\kappa)$ depending on C_K and κ. Finally,

$$\|g\|_{W^{1,\infty}(\mathbb{T})} \leqslant \alpha_a \max(1, (c(K;\kappa))^a) [\varepsilon(\|g\|_{L^\infty(K)})]^a.$$

The proof for $n \geqslant 2$ follows, by induction. □

4. Proofs of stability properties for (PR)

Proof of theorem 3. It follows from theorem 2 that, for $j = 1, 2$, the analytic functions g_j such that $g_{j|_K} = u_j + \mathrm{i}\int\phi$ belong to $W^{2,2}(T)$, whence also $g = g_1 - g_2$, which satisfies $g_{|_K} = u_1 - u_2$. Theorem 2 also ensures that $\|g\|_{W^{2,2}(T)} \leqslant \kappa$ for some $\kappa > 0$ which does not depend on q_1, q_2.

Using (9), we get

$$q_1 - q_2 = \frac{1}{\operatorname{Re} g_2}\frac{\partial \operatorname{Im} g_2}{\partial\theta} - \frac{1}{\operatorname{Re} g_1}\frac{\partial \operatorname{Im} g_1}{\partial\theta} = -\frac{1}{\operatorname{Re} g_1}\frac{\partial \operatorname{Im} g}{\partial\theta} + \frac{\partial \operatorname{Im} g_2}{\partial\theta}\frac{\operatorname{Re} g}{\operatorname{Re} g_1 \operatorname{Re} g_2}.$$

Now, lemma 1 ensures that $u_j = \operatorname{Re} g_j \geqslant \varrho > 0$ in $\overline{\mathbb{D}}$ for some $\varrho > 0$ whenever ϕ is non-negative. Also, because $\|g_2\|_{W^{2,2}(\mathbb{T})} \leqslant \kappa$, it follows from Gagliardo–Nirenberg inequalities, see e.g. [12, corollary IX.13], that there exists $M = M_\gamma > 0$ such that

$$\left\|\frac{\partial \operatorname{Im} g_2}{\partial\theta}\right\|_{L^\infty(\gamma)} \leqslant \|g_2\|_{W^{1,\infty}(\gamma)} \leqslant M\|g_2\|_{W^{2,2}(\gamma)} \leqslant M\|g_2\|_{W^{2,2}(\mathbb{T})} \leqslant M\kappa.$$

We then get

$$\|q_1 - q_2\|_{L^2(\gamma)} \leqslant \frac{1}{\varrho}\|g\|_{W^{1,2}(\gamma)} + \frac{M\kappa}{\varrho^2}\|g\|_{L^2(\gamma)} \leqslant \left(\frac{M\kappa}{\varrho^2} + \frac{1}{\varrho}\right)\|g\|_{W^{1,2}(\gamma)}. \quad (20)$$

Assume now that $D = \mathbb{D}$. From lemma 2, $g \in \mathcal{H}^{2,2}$. Hence, if $\|g\|_{W^{1,2}(K)} \leqslant \mathrm{e}^{-1/\rho}$ and $\|g\|_{L^2(K)} \leqslant \tau$, we get from corollary 3 (with $n = 1$ and $\max(\kappa, 1)$ in place of κ) that

$$\|g\|_{W^{1,2}(\gamma)} \leqslant \|g\|_{W^{1,2}(\mathbb{T})} \leqslant \alpha_1\varepsilon(\|g\|_{W^{1,2}(K)}) = \alpha_1\varepsilon(\|g\|_{W^{1,2}(K)}),$$

and similarly

$$\|g\|_{W^{1,2}(\gamma)} \leqslant \alpha_1\varepsilon(\|g\|_{L^2(K)}).$$

Finally, (2) follows from (20), with

$$C = \alpha_1\left(\frac{M\kappa}{\varrho^2} + \frac{1}{\varrho}\right).$$

Next, if $D \neq \mathbb{D}$ but is a Jordan domain with $C^{1,\beta}$ boundary T, recall that the Kellogg–Warschawski theorem [23, theorem 3.6] ensures that there exists a conformal mapping $\mathcal{C}$ from D into the unit disc $\mathbb{D}$ having a $C^{1,\beta}$ extension to $\overline{D}$; in particular, $\mathcal{C}$ is bounded on T and $\mathcal{C}' \neq 0$ in $\overline{D}$, from [23, theorem 3.5]. It also follows from lemma 2 that the function defined on $\mathbb{D}$ by $g_{\mathbb{D}} = g \circ \mathcal{C}^{-1}$ belongs to $\mathcal{H}^{2,2}$. Let $K_{\mathbb{T}} = \mathcal{C}(K) \subset \mathbb{T}$ and put

$$m_{\mathcal{C}} = m(\mathcal{C}, T) = \max\left(\|\mathcal{C}'\|_{L^\infty(T)}, \left\|\frac{1}{\mathcal{C}'}\right\|_{L^\infty(T)}\right).$$

Then, straightforward computations show that $\|g_{\mathbb{D}}\|_{W^{1,2}(\mathbb{T})} \leqslant m_{\mathcal{C}}\|g\|_{W^{1,2}(T)}$, $\|g_{\mathbb{D}}\|_{W^{n,2}(K_{\mathbb{T}})} \leqslant m_{\mathcal{C}}\|g\|_{W^{n,2}(K)}$, for $n = 0, 1$, and $\|g\|_{W^{1,2}(T)} \leqslant m_{\mathcal{C}}\|g_{\mathbb{D}}\|_{W^{1,2}(\mathbb{T})}$. Let then $\rho = \rho(K, T, \mathcal{C}) > 0$ be such that

$$\rho \leqslant \frac{\inf_K |\mathcal{C}'|}{\|\mathcal{C}'\|_{L^\infty(K)}}\frac{\mu(K)}{\mu(T)}.$$

Again, we directly see that $\rho \leqslant \rho_{\mathbb{T}} = \mu(K_{\mathbb{T}})/2\pi$, and the hypothesis $\|g\|_{W^{1,2}(K)} \leqslant \mathrm{e}^{-1/\rho}$ ensures that

$$\|g_{\mathbb{D}}\|_{W^{1,2}(K_{\mathbb{T}})} \leqslant m_{\mathcal{C}}\,\mathrm{e}^{-1/\rho_{\mathbb{T}}}.$$

Applying then corollary 3 to $g_{\mathbb{D}}/m_{\mathcal{C}}$, it is easily checked that

$$\|g_{\mathbb{D}}\|_{W^{1,2}(\mathbb{T})} \leqslant m_{\mathcal{C}}\alpha_1\varepsilon\left[\frac{\|g_{\mathbb{D}}\|_{W^{1,2}(K_{\mathbb{T}})}}{m_{\mathcal{C}}}\right] \leqslant m_{\mathcal{C}}\alpha_1\varepsilon(\|g\|_{W^{1,2}(K)}),$$

ε being non-decreasing in $(0, \mathrm{e}^{-1/\rho})$. Now, corollary 3 asserts that there exists $\tau = \tau(K_{\mathbb{T}}; \kappa)$ such that if $\|g\|_{L^2(K)} \leqslant \tau/m_{\mathcal{C}}$, then

$$\|g_{\mathbb{D}}\|_{W^{1,2}(\mathbb{T})} \leqslant m_{\mathcal{C}}\alpha_1\varepsilon(\|g\|_{L^2(K)}),$$

Finally,

$$\|g\|_{W^{1,2}(\gamma)} \leqslant \|g\|_{W^{1,2}(T)} \leqslant m_{\mathcal{C}}\|g_{\mathbb{D}}\|_{W^{1,2}(\mathbb{T})} \leqslant m_{\mathcal{C}}^2\alpha_1\varepsilon(\|g\|_{W^{1,2}(K)}),$$

and (20) implies that

$$\|q_1 - q_2\|_{L^2(\gamma)} \leqslant \max(\kappa, 1)m_{\mathcal{C}}^2\alpha_1\left(\frac{M\kappa}{\varrho^2} + \frac{1}{\varrho}\right)\varepsilon(\|g\|_{W^{1,2}(K)}),$$

which is (2) with

$$C = \max(\kappa, 1)m_{\mathcal{C}}^2\alpha_1\left(\frac{M\kappa}{\varrho^2} + \frac{1}{\varrho}\right).$$

□

Proof of corollary 1. The estimate (3) follows from corollary 4 with $n = 1$. Indeed, by hypothesis, $g \in W^{3,2}(T) \subset W^{2,\infty}(T)$ and satisfies the assumptions of corollary 4, whence

$$\|q_1 - q_2\|_{L^\infty(\gamma)} \leqslant \left(\frac{\kappa}{\varrho^2} + \frac{1}{\varrho}\right)\|g\|_{W^{1,\infty}(T)}$$

leads to the conclusion, with $C_a = \alpha_a(\frac{\kappa}{\varrho^2} + \frac{1}{\varrho})$. □

Proof of corollary 2. Inequality (4) can be deduced either from the order n version of theorem 2 and corollary 3 as in the proof of theorem 3, or by induction from theorem 3 itself with a proper choice of the constants. As to (5), it follows by induction from corollary 1 and proposition 2. □

5. Conclusion

We have derived a class of global logarithmic stability properties for the inverse problem (PR) of determination of a boundary Robin coefficient arising in Laplace equation from results concerning analytic functions and Hardy spaces.

Such tools have been applied to some other 2D inverse problems involving harmonic functions. In [7, 10], the geometric inverse problems of crack recovery is approached with tools from best analytic or meromorphic approximation, both for complete and incomplete boundary data whereas the Robin inverse problem has also been solved using the same kind of tools in [16]. The inverse source problem, as well as the Cauchy problem of determining an unknown part of the boundary from available data on the complementary part, are currently under study with these techniques [4].

Acknowledgments

The authors wish to thank the reviewers, whose remarks and suggestions allowed them to improve their main results. The LAMSIN's researchers work is supported by the Tunisian Secretary of State for Research, within the LAB-STI-02 programme, and by INRIA through its associate research teams programme.

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