ASYMPTOTIC ESTIMATES FOR INTERPOLATION AND CONSTRAINED APPROXIMATION IN H^2 BY DIAGONALIZATION OF TOEPLITZ OPERATORS

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Sharp convergence rates are provided for interpolation and approximation schemes in the Hardy space H^2 that use band-limited data. By means of new explicit formulae for the spectral decomposition of certain Toeplitz operators, sharp estimates for Carleman and Krein–Nudel'man approximation schemes are derived. In addition, pointwise convergence results are obtained. An illustrative example based on experimental data from a hyperfrequency filter is provided.

1 Notation

Let $\mathbb{T}$ denote the unit circle and $\mathbb{D}$ the open unit disk. We write $\mathbb{T} = I \cup J$, the union of two disjoint arcs, say one of which is open for definiteness. Without loss of generality, we can take $I = (e^{-ia}, e^{ia})$ and $J = [e^{ia}, e^{i(2\pi-a)}]$, where $0 < a < \pi$.

For an interval $E \subset \mathbb{T}$ or $E \subset \mathbb{R}$ and $1 \leq p \leq \infty$, we denote by $L^p(E)$ the familiar Lebesgue space and by $\|.\|_{L^p(E)}$ the corresponding norm; the symbol $(\ , \ )_{L^2(E)}$ indicates the scalar product in $L^2(E)$. The Sobolev space $\mathcal{W}^{1,p}(E)$ consists of functions in $L^p(E)$ having a derivative in the distributional sense that belongs to $L^p(E)$; since E is 1-dimensional in our case, a function belongs to $\mathcal{W}^{1,p}(E)$ if, and only if, it coincides a.e. on E with some absolutely continuous function whose derivative lies in $L^p(E)$ (see, for example, [11, thm VIII.2]). When k is an integer strictly greater than 1, the space $\mathcal{W}^{k,p}(E)$ is defined inductively to consist of functions in $L^p(E)$ whose distributional derivative lies in $\mathcal{W}^{k-1,p}(E)$. Whenever f is defined on some open subset of $\mathbb{T}$, we let f' be its ordinary derivative with respect to θ. More generally the superscript $'$ denotes the derivative for functions of a real variable.

We designate by H^2 the Hardy space with exponent 2 of the unit disk, consisting of functions in $L^2(\mathbb{T})$ whose Fourier coefficients of strictly negative index do vanish. Such functions have a Poisson extension in $\mathbb{D}$ which is not just harmonic but in fact holomorphic, and one recovers the function from its extension by taking non-tangential limits a.e. on $\mathbb{T}$ (see *e.g.* [15, 22, 23]). For that reason, with a slight abuse of notation, we regard H^2 both as a subset of $L^2(\mathbb{T})$ and as a Hilbert space of holomorphic functions on $\mathbb{D}$.

It is well-known that $\log |g|$ belongs to $L^1(\mathbb{T})$ whenever g lies in H^2 and g is not the zero function. This entails that an H^2-function is uniquely defined by the values it assumes on a subset of $\mathbb{T}$ of positive

Lebesgue measure. Conversely, whenever $m \in L^2(\mathbb{T})$ is a positive function such that $\log m \in L^1(\mathbb{T})$, the function

$$\psi(z) = \exp\left\{\frac{1}{2\pi}\int_{\mathbb{T}} \frac{e^{it}+z}{e^{it}-z}\log m(t)\,dt\right\}, \ z \in \mathbb{D} \tag{1}$$

lies in H^2 and is called the (normalized) *outer function* associated with m [15, 22, 23]; here, and elsewhere, for $\Gamma \subset \mathbb{T}$, the notation $\int_\Gamma$ indicates that we integrate over those t with $e^{it} \in \Gamma$. Granted the normalization condition $\psi(0) > 0$, the outer function associated with m is characterized by two facts, namely:

(i) $|\psi| = m$ a.e. on $\mathbb{T}$,

(ii) among H^2-functions that satisfy (i), ψ is largest-in-modulus pointwise on $\mathbb{D}$.

Intuitively, outer functions should be regarded as those Hardy functions having a well-defined logarithm on $\mathbb{T}$.

For $E \subset \mathbb{T}$, we write $H^2_{|_E}$ to mean the space of traces on E of H^2 functions. More generally, the subscript $|_E$ indicates restriction to E.

We denote by $\bar{H}^2_0$ the orthogonal complement of H^2 in $L^2(\mathbb{T})$, consisting of functions whose Fourier coefficients of non-negative index vanish. Subsequently, we let $P_{H^2}\ : L^2(\mathbb{T}) \to H^2$ be the orthogonal projection, and

$$\phi: \quad H^2 \quad \to H^2 \tag{2}$$

$$g \quad \mapsto P_{H^2}(\chi_J g) \tag{3}$$

be the Toeplitz operator with symbol χ_J, the characteristic function of J. Since

$$(\phi g, h)_{L^2(\mathbb{T})} = (P_{H^2}(\chi_J g), h)_{L^2(\mathbb{T})} = (\chi_J g, h)_{L^2(\mathbb{T})} = (g, h)_{L^2(J)}, \quad g, h \in H^2,$$

it is clear that ϕ is a strictly positive self-adjoint operator; in fact, it has no point spectrum and its spectrum is $[0, 1]$ (see [22]).

The Landau notations big O and little o will be given their standard meaning for comparison of functions, namely $f = O(g)$ as $x \to x_0$ means that $\limsup_{x\to x_0} |f(x)/g(x)| < \infty$ and $f = o(g)$ as $x \to x_0$ means that $\lim_{x\to x_0} f(x)/g(x) = 0$. The notation $f \simeq g$ as $x \to x_0$ will be used to express the property that $\lim_{x\to x_0} f(x)/g(x) = 1$.

2 Introduction

In [2, 6, 8, 17, 24], a family of bounded extremal problems was studied that generalizes classical dual extremal problems in H^p to the case where the approximation is sought on a proper subset of $\mathbb{T}$. Existence and uniqueness results are available there, together with a characterization of solutions leading to convergent numerical algorithms. In this paper, we shall be concerned exclusively with $p = 2$, in which case the bounded extremal problem in question can be stated as follows:

(BEP) *given $f \in L^2(I)$, $\Psi \in L^2(J)$, and $M > 0$, find $g = g_\Psi \in H^2$ to minimize $\|f - g\|_{L^2(I)}$ under the constraint $\|\Psi - g\|_{L^2(J)} \le M$.*

This question was originally considered in [17] when $f = 0$, in [2] when $\Psi = 0$, and generally in [6] where the connection to Carleman's interpolation formulas [3, 21] was also stressed. An extension to more abstract function spaces has been carried out in [19]. We refer to [13] for a recent survey of this and related approximation problems.

Apart from their theoretical interest, such problems have several physical motivations. For example, they occur in signal deconvolution and linear systems identification from partial frequency-response measurements [2, 8, 16, 20], as well as in the study of inverse 2-D Dirichlet–Neumann problems such as those occurring in fault detection [7]. More generally, the question of approximating a function on an arc by one which is analytic in a prescribed region of the plane arises in many inverse problems involving 1-D Fourier transforms or 2-D Laplacians. Typically, one could regard f as the measured or designed behaviour on I of some H^2-function, Ψ as a reference behaviour for that function on J, and M as a tolerance on the unmodelled energy one is willing to allow off I in order to have a better fit on I between the data f and the model g_Ψ.

In [2, 6, 19], it is established that there always exists a unique solution g_Ψ to (BEP); moreover $\|\Psi - g_\Psi\|_{L^2(J)} = M$, unless f is the trace on I of some H^2-function h such that $\|h - \Psi\|_{L^2(J)} < M$ in which case $g_\Psi = h$ of course. In the present paper, we study the decrease of $\|f - g_\Psi\|_{L^2(I)}$ relative to the increase of M. We shall distinguish according whether $f \notin H^2_{|_I}$ or $f \in H^2_{|_I}$, the two situations being closely related but quite different in character.

Approximation: When $f \notin H^2_{|_I}$, we refer to (BEP) as *the approximation problem.* Under this assumption $\|f - g_\Psi\|_{L^2(I)}$ goes to zero *if, and only if* M goes to infinity; this follows easily from the density of $H^2_{|_I}$ in $L^2(I)$ and the weak-compactness of balls in H^2 [2]. In this case it can be proved (see [6]) that $\|\Psi - g_\Psi\|_{L^2(J)} = M$ so that, by uniqueness of the solution, $\|f - g_\Psi\|_{L^2(I)}$ is a strictly decreasing function of M for fixed f and Ψ, that may as well be inverted to regard M as strictly decreasing function of $\|f - g_\Psi\|_{L^2(I)}$. The approximation problem is the one encountered in practice. Indeed, if one thinks again of f as the result of certain measurements or computations to represent an H^2-function on I, the unavoidable experimental or numerical errors will prevent f from ever being exactly the trace of an H^2-function. Therefore the modelling error $\|f - g_\Psi\|_{L^2(I)}$ may become small only if M goes large, and a trade-off has to be made in which the increase of M relative to the decrease of $\|f - g_\Psi\|_{L^2(I)}$ plays a central role that motivates the present study.

Specifically, letting for simplicity $e = \|f - g_\Psi\|^2_{L^2(I)}$ denote the approximation error, we shall obtain asymptotic formulas for M as a function of e when the latter goes to zero, that are essentially sharp with respect to some Sobolev-type assumptions for f on I (*cf.* Corollaries 4.5 and 4.6). We also treat the situation where f is a meromorphic function in the disk of the form h/q with $h \in H^2$ and q a trigonometric polynomial. This is a case where f is ultra-smooth, not only on I but also in a 2-dimensional neighborhood of it, and a very important one in practice since it comprises rational functions, in particular trigonometric polynomials. In this connection, it is significant that the increase of M is *much slower* than before. As a byproduct of the analysis, we also get that $g_\Psi(\lambda)$ converges pointwise *a.e.* on I to f when f has absolutely continuous derivative. Upper estimates of this kind were obtained previously in [5], but they were rather pessimistic in view of Theorem 4.3.

Interpolation: When $f \in H^2_{|_I}$, we refer to (BEP) as *the interpolation problem.* In this case, for simplicity, we allow ourselves a slight abuse of notation in that we will continue to denote by f the H^2-function defined on the whole of $\mathbb{T}$. With this convention, $\|f - g_\Psi\|_{L^2(I)}$ decreases strictly to zero as M increases to

$\|f-\Psi\|_{L^2(J)}$ and vanishes identically for $M \geq \|f-\Psi\|_{L^2(J)}$; this is again a straightforward consequence of the weak-compactness of balls in H^2. From a constructive point of view, the interpolation problem is not so interesting since the slightest error in the numerical representation of f on I will destroy its analytic character and bring us back to an approximation problem whose answer will depend on Ψ and M in a crucial manner. This is but one way of regarding the classical ill-posedness of recovering analytic functions from incomplete boundary data [18]. However, the interpolation problem is interesting from a mathematical viewpoint because the set of solutions for $M < \|f-\Psi\|_{L^2(J)}$ coincides with an approximating family introduced in [21] which is itself an outgrowth of classical recovery schemes dating back to Carleman [3]. This connection, noted in [6], is perhaps unexpected since [21] is not concerned with optimality properties of the family in question. Our contribution here will be to show that $\|f-g_\Psi\|_{L^2(I)}$ tends to zero exponentially fast as M increases to $\|f-\Psi\|_{L^2(J)}$, and subsequently that g_Ψ converges to f pointwise a.e. on $\mathbb{T}$ if f has an absolutely continuous derivative there. Because g_Ψ was merely known to converge in H^2 so far, this yields a new piece of information on a rather old interpolation scheme.

The present paper dwells on the fact that the solution to (BEP) can be expressed in terms of a real parameter $\lambda \in (-1,+\infty)$ playing the role of a Lagrange multiplier, *cf.* [2, 6, 19]. More precisely, if we let

$$\tilde{f} = \begin{cases} f & \text{on } I \\ 0 & \text{on } J \end{cases}, \qquad \tilde{\Psi} = \begin{cases} 0 & \text{on } I \\ \Psi & \text{on } J \end{cases}, \tag{4}$$

and if f is not the trace on I of a H^2-function (again denoted by f) such that $\|f-\Psi\|_{L^2(J)} < M$, then the solution g_Ψ to (BEP) assumes the form

$$g_\Psi = g_\Psi(\lambda) = (1+\lambda\,\phi)^{-1}\,P_{H^2}\left(\tilde{f} + (1+\lambda)\tilde{\Psi}\right) \tag{5}$$

where ϕ is the Toeplitz operator defined in (2) and $\lambda \in (-1,+\infty)$ is some real number such that $\|\Psi - g_\Psi(\lambda)\|_{L^2(J)} = M$. Although λ does not appear in the statement of the problem, (BEP) is most conveniently studied if we use (5) to *define* $g_\Psi(\lambda)$ as a function of $\lambda \in (-1,+\infty)$, and if we introduce

$$e_\Psi(\lambda) = \|f - g_\Psi(\lambda)\|^2_{L^2(I)}, \qquad M_\Psi(\lambda) = \|\Psi - g_\Psi(\lambda)\|_{L^2(J)}.$$

The first technical observation to be made is that $e_\Psi(\lambda)$ and $M^2_\Psi(\lambda)$ are *real analytic* functions of λ. For instance, if we write

$$\begin{aligned} M^2_\Psi(\lambda) &= \|\Psi\|^2_{L^2(J)} - 2\mathrm{Re}\Big\{\big(g_\Psi(\lambda),\tilde{\Psi}\big)_{L^2(\mathbb{T})}\Big\} + (\chi_J g_\Psi(\lambda), g_\Psi(\lambda))_{L^2(\mathbb{T})} \\ &= \|\Psi\|^2_{L^2(J)} - 2\mathrm{Re}\Big\{\big(g_\Psi(\lambda), P_{H^2}(\tilde{\Psi})\big)_{L^2(\mathbb{T})}\Big\} + (\phi g_\Psi(\lambda), g_\Psi(\lambda))_{L^2(\mathbb{T})}, \end{aligned}$$

the real analytic character of $M^2_\Psi(\lambda)$ follows at once from (5) and the spectral theorem as applied to ϕ; a similar argument works for $e_\Psi(\lambda)$ if one takes into account the elementary identity

$$P_{H^2}(\chi_I g) = (1-\phi)\,g, \qquad g \in H^2. \tag{6}$$

Note also that neither M_Ψ nor e_Ψ can be the constant function except if $\tilde{f} + \tilde{\Psi} \in H^2$, for we saw that $\|f - g_\Psi\|_{L^2(I)}$ strictly decreases as M increases while $g_\Psi = g_\Psi(\lambda)$ for some $\lambda \in (0,+\infty)$ by (5), unless $f \in H^2_{|_I}$ and $M \geq \|f-\Psi\|_{L^2(J)}$.

Assuming that $\tilde{f}+\tilde{\Psi} \notin H^2$, our second observation is that $e(\lambda)$ strictly increases with λ and that $M_\Psi^2(\lambda)$ strictly decreases. To see this, suppose that λ_1 and λ_2 are two unequal parameters such that $g_\Psi(\lambda_1) = g_\Psi(\lambda_2) = g_\Psi$, say. Then $(1+\lambda_j\phi)g_\Psi = P_{H^2}\left(\tilde{f}+(1+\lambda_j)\tilde{\Psi}\right)$ for $j=1,\,2$, so on subtracting we obtain $(\lambda_1-\lambda_2)\phi g_\Psi = P_{H^2}((\lambda_1-\lambda_2)\tilde{\Psi})$. Thus $P_{H^2}(\chi_J g_\Psi - \tilde{\Psi}) = 0$. Since a nonzero anti-analytic function cannot vanish on a set of positive measure, this implies that $g_\Psi = \Psi$ on J, and so $M=0$. Otherwise, the uniqueness of the solution for each M implies that $e_\Psi(\lambda)$ and $M_\Psi^2(\lambda)$ are strictly monotonic functions of λ.

The strict monotonicity that we just observed implies, if f is not the trace on I of a H^2-function such that $\|f-\Psi\|_{L^2(J)} < M$, that λ in (5) is *uniquely* determined by the requirement that $M_\Psi(\lambda) = M$. Of course the correct guess for λ is not known *a priori*, and the constructive approach to (BEP) proposed in [2] relies on iterative applications of (5) where the Lagrange multiplier is adjusted according to a dichotomy procedure that makes it converge to the right value. The situation when $f \in H^2$ satisfies $\|f-\Psi\|_{L^2(J)} < M$, which was left out of consideration, can be recaptured by letting $\lambda = -1$ in (5), for (6) shows that $P_{H^2}(\tilde{f}) = (1-\phi)f$ and setting $\lambda=-1$ in (5) yields then $g_\Psi = f$ which is indeed the solution to (BEP) in this case.

To recap, given $\lambda > -1$ we know that $g_\Psi(\lambda)$ is the solution to (BEP) corresponding to $M = M(\lambda)$, and also that $e_\Psi^{1/2}(\lambda)$ is the value of the problem. If $f \notin H^2_{|_I}$, every instance of (BEP) gets associated in this manner to some unique value of λ. If $f \in H^2_{|_I}$, only those instances of (BEP) such that $M < \|\Psi - f\|_{L^2(J)}$ can be recast in this fashion while the remaining ones are recovered in the limiting case $\lambda=-1$. However, we shall no longer be concerned with (BEP) in the trivial case where $f \in H^2_{|_I}$ and $M \geq \|\Psi - f\|_{L^2(J)}$, so the parametrization of solutions in terms of $\lambda \in (-1,+\infty)$ is well-adapted to our needs. In any case, we have that

$$\lim_{\lambda\to-1^+} e_\Psi(\lambda) = 0. \tag{7}$$

If we are considering the approximation problem, that is to say if $f \notin H^2_{|_I}$, then it also holds that

$$\lim_{\lambda\to-1^+} M_\Psi(\lambda) = +\infty. \tag{8}$$

If we are considering the interpolation problem, in other words if f is the trace on I of an H^2-function (still denoted by f), then

$$\lim_{\lambda\to-1^+} M_\Psi(\lambda) = \|f-\Psi\|_{L^2(J)}. \tag{9}$$

The general approach we take to the asymptotic analysis of the approximation problem is to estimate the rate of convergence in (7) and (8) and then eliminate λ to obtain an inequality between $\|f-g_\Psi\|_{L^2(I)}$ and M. When dealing with the interpolation problem, we estimate the rate of convergence in (7) and (9) in a similar manner, but then take advantage of a singular integral representation of g_Ψ of Carleman type, where λ is naturally connected to the exponent of the kernel, in order to establish the convergence properties that we seek.

Let us stress once again that the approximation problem may be regarded as a substitute for interpolation in practical situations, that allows one to discriminate rather efficiently between close-to-analytic data and far-from-analytic ones. This way (BEP) can be used as a tool in modelling practice, and we shall examplify this on real data from a hyperfrequency filter provided to the authors by the French National Space Agency (CNES-Toulouse) and processed using the software *Hyperion* developed at INRIA-Sophia. In fact, the need for solving such problems in harmonic identification originally motivated the present investigations.

Also, the function Ψ in (BEP) provides some flexibility in applications, but plays no significant role in the analysis to come. In fact, results will be proved first when $\Psi = 0$, and then carried over over to $\Psi \neq 0$ *via* the formula:

$$g_\Psi(\lambda) = g_0(\lambda) + (1+\lambda)\,(1+\lambda\,\phi)^{-1}\,P_{H^2}\tilde{\Psi} \tag{10}$$

which is an immediate consequence of (5). For that reason, we will often drop the subscript 0 and write

$$e(\lambda) = e_0(\lambda) = \|f - g_0(\lambda)\|^2_{L^2(I)}\,,$$

$$M(\lambda) = M_0(\lambda) = \|g_0(\lambda)\|_{L^2(J)}\,,$$

where

$$g_0(\lambda) = (1+\lambda\,\phi)^{-1}\,P_{H^2}\tilde{f} \tag{11}$$

is the solution associated to $\Psi = 0$ through (5).

Our working tool will be the constructive diagonalization procedure for Toeplitz operators [22] as applied to the following formulas obtained in [2]:

$$M^2(\lambda) = (\phi(1+\lambda\phi)^{-2}P_{H^2}\tilde{f}, P_{H^2}\tilde{f})_{L^2(\mathbb{T})}, \tag{12}$$

$$e'(\lambda) = -(\lambda+1)\,(M^2)'(\lambda)\,. \tag{13}$$

Differentiability is understood here in the strong sense: we saw that $e(\lambda)$ and $M^2(\lambda)$ are smooth (even real analytic) functions of $\lambda \in (0, +\infty)$.

The outline of the paper is as follows. In Section 3, we recall the diagonalization procedure from [22] which exhibits an explicit unitary transformation between H^2 and $L^2(0,1)$ transforming a Toeplitz operator into a multiplication operator. In Section 4, we apply this constructive spectral theory to formula (12) for a Sobolev class of functions f in order to get the asymptotic estimates of Theorem 4.3 for e_Ψ and M_Ψ as λ approaches -1; their sharpness is discussed in Remark 4.4. Next, we consider in Section 5 the case where f is the trace on I of a meromorphic function, using the residue theorem to compute the effect of the concrete diagonalization procedure on $\tilde{f}$.

Finally, we restrict our attention in Section 6 to the interpolation problem, for which stronger asymptotics hold as derived in Subsection 6.1; pointwise convergence results are derived in Subsection 6.2, and a numerical example is shown in Section 7. Concluding remarks are made in Section 8.

3 Concrete spectral theory

The cornerstone of the present work is that formula (12) can be re-expressed using the spectral measure of ϕ. More precisely, following the concrete spectral theory and the diagonalization procedure for self–adjoint Toeplitz operators of multiplicity 1 given in [22, ch.3], we see that ϕ is unitarily equivalent to multiplication M_x by the independent variable x on $L^2([0,1], d\rho)$, where $d\rho(x) = C dx$ with $C = \sin a/\pi$. In fact, there exists a unitary transformation $V : H^2 \to L^2([0,1], d\rho)$ such that

$$V\,\phi\,V^{-1} = M_x\,, \tag{14}$$

which acts on Cauchy kernels $k_\alpha(z) = 1/(1-\bar{\alpha}z)$ as

$$(Vk_\alpha)(x) = [\overline{\psi_x(\alpha)}(1-\bar{\alpha}e^{ia})^{1/2}(1-\bar{\alpha}e^{-ia})^{1/2}]^{-1}, \tag{15}$$

where, for $0 < x < 1$, we let ψ_x be the unique outer function (cf. (1)) such that $\psi_x(0) > 0$ and

$$|\psi_x|^2 = \begin{cases} x & \text{on } I, \\ 1-x & \text{on } J, \end{cases}$$

and where the principal branch of the square root, namely the one which is positive for positive arguments, is used in (15).

We now generalize formula (15) as follows:

THEOREM 3.1 *For every $h \in L^2(\mathbb{T})$ such that*

$$(1 - e^{-i\theta}e^{ia})^{-1/2}(1 - e^{-i\theta}e^{-ia})^{-1/2}\, h(e^{i\theta}) \in L^1(\mathbb{T}), \tag{16}$$

we have that, for a.e. $x \in (0,1)$,

$$V\,(P_{H^2}\,h)(x) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{h(e^{i\theta})\,d\theta}{\overline{\psi_x(e^{i\theta})}\,(1 - e^{-i\theta}e^{ia})^{1/2}(1 - e^{-i\theta}e^{-ia})^{1/2}}\,. \tag{17}$$

PROOF. First, let h be a trigonometric polynomial. Then h extends analytically across $\mathbb{T}$, and by the Cauchy formula we get for $r > 1$:

$$(P_{H^2}\,h)(z) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{h(r\,e^{i\theta})\,d\theta}{1 - \frac{z\,e^{-i\theta}}{r}}\,, \quad |z| < r\,.$$

Thus, if we divide $[0, 2\pi)$ into n intervals $[\theta_k, \theta_{k+1})$ of equal length, $(P_{H^2}\,h)(z)$ is equal for $|z| \le 1$ to the uniform limit as $n \to \infty$ of the following Riemann sum:

$$\frac{1}{2\pi}\sum_{k=0}^{n-1} \frac{h(r\,e^{i\theta_k})(\theta_{k+1} - \theta_k)}{1 - \frac{z\,e^{-i\theta_k}}{r}}\,.$$

Since $V : H^2 \to L^2([0,1], d\rho)$ is an isometry, $V\,(P_{H^2}\,h)$, when viewed as a function of $x \in [0,1]$, is equal by formula (15) to the $L^2([0,1], d\rho)$ limit of

$$\frac{1}{2\pi}\sum_{k=0}^{n-1} h(r\,e^{i\theta_k})(\theta_{k+1} - \theta_k)\,\left(V\,k_{e^{i\theta_k}/r}\right)(x)$$

$$= \frac{1}{2\pi}\sum_{k=0}^{n-1} \frac{h(r\,e^{i\theta_k})\,(\theta_{k+1} - \theta_k)}{\overline{\psi_x(e^{i\theta_k}/r)}(1 - e^{-i\theta_k}e^{ia}/r)^{1/2}(1 - e^{-i\theta_k}e^{-ia}/r)^{1/2}}\,.$$

As the L^2 limit is certainly equal to the pointwise limit when the latter exists and since ψ_x is continuous on the circle $|z| = 1/r$, for each $r > 1$ and $x \in (0,1)$, we get by taking the limit of the above Riemann sum that

$$\left(V\,(P_{H^2}\,h)\right)(x) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{h(r\,e^{i\theta})\,d\theta}{\overline{\psi_x(e^{i\theta}/r)}(1 - e^{-i\theta}e^{ia}/r)^{1/2}(1 - e^{-i\theta}e^{-ia}/r)^{1/2}}\,.$$

Letting $r \to 1$ proves the theorem for trigonometric polynomials by dominated convergence since $|\psi_x|$ is uniformly bounded away from zero in $\mathbb{D}$ for fixed $x \in (0,1)$. When h is continuous, it is the uniform limit on $\mathbb{T}$ of a sequence of trigonometric polynomials; then the convergence holds both in L^2 and under the integral sign in the right hand-side of (17) for fixed $x \in (0,1)$, the use of dominated convergence being justified by the boundedness of $1/|\psi_x|$ in $\mathbb{D}$ and by hypothesis (16). This proves the result for continuous functions h. If h is merely bounded, we can find a family of continuous functions converging boundedly pointwise *a.e.* to h by Lusin's theorem and the Borel–Cantelli lemma [14, Lemma VIII.3.1]); by Lebesgue's dominated convergence theorem, such a sequence tends to h in L^2 and still the right hand-side of (17) is preserved in the limit, which proves the result for bounded functions. Finally, under the hypotheses of the theorem, we approximate h by the sequence of bounded functions $\chi_{[0,n]}(|h|)\, h$ and appeal to dominated convergence again. ■

Since functions h satisfying (16) are dense in $L^2(\mathbb{T})$, Theorem 3.1 gives a rather explicit description of how V operates on a dense subspace of H^2 comprising, say all continuous functions there, and this is all we shall need to proceed with the estimates we have in mind for (BEP). Nevertheless, it is natural to ask how one computes $V(P_{H^2}\, h)$ for any $h \in L^2(\mathbb{T})$. For this, he may approximate h in $L^2(\mathbb{T})$ by a sequence h_n satisfying (16), and the corresponding limit in the right-hand side of (17) will hold in $L^2([0,1], d\rho)$ although not necessarily pointwise on x. In this respect, the definition of V is reminiscent of the Fourier transform of a function H, which is defined pointwise as

$$\mathcal{F}(H)(y) = \int_{-\infty}^{+\infty} H(\sigma) e^{-iy\sigma}\, d\sigma$$

if $H \in L^1(\mathbb{R})$, and as the $L^2(\mathbb{R})$-limit, when $A \to +\infty$, of $\mathcal{F}(H\chi_{[-A,A]})$ if $H \in L^2(\mathbb{R})$. This analogy is actually no accident, for there is an explicit link between V and $\mathcal{F}$ which lies at the heart of many computations in the present paper. To state the result conveniently, let us introduce two functions:

$$\omega_I : (-a, a) \to \mathbb{R}, \qquad \omega_I(\theta) = \frac{\log 2}{2\pi} \log \frac{1 - \cos(\theta + a)}{1 - \cos(\theta - a)}\,, \tag{18}$$

$$\omega_J : (a, 2\pi - a) \to \mathbb{R}, \qquad \omega_J(\theta) = \frac{\log 2}{2\pi} \log \frac{1 - \cos(\theta - a)}{1 - \cos(\theta + a)}\,.$$

Note that $\omega_I : I \to \mathbb{R}$ and $\omega_J : J \to \mathbb{R}$ are increasing diffeomorphisms since their derivatives are respectively

$$\omega_I'(\theta) = \frac{\log 2}{2\pi} \frac{2\sin a}{\cos\theta - \cos a}, \qquad \omega_J'(\theta) = \frac{\log 2}{2\pi} \frac{2\sin a}{\cos a - \cos\theta}. \tag{19}$$

Let us also fix the following notation, that will be in use throughout the paper:

$$\gamma(x) = \frac{\log x - \log(1-x)}{2\log 2}, \qquad x \in (0,1). \tag{20}$$

THEOREM 3.2 *To any measurable function $h : \mathbb{T} \to \mathbb{C}$, associate two functions $H_I, H_J : \mathbb{R} \to \mathbb{C}$ by:*

$$H_I(\sigma) = \frac{h(e^{i\theta})}{2\pi\omega_I'(\theta)(1 - e^{-i\theta}e^{ia})^{1/2}(1 - e^{-i\theta}e^{-ia})^{1/2}}\,, \qquad \theta = \omega_I^{-1}(\sigma)\,, \tag{21}$$

$$H_J(\sigma) = \frac{h(e^{i\theta})}{2\pi\omega_J'(\theta)(1 - e^{-i\theta}e^{ia})^{1/2}(1 - e^{-i\theta}e^{-ia})^{1/2}}\,, \qquad \theta = \omega_J^{-1}(\sigma)\,. \tag{22}$$

Then $H_I, H_J \in L^2(\mathbb{R})$ if, and only if, $h \in L^2(\mathbb{T})$, and in this case

$$V(P_{H^2}\, h)(x) = \frac{1}{\sqrt{x}}\, \mathcal{F}(H_I)(-\gamma(x)) - \frac{1}{\sqrt{1-x}}\, \mathcal{F}(H_J)(-\gamma(x)) \qquad a.e.\ x \in (0,1). \tag{23}$$

PROOF. By the chain rule, we get from (21), (22) that

$$\int_{-\infty}^{+\infty} |H_I(\sigma)|^2\, d\sigma = \int_I \frac{|h(\theta)|^2}{4\pi^2|(1-e^{-i\theta}e^{ia})(1-e^{-i\theta}e^{-ia})|\omega_I'(\theta)}\, d\theta,$$

$$\int_{-\infty}^{+\infty} |H_J(\sigma)|^2\, d\sigma = \int_J \frac{|h(\theta)|^2}{4\pi^2|(1-e^{-i\theta}e^{ia})(1-e^{-i\theta}e^{-ia})|\omega_J'(\theta)}\, d\theta.$$

If we take into account the identity:

$$(1-e^{-i\theta}e^{ia})(1-e^{-i\theta}e^{-ia}) = 2e^{-i\theta}(\cos\theta - \cos a),$$

we see from (19) that $\|h\|^2_{L^2(\mathbb{T})}$ is equal, up to a multiplicative constant, to $\|H_I\|^2_{L^2(\mathbb{R})} + \|H_J\|^2_{L^2(\mathbb{R})}$, thereby showing that the former is finite if, and only if, the latter is. In addition, this entails by density that it is enough to establish (23) when h satisfies (16). In this case, starting from (17), the result is obtained as follows. Using the notation introduced in (20), we compute from the definition of an outer function with prescribed modulus given in (1) that

$$\psi_x(e^{i\theta}) = \sqrt{x}\,\Upsilon^{\gamma(x)}(e^{i\theta}) = \sqrt{x}\,\exp(\gamma(x)\,\log\Upsilon(e^{i\theta}))\,,\quad \text{a.e. on } \mathbb{T}\,,$$

where Υ is the outer function such that

$$|\Upsilon(e^{i\theta})| = \begin{cases} 1 & \text{a.e. on } I, \\ 1/2 & \text{a.e. on } J. \end{cases}$$

Denote by $\tilde{\omega}_I$, $\tilde{\omega}_J$ the argument of Υ on I, J, respectively. It coincides with ω_I on I, with ω_J on J. Indeed, for $e^{i\theta} \in I$, we have:

$$\tilde{\omega}_I(e^{i\theta}) = -\frac{\log 2}{2\pi}\,\mathrm{Im}\int_a^{2\pi - a} \frac{e^{it}+e^{i\theta}}{e^{it}-e^{i\theta}}\, dt\,.$$

A direct computation gives

$$\tilde{\omega}_I(e^{i\theta}) = \frac{\log 2}{\pi}\log\left(e^{ia}\frac{e^{i\theta}-e^{-ia}}{e^{i\theta}-e^{ia}}\right) = \frac{\log 2}{2\pi}\log\left|e^{ia}\frac{e^{i\theta}-e^{-ia}}{e^{i\theta}-e^{ia}}\right|^2 = \omega_I(e^{i\theta})\,,\quad e^{i\theta}\in I, \tag{24}$$

from the definition (18), the quantity $e^{ia}\,(e^{i\theta}-e^{-ia})\,/\,(e^{i\theta}-e^{ia})$ being real valued there. Similarly, we see that $\tilde{\omega}_J = \omega_J$. Let us now rewrite ψ_x in polar form:

$$\psi_x(e^{i\theta}) = \begin{cases} \sqrt{x}\,\exp(i\,\gamma(x)\,\omega_I(e^{i\theta}))\,, & \text{a.e. on } I\,, \\ \sqrt{1-x}\,\exp(i\,\gamma(x)\,\omega_J(e^{i\theta}))\,, & \text{a.e. on } J\,, \end{cases} \tag{25}$$

If we set for simplicity

$$H(\theta) = \frac{h(e^{i\theta})}{(1-e^{-i\theta}e^{ia})^{1/2}(1-e^{-i\theta}e^{-ia})^{1/2}}\,,\ e^{i\theta}\in I\,,$$

then $H \in L^1(I)$ by (16). From

$$V(P_{H^2}\,\chi_I\,h)(x) = \frac{1}{2\pi}\int_I \frac{h(e^{i\theta})\psi_x(e^{i\theta})\,d\theta}{x(1-e^{-i\theta}e^{ia})^{1/2}(1-e^{-i\theta}e^{-ia})^{1/2}}\,,\quad x\in(0,1),$$

together with (25), we obtain

$$V(P_{H^2}\,\chi_I\,h)(x) = \frac{1}{2\pi}\int_I \frac{H(\theta)\exp[i\gamma(x)\omega_I(\theta)]\,d\theta}{\sqrt{x}}\,.$$

Performing the change of variable $\sigma = \omega_I(\theta)$ and the analogous calculation on J leads to (23). ■

4 Approximation in a Sobolev class

We now return to the approximation problem (BEP) and we shall apply the results of the previous section to formula (12). For $f \in L^2(I)$ and $\tilde{f}$ as defined in (4), we let

$$v = V\,P_{H^2}\tilde{f}\,, \tag{26}$$

where V was introduced in (14).

PROPOSITION 4.1 *Suppose that f satisfies*

$$(1-e^{-i\theta}e^{ia})^{-1/2}(1-e^{-i\theta}e^{-ia})^{-1/2}\,f(e^{i\theta}) \in L^1(I)\,, \tag{27}$$

and

$$(1-e^{-i\theta}e^{ia})^{1/2}(1-e^{-i\theta}e^{-ia})^{1/2}\,f(e^{i\theta}) \in \mathcal{W}^{1,1}(I)\,. \tag{28}$$

Then

$$\lim_{x\to 1^-} v(x)\log(1-x) = 0. \tag{29}$$

PROOF. First, (27) and (28) imply that $f \in L^2(I)$, since $\mathcal{W}^{1,1}(I) \subset L^\infty(I)$ and

$$\begin{aligned}\|f\|^2_{L^2(I)} \;\leq\; & \|(1-e^{-i\theta}e^{ia})^{1/2}(1-e^{-i\theta}e^{-ia})^{1/2}\,f\|_{L^\infty(I)} \\ & \times\|(1-e^{-i\theta}e^{ia})^{-1/2}(1-e^{-i\theta}e^{-ia})^{-1/2}\,f\|_{L^1(I)}\,.\end{aligned}$$

Thus $\tilde{f}$ satisfies the hypotheses of Theorem 3.1, and it follows from the latter and from the definition of ψ_x that

$$v(x) = \frac{1}{2\pi}\int_I \frac{f(e^{i\theta})\psi_x(e^{i\theta})\,d\theta}{x(1-e^{-i\theta}e^{ia})^{1/2}(1-e^{-i\theta}e^{-ia})^{1/2}}\,,\quad x \in (0,1). \tag{30}$$

As in the proof of Theorem 3.2, setting

$$F(\theta) = \frac{f(e^{i\theta})}{(1-e^{-i\theta}e^{ia})^{1/2}(1-e^{-i\theta}e^{-ia})^{1/2}}\,, \tag{31}$$

it holds that $F \in L^1(I)$ by (27) and

$$v(x) = \frac{1}{2\pi}\int_I \frac{F(\theta)\exp[i\gamma(x)\omega_I(\theta)]\,d\theta}{\sqrt{x}}\,. \tag{32}$$

Again we let $\sigma = \omega_I(\theta)$ and conclude that

$$v(x) = \frac{1}{\sqrt{x}}\,\mathcal{F}(G)(-\gamma(x))\,, \tag{33}$$

where

$$G(\sigma) = \frac{F(\theta)}{2\pi\omega_I'(\theta)}\,, \qquad \theta = \omega_I^{-1}(\sigma)\,. \tag{34}$$

To unwind the definition of G, we observe that

$$(1-e^{-i\theta}e^{ia})(1-e^{-i\theta}e^{-ia}) = 2e^{-i\theta}(\cos\theta - \cos a),$$

and we obtain from (34) in conjunction with (31) and (19) that

$$\begin{aligned} G(\omega_I(\theta)) &= \frac{f(e^{i\theta})(\cos\theta-\cos a)}{2(\log 2)(\sin a)(1-e^{-i\theta}e^{ia})^{1/2}(1-e^{-i\theta}e^{-ia})^{1/2}} \\ &= \frac{f(e^{i\theta})e^{i\theta}(1-e^{-i\theta}e^{ia})^{1/2}(1-e^{-i\theta}e^{-ia})^{1/2}}{4(\log 2)(\sin a)}. \end{aligned} \tag{35}$$

Now, by (34) and the chain rule, we have

$$\int_{-\infty}^{+\infty} |G(\sigma)|\, d\sigma = \int_I |F(\theta)|\, d\theta \quad \text{ and } \quad \int_{-\infty}^{+\infty} |G'(\sigma)|\, d\sigma = \int_I \Big|\frac{d}{d\theta} G(\omega_I(\theta))\Big|\, d\theta.$$

Consequently, we see from (31) and (35) that G belongs to $\mathcal{W}^{1,1}(\mathbb{R})$ if, and only if, f satisfies (27) and (28). Moreover, since the Fourier transform converts differentiation into multiplication by the independent variable, it follows from (33) that

$$\gamma(x)\sqrt{x}v(x) = -\mathcal{F}(G')(-\gamma(x))$$

and, in view of (20), we obtain

$$|\log(1-x)\, v(x)| \simeq 2\log 2\, |\mathcal{F}(G')(-\gamma(x))|\,, \quad \text{as } x \to 1^-;$$

however, the Fourier transform of an $L^1(\mathbb{R})$ function is continuous on $\mathbb{R}$ and goes to 0 at $\pm\infty$ by the Riemann–Lebesgue lemma [23, thm 9.6], thereby establishing (29). ∎

Estimates for $M(\lambda)$ and $e(\lambda)$ will follow from Proposition 4.1 and the following lemma.

LEMMA 4.2 *There exist absolute constants $\mu_0 > 1$, $C_1 > 0$, and $C_2 > 0$ such that, for any increasing function $\varepsilon : (0, 1/2) \to \mathbb{R}^+$, we have:*

$$\int_0^{1/2} \frac{\varepsilon(x)\, dx}{(1+\mu x)^2 \log^2 x} \le \frac{C_1\,\varepsilon\left(\log^3 \mu/\mu\right)}{\mu \log^2 \mu} + \frac{C_2 \varepsilon(1/2)}{\mu \log^3 \mu}$$

as soon as $\mu \ge \mu_0$.

PROOF. If β is such that

$$1/\mu < \beta < 1/2, \tag{36}$$

then

$$\int_0^{\beta} \frac{\varepsilon(x)\, dx}{(1+\mu x)^2 \log^2 x} \le (\varepsilon(\beta)/\log^2\beta) \int_0^{\beta} \frac{dx}{(1+\mu x)^2} = \frac{\beta\varepsilon(\beta)}{(1+\mu\beta)\log^2\beta}\,;$$

also

$$\int_{\beta}^{1/2} \frac{\varepsilon(x)\, dx}{(1+\mu x)^2 \log^2 x} \le \sup_{\beta \le x \le 1/2} \left(x\varepsilon(x)/(1+\mu x)^2\right) \int_{\beta}^{1/2} \frac{dx}{x \log^2 x} \le \frac{\beta\varepsilon(1/2)}{(1+\mu\beta)^2 \log 2}.$$

Taking $\beta = \log^3 \mu/\mu$, it is easily checked that we satisfy (36) as soon as, say $\mu \geq 17$, and then a short computation shows that $|\log \beta| > \log \mu/10$. From this, the required estimate follows immediately by adding up the two inequalities above. ■

We are now able to state and prove the main result of this section:

THEOREM 4.3 *If f satisfies* (27) *and* (28), *then as $\lambda \searrow -1$,*

$$M_\Psi^2(\lambda) = o\Big((\lambda+1)^{-1}\log^{-2}(\lambda+1)\Big), \tag{37}$$

while

$$e_\Psi(\lambda) = o\Big(|\log^{-1}(1+\lambda)|\Big). \tag{38}$$

PROOF. Let again $v = V P_{H^2}\tilde{f}$. From (14), it follows easily by a continuity argument that, for any continuous $H : [0,1] \to \mathbb{R}$, one has

$$(H(\phi)P_{H^2}\tilde{f}, P_{H^2}\tilde{f})_{L^2(\mathbb{T})} = C\int_0^1 H(t)|v(t)|^2\,dt\,.$$

Therefore, we get from (12) that

$$M^2(\lambda) = C\int_0^1 \frac{t}{(1+\lambda t)^2}|v(t)|^2\,dt\,. \tag{39}$$

For λ near to -1, the behaviour of the integrand near $t = 1$ dominates; to help us derive an estimate, we introduce two auxiliary functions, namely

$$\varrho(y) = |v(1-y)\log y| \qquad \text{for } 0 < y < 1, \tag{40}$$

and

$$\varepsilon(y) = \sup_{0<x\leq y} \varrho^2(x). \tag{41}$$

By (39), we can write

$$M^2(\lambda) = C\int_0^1 \frac{t\,\varrho^2(1-t)\,dt}{(1+\lambda t)^2\log^2(1-t)}. \tag{42}$$

Putting $y = 1-t$ and $\lambda = -1 + 1/\mu$, we obtain

$$M^2(\lambda) = \mu^2 C\int_0^1 \frac{(1-y)\varrho^2(y)\,dy}{(1+(\mu-1)y)^2\log^2 y}, \tag{43}$$

thus *a fortiori*

$$M^2(\lambda) \leq \mu^2\frac{C}{2}\int_0^{1/2} \frac{\varepsilon(y)\,dy}{(1+(\mu-1)y)^2\log^2 y} + O(1) \quad \text{as } \lambda \searrow -1.$$

We now apply Lemma 4.2 with ε as in (41) and μ replaced by $\mu - 1$. Recalling that $\lambda + 1 = \mu^{-1}$, this yields

$$M^2(\lambda) \leq \frac{C_1\,\varepsilon\Big(|1+\lambda^{-1}|\,\big|\log^3|1+\lambda^{-1}|\big|\Big)}{(-\lambda)(\lambda+1)\log^2|1+\lambda^{-1}|} + \frac{C_2\,\varepsilon(1/2)}{(-\lambda)(\lambda+1)\,\big|\log^3|1+\lambda^{-1}|\big|}, \tag{44}$$

$$\leq \frac{C_3\,\varepsilon\Big(|1+\lambda^{-1}|\;\big|\log^3|1+\lambda^{-1}|\big|\Big)}{(\lambda+1)\log^2(\lambda+1)} + \frac{C_4\,\varepsilon(1/2)}{(\lambda+1)\,\big|\log^3(\lambda+1)\big|} \tag{45}$$

for some absolute constants $C_3, C_4 > 0$, as soon as $\lambda + 1 < (\mu_0+1)^{-1}$ with μ_0 as in Lemma 4.2.

Now we turn our attention to the behaviour of $e(\lambda)$. Using (13) and differentiating (39) under the integral sign, we get

$$e'(\rho) = 2C(\rho+1)\int_0^1 \frac{t^2}{(1+\rho t)^3}|v(t)|^2\,dt$$

and, since $e(-1)=0$, integrating by parts with respect to ρ while appealing to Fubini's theorem gives us after a short computation:

$$e(\lambda) = \int_{-1}^{\lambda} e'(\rho)\,d\rho = C(\lambda+1)^2\int_0^1 \frac{t^2}{(1-t)(1+\lambda t)^2}|v(t)|^2\,dt\,. \tag{46}$$

Using (40), this can be rewritten as

$$e(\lambda) = C(\lambda+1)^2\int_0^1 \frac{t^2\,\varrho^2(1-t)\,dt}{(1+\lambda t)^2(1-t)\log^2(1-t)}\,. \tag{47}$$

To get an upper estimate, we restrict ourselves to $-1<\lambda<0$ which is possible since λ will tend to -1 from above, and we split the integral into $\int_0^{-\lambda}$ and $\int_{-\lambda}^1$ that we evaluate separately.
As to the first term, since $0<(\lambda+1)/(1-t)\leq 1$ for $t\leq -\lambda$, we get

$$\begin{aligned} C\,(\lambda+1)^2 &\quad \int_0^{-\lambda} \frac{t^2\,\varrho^2(1-t)\,dt}{(1+\lambda t)^2(1-t)\log^2(1-t)} \\ \leq &\quad C\,(\lambda+1)\int_0^1 \frac{t\,\varrho^2(1-t)\,dt}{(1+\lambda t)^2\log^2(1-t)} = (\lambda+1)M^2(\lambda)\,, \end{aligned}$$

where the last equality follows from (42).
As to the second term, we observe that $0\leq(\lambda+1)/(1+\lambda t)\leq 1$ whence

$$(\lambda+1)^2\int_{-\lambda}^1 \frac{t^2\,\varrho^2(1-t)\,dt}{(1+\lambda t)^2(1-t)\log^2(1-t)} \leq \varepsilon(1+\lambda)\int_{-\lambda}^1 \frac{dt}{(1-t)\log^2(1-t)} = \frac{\varepsilon(\lambda+1)}{|\log(\lambda+1)|}$$

where the second inequality uses (41). Altogether, we have that

$$e(\lambda) \leq (\lambda+1)M^2(\lambda) + \frac{C\,\varepsilon(\lambda+1)}{|\log(\lambda+1)|}\,, \tag{48}$$

and since $\varepsilon(y)\to 0$ when $y\to 0^+$ by Proposition 4.1, the estimates (45) and (48) establish the desired result for $\Psi = 0$.

The general case where $\Psi\in L^2(J)$ now follows easily. Indeed, we get from (10) and the self-adjointness of ϕ that

$$\begin{aligned} \|g_0-g_\Psi\|^2_{L^2(I)} &= (\lambda+1)^2\left((1+\lambda\,\phi)^{-1}\,P_{H^2}\tilde{\Psi},(1+\lambda\,\phi)^{-1}P_{H^2}\tilde{\Psi}\right)_{L^2(I)} \\ &= (\lambda+1)^2\left((1+\lambda\,\phi)^{-2}\;P_{H^2}\tilde{\Psi},\,P_{H^2}\tilde{\Psi}\right)_{L^2(I)}\,, \end{aligned}$$

whence

$$\|g_0 - g_\Psi\|^2_{L^2(I)} = (\lambda+1)^2 \left(\chi_I\,(1+\lambda\,\phi)^{-2}\; P_{H^2}\tilde{\Psi},\; P_{H^2}\tilde{\Psi}\right)_{L^2(\mathbb{T})}.$$

Now, we can apply P_{H^2} to the left argument of the above scalar product without changing its value, because the right argument lies in H^2. Noting that $P_{H^2}(\chi_I u) = (1-\phi)u$ whenever $u \in H^2$, this yields

$$\|g_0 - g_\Psi\|^2_{L^2(I)} = (\lambda+1)^2 \left((1-\phi)\,(1+\lambda\,\phi)^{-2}\; P_{H^2}\tilde{\Psi}, P_{H^2}\tilde{\Psi}\right)_{L^2(\mathbb{T})}.$$

Using the relation

$$1-\phi = (1+\lambda\,\phi) - (\lambda+1)\,\phi\,, \tag{49}$$

together with the obvious upper bound:

$$\|(1+\lambda\,\phi)^{-1}\| \le 1/(\lambda+1) \quad \text{for } -1<\lambda\le 0\,, \tag{50}$$

it follows that

$$\|g_0 - g_\Psi\|^2_{L^2(I)} = O(\lambda+1) \qquad \text{as } \lambda\searrow -1.$$

But the triangular inequality implies that

$$e_\Psi^{1/2}(\lambda) \le e^{1/2}(\lambda) + \|g_0 - g_\Psi\|_{L^2(I)}\,,$$

so by the previous part of the proof

$$e_\Psi^{1/2}(\lambda) = o\left(\frac{1}{|\log(1+\lambda)|^{1/2}}\right) + O\left((\lambda+1)^{1/2}\right) = o\left(\frac{1}{|\log(1+\lambda)|^{1/2}}\right)$$

when $\lambda\searrow -1$ as was to be shown. Also,

$$M_\Psi(\lambda) \le M(\lambda) + \|g_0 - g_\Psi\|_{L^2(J)} = M(\lambda) + \|(\lambda+1)\,(1+\lambda\,\phi)^{-1}\; P_{H^2}\tilde{\Psi}\|_{L^2(J)}\,,$$

and by (50) the last term in the right hand-side remains bounded when $\lambda\searrow -1$. Therefore the estimate for $M(\lambda)$ remains valid for $M_\Psi(\lambda)$ and the proof is complete. ■

REMARK 4.4 A discussion of the sharpness of these estimates is appropriate at this point. When speaking of the sharpness of (37) and (38), we mean that whenever ε_1 and ε_2 are positive functions such that

$$\varepsilon_1(\lambda) = o\Big((\lambda+1)^{-1}\log^{-2}(\lambda+1)\Big) \;\text{ and }\; \varepsilon_2(\lambda) = o\Big(|\log^{-1}(1+\lambda)|\Big) \quad \text{as } \lambda\searrow -1,$$

then there exists f satisfying (27) and (28) such that $M^2_\Psi(\lambda) \ge \varepsilon_1(\lambda)$ and $e_\Psi(\lambda) \ge \varepsilon_2(\lambda)$ as soon as $\lambda+1$ is small enough. By the estimates given at the end of the previous proof, the actual choice of Ψ is irrelevant in this definition of sharpness; hence we consider $\Psi = 0$ only.

Observe, since ε is decreasing, that for $\mu \ge 5$

$$\int_0^1 \frac{(1-y)\varepsilon(y)\,dy}{(1+(\mu-1)y)^2\log^2 y} \ge \frac{(1-2/(\mu-1))\varepsilon(1/(\mu-1))}{\log^2(\mu-1)} \int_{1/(\mu-1)}^{2/(\mu-1)} \frac{dy}{(1+(\mu-1)y)^2}$$

$$\geq \frac{\varepsilon(1/\mu)}{12(\mu-1)\log^2(\mu-1)}, \tag{51}$$

and

$$\int_0^1 \frac{(1-y)\varepsilon(y)\,dy}{(1+(\mu-1)y)^2 y\log^2 y} \geq \frac{\varepsilon(1/\mu^2)}{(1+(\mu-1)/\mu)^2}\int_{1/\mu^2}^{1/\mu}\frac{dy}{y\log^2 y} \geq \frac{\varepsilon(1/\mu^2)}{8\log\mu}. \tag{52}$$

If ϱ defined in (40) happens to be increasing near 0 *so that* $\varepsilon(y)=\varrho^2(y)$ for y small enough, then (43) and (51) will imply

$$M^2(\lambda) \geq \frac{c_1\,\varepsilon(\lambda+1)}{(\lambda+1)\log^2(\lambda+1)}, \tag{53}$$

for some absolute constant $c_1>0$ as soon as $\lambda+1=\mu^{-1}$ is small enough; analogously, we get in this case from (47) and (52) that

$$e(\lambda) \geq \frac{c_2\,\varepsilon((\lambda+1)^2)}{|\log(\lambda+1)|} \tag{54}$$

for some absolute $c_2>0$ as soon as $\lambda+1$ is small enough. The lower estimates (53) and (54) will establish the sharpness of the upper bounds (37) and (38) if we can show that f may be chosen to satisfy (27) and (28) in such a way that $\varrho^2(1-x)=|\log(1-x)\,v(x)|^2$ converges to zero arbitrarily slowly as $x\to 1$ and in addition monotonically for $x_0<x<1$ and some $x_0>0$.

We claim that this is possible. Indeed, we already observed when f and G are related by (31) and (34) that (27) and (28) together are equivalent to the condition $G\in\mathcal{W}^{1,1}(\mathbb{R})$. Therefore it is enough to prove that $|\mathcal{F}(G')|$ can tend to zero arbitrarily slowly at infinity and in a monotonic way there. Now, the Riemann–Lebesgue lemma is known to be sharp, in that *every* continuous even function κ on $\mathbb{R}$ that is convex on $(0,\infty)$ and decreasing monotonically to zero, is the Fourier transform of a function in $L^1(\mathbb{R})$ (see, for example, [25, thm.124]). Note for later use that κ is then absolutely continuous with bounded derivative. Since only the behaviour near infinity is of interest here, we may suppose that κ is linear on $[0,1]$. By adding to κ a continuous piecewise linear even function of compact support, whose inverse Fourier transform will lie in $L^1(\mathbb{R})$, we may obtain a function κ_0 such that:

(i) $\kappa_0(y)=0$ on some neighborhood of 0,

(ii) $\kappa_0(y)=\kappa(y)$ for $|y|$ sufficiently large,

(iii) the inverse Fourier transform of κ_0, say G_1 lies in $L^1(\mathbb{R})$,

(iv) κ_0 is bounded and absolutely continuous with bounded derivative on $\mathbb{R}$.

Let $G\in L^2(\mathbb{R})$ be the function whose Fourier transform is $\kappa_0(y)/y$. It certainly exists since $\kappa_0(y)/y\in L^2(\mathbb{R})$ by (i) and (iv). Taking the derivative in the sense of distributions, we get $\mathcal{F}(G')(y)=i\kappa_0(y)$, which implies that in fact $G'=iG_1$ lies in $L^1(\mathbb{R})$. Since

$$\mathcal{F}(-itG(t))(y)=\frac{d}{dy}(\kappa_0(y)/y)\in L^2(\mathbb{R})$$

(because κ_0 and κ_0' are bounded), we have $tG(t)\in L^2(\mathbb{R})$. Writing $S=\mathbb{R}\setminus[-1,1]$, we have

$$\|G(t)\|_{L^1(S)} \leq \|tG(t)\|_{L^2(S)}\|t^{-1}\|_{L^2(S)} < \infty,$$

by the Cauchy–Schwarz inequality, hence $G\in L^1(\mathbb{R})$ since G is continuous. Altogether $G\in\mathcal{W}^{1,1}(\mathbb{R})$ and $|\mathcal{F}(G')(y)|=\kappa(y)$ when $|y|$ is large enough. Finally, since any positive continuous function $[0,\infty)\to\mathbb{R}^+$

tending to zero at infinity is majorized by a convex continuous function decreasing to zero (a piecewise linear one is easily constructed), we can assume that $\kappa(y)$ goes to zero arbitrarily slowly at infinity *which proves the claim.* Thus (37) and (38) are, indeed, sharp.

Following on from Theorem 4.3 and the previous discussion, we can eliminate the parameter λ between (37) and (38) and obtain in (BEP) an upper bound for M_Ψ in terms of e_Ψ which is sharp with respect to the considered class of functions.

COROLLARY 4.5 *If in* (BEP) f *satisfies the assumptions of Theorem* 4.3, *then to each* $K_1 > 0$ *there is* $K_2 = K_2(f) > 0$ *such that*

$$M_\Psi^2 \leq K_2\, e_\Psi^2 \, \exp\{K_1 e_\Psi^{-1}\}\,. \tag{55}$$

In the above statement, the factor e_Ψ^{-1} *in the exponent cannot be replaced by* $h(e_\Psi)$ *for any function* $h : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $h(x) = o(1/x)$ *as* $x \searrow 0$.

PROOF. By the estimates at the end of Theorem 4.3 and Remark 4.4, we may assume without loss of generality that $\Psi = 0$. The relation $e \to 0$ being equivalent to $\lambda \searrow -1$, it follows from (37) and (38) that

$$M^2(\lambda) \leq \frac{K_1}{(\lambda+1)\log^2(\lambda+1)}\,, \tag{56}$$

and

$$e \leq \frac{K_1}{|\log(\lambda+1)|} \tag{57}$$

as soon as e is small enough. If we set for simplicity $E = 1/|\log(\lambda+1)|$, we can rewrite (56) as $M^2 \leq K_1 E^2 \exp 1/E$ and (57) as $e/K_1 \leq E$. However, for sufficiently small $x > 0$ the function $x \mapsto x^2 \exp 1/x$ is decreasing, and hence for sufficiently small $E > 0$ we have

$$M^2 \leq K_1^{-1}\, e^2 \, \exp\{K_1/e\}. \tag{58}$$

Since (58) is valid for all e small enough and M decreases as e increases, we may adjust K_2 so that (55) holds for all e.

To show that the exponent e^{-1} cannot be replaced by some $o(1/e)$, suppose on the contrary that whenever f satisfies (27) and (28), then to then to each $K_1 > 0$ there is $K_2 = K_2(f) > 0$ such that

$$M^2 \leq K_2\, e^2 \, \exp\{K_1 \eta(e)/e\} \tag{59}$$

for some function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{x\to 0^+} \eta(x) = 0$. We may assume that $\eta(x) \geq x^2$ and also that η is increasing upon replacing it by $\sup_{0<y\leq x} \eta(y)$. By the sharpness of (38) discussed in Remark 4.4, we may choose f such that

$$\frac{\eta^{1/2}(|\log^{-1}(1+\lambda)|)}{|\log(1+\lambda)|} < e(\lambda) < \frac{1}{|\log(1+\lambda)|} \tag{60}$$

as soon as λ is close enough to -1; in addition we may ensure that the associated function ϱ defined in (40) in monotonic near 0. Now, by the monotonicity of η we have that $\eta(e) \leq \eta(|\log^{-1}(1+\lambda)|)$, and inserting the above majorizations in (59) yields

$$M^2 \quad \leq \quad K_2\, \frac{\exp\{K_1 \eta^{1/2}(|\log^{-1}(1+\lambda)|)\,|\log(1+\lambda)|\}}{\log^2(1+\lambda)}$$

$$= \frac{K_2}{(1+\lambda)^{K_1 \eta^{1/2}(|\log^{-1}(1+\lambda)|)} \log^2(1+\lambda)}$$

as soon as λ is close enough to -1. In view of (53) which is valid when $\lambda \searrow -1$ by the monotonicity of $\varrho(y)$ for small y, we deduce that

$$\varepsilon(\lambda+1) \leq \frac{K_2}{c_1}(1+\lambda)^{1-K_1\eta^{1/2}(|\log^{-1}(1+\lambda)|)}$$

hence $\varepsilon(\lambda+1) = o((\lambda+1)^\alpha)$ for every $\alpha < 1$ as $\lambda \to -1$. Comparing this with (48) and (37) we see that $e = o(\log^{-2}(\lambda+1))$, but since $\eta(x) \geq x^2$ we also get from (60) that $\log^{-2}(\lambda+1) < e$, a contradiction that completes the proof. ■

If f is actually smoother than stated in Theorem 4.3, the estimate (55) can be improved. For example, one has the following result:

COROLLARY 4.6 *Suppose that f satisfies* (27) *and moreover that*

$$(1-e^{-i\theta}e^{ia})^{3/2}(1-e^{-i\theta}e^{-ia})^{3/2}\, f(e^{i\theta}) \in \mathcal{W}^{2,1}(I)\,. \tag{61}$$

Then to each constant $K_3 > 0$ there is $K_4 = K_4(f) > 0$ such that

$$M_\Psi^2 \leq K_4\, e_\Psi^{3/2} \exp\sqrt{K_3/e_\Psi}\,. \tag{62}$$

PROOF. It is easy to check that (27) and (61) together imply that G defined in (34) lies in $\mathcal{W}^{2,1}(\mathbb{R})$. Thus $v(x)$ is $o(\gamma^{-2}(x))$ by (33), and therefore $\varepsilon(y)$ defined in (41) is $o(|\log^{-1}(\lambda+1)|)$. From (45) and (48) we now see that (37) and (38) sharpen to

$$M_\Psi^2(\lambda) = o((\lambda+1)^{-1}|\log^{-3}(\lambda+1)|)\,,$$

$$e_\Psi(\lambda) = o(\log^{-2}(1+\lambda)),$$

and from that point the proof follows a course similar to that of Corollary 4.5. ■

A numerical illustration of the estimates given by Theorem 4.3 is provided in Section 7.

5 Approximation of traces of meromorphic functions

Roughly speaking, we found in the previous section that the smoother f on I, the slower the increase of M_Ψ as e_Ψ goes to zero. It is natural to ask whether these estimates can be further improved if f extends smoothly in two dimensions, in particular when it is analytic in some annulus containing $\mathbb{T}$. In this section, we shall consider the case where f is of the form h/q_N, where $h \in H^2$ and q_N is a polynomial of degree N having all its roots in $\mathbb{D}$. This is especially interesting from the point of view of applications, since many f in practice would be represented as trigonometric polynomials. We begin with an improvement of Proposition 4.1 when f is rational.

PROPOSITION 5.1 *Assume that f is the trace on I of a rational function p_{N-1}/q_N where p_{N-1} and q_N are algebraic polynomials of degree $N-1$ and N respectively, and where the zeros $\xi_1, \cdots, \xi_N$ of q_N lie in some compact subset $\mathcal{K} \subset \mathbb{D}$. Then, v being as in* (26), *it holds that*

$$\frac{v(x)}{N(1-x)^{1/2}} = O(1) \qquad \text{as } x \to 1^-, \tag{63}$$

where the $O(1)$ holds uniformly with respect to the $\xi_j \in \mathcal{K}$.

PROOF. We get from (24) and (32) that

$$\sqrt{x}v(x) = \frac{1}{2i\pi}\int_I \frac{f(\xi)}{(\xi - e^{ia})^{1/2}(\xi - e^{-ia})^{1/2}} \exp\left\{i\gamma(x)\frac{\log 2}{\pi}\log\left(e^{ia}\frac{\xi - e^{-ia}}{\xi - e^{ia}}\right)\right\} d\xi,$$

which is understood as a line integral on $I \subset \mathbb{T}$ oriented in the counterclockwise direction. Put

$$H_x(\xi) = \exp\left\{i\gamma(x)\frac{\log 2}{\pi}\log\left(e^{ia}\frac{\xi - e^{-ia}}{\xi - e^{ia}}\right)\right\}$$

and

$$B(\xi) = \frac{f(\xi)}{(\xi - e^{ia})^{1/2}(\xi - e^{-ia})^{1/2}}.$$

With the notation of Section 3, it holds that $H_x(\xi) = \psi_x(\xi)/\sqrt{x}$, $B(e^{i\theta}) = e^{-i\theta}F(\theta)$. The function H_x is analytic and bounded in $\overline{\mathbb{C}} \setminus I$ while B is meromorphic in $\overline{\mathbb{C}} \setminus I$ with poles $\xi_1, \cdots, \xi_N$ in $\mathbb{D}$, and vanishes with order 2 at infinity. By Cauchy's theorem, it holds that

$$0 = \frac{1}{2i\pi}\int_{\mathbb{T}} H_x(\xi)B(\xi)\,d\xi = \sum_{j=1}^{N}\mathrm{Res}_{\xi_j}(H_xB) + \frac{1}{2i\pi}\int_I \left(H_x^+(\xi)B^+(\xi) - H_x^-(\xi)B^-(\xi)\right) d\xi,$$

where the symbol Res_{ξ_j} indicates the residue at ξ_j and the subscript $\pm$ indicates the determination of a function on the positive or negative side of the oriented cut I. As it is easily checked that

$$H_x^-(\xi) = \exp\left(2\gamma(x)\log 2\right)\, H_x^+(\xi), \qquad \text{while} \quad B^-(\xi) = -B^+(\xi),$$

and since by definition

$$\sqrt{x}v(x) = \frac{1}{2i\pi}\int_I H_x^+(\xi)B^+(\xi)\,d\xi,$$

we deduce by taking into account the definition of $\gamma(x)$ that

$$v(x) = -\frac{1-x}{x}\sum_{j=1}^{N}\mathrm{Res}_{\xi_j}(H_xB).$$

Observe that the argument of $e^{ia}(\xi_j - e^{-ia})(\xi_j - e^{ia})^{-1}$ lies within $(0, -\pi)$, uniformly with respect to $\xi_j \in \mathcal{K} \subset \mathbb{D}$. Using this, one checks that each residue is bounded up to some multiplicative constant by its multiplicity times $\sqrt{x/(1-x)}$ (this is straightforward for simple poles, and multiple poles can be handled by an easy limiting argument). The result now follows. ■

We now derive the analogue of Theorem 4.3.

THEOREM 5.2 *If f is of the form h/q_N with $h \in H^2$ and q_N a polynomial of degree N whose roots all lie in $\mathbb{D}$ at a distance $d > 0$ from $\mathbb{T}$. Then, as $\lambda \searrow -1$, we have that*

$$M_\Psi^2(\lambda) = O\left(N^2\,|\log(\lambda+1)|\right), \tag{64}$$

and

$$e_\Psi(\lambda) = O\left(N^2(1+\lambda)\right), \tag{65}$$

where the symbols O hold uniformly with respect to d and $\|f\|_{L^2(\mathbb{T})}$.

PROOF. By division, we can write $f = u + p_{N-1}/q_N$ with $u \in H^2$ and p_{N-1} a polynomial of degree $N-1$. The H^2 norm of u is uniformly majorized with d and $\|f\|_{L^2(\mathbb{T})}$, so u will play no role in the asymptotic behaviour of $M(\lambda)$ and $e(\lambda)$, and we may as well assume that $f = p_{N-1}/q_N$ and apply Proposition 5.1. A straightforward majorization of (39) and (46) using (63) gives us the result.

COROLLARY 5.3 *If f is of the form h/q_N with $h \in H^2$ and q_N a polynomial of degree N whose roots all lie in $\mathbb{D}$ at a distance $d > 0$ from $\mathbb{T}$. Then*

$$M_\Psi^2 = O\left(N^2 \,|\log e_\Psi|\right) , \tag{66}$$

and the symbol O holds uniformly with respect to $\|f\|_{L^2(\mathbb{T})}$ and d, the estimate being sharp in the considered class of functions.

PROOF. The uniform estimate follows from (64) and (65). It is sharp because when f is a polynomial of degree N in $1/z$, the proof of Proposition 5.1 yields a sharp estimate. ■

6 Interpolation

When $f \in H^2_{|_I}$ and $\Psi = 0$, the set of all solutions to (BEP) as M ranges from $\|P_{H^2}\tilde{f}\|_{L^2(J)}$ to $\|f\|_{L^2(J)}$ defines *via* equation (11) a family of functions $g_0(\lambda)$ indexed by $\lambda \in (-1, 0)$. In [6], it was shown to coincide with the family of Carleman-type interpolants studied in [21] and described also in [3, 20]. It is remarkable, by the way, that the latter has the extremal property of solving for (BEP) whereas it was originally built for recovery purposes rather than those of approximation. In this section, the singular Cauchy integrals expressing Carleman interpolants will team up with our functional-analytic approach to (BEP) to produce new information on the convergence of this classical interpolation scheme.

We shall consistently assume that $f \in H^2_{|_I} \setminus \{0\}$; thus it extends uniquely to some nonzero H^2-function defined on the whole of $\mathbb{T}$ that, with a slight abuse of notation, we shall still denote by f. Moreover, since we only consider the case where $\psi = 0$ in (BEP), we shall set for simplicity $g_\lambda = g_0(\lambda)$ and this will simplify the notation into $g_\lambda(z)$ or $g_\lambda(e^{it})$ when evaluating this function at $z \in \mathbb{D}$ or at $e^{it} \in \mathbb{T}$. Now, using (6) and (49), formula (11) becomes

$$g_\lambda = f - (\lambda + 1)\,(1 + \lambda\,\phi)^{-1}\,\phi\,f\,. \tag{67}$$

This expression for the solution to (BEP) when $\Psi = 0$ and $f \in H^2_{|_I}$, combined with the concrete spectral theory of Section 3, will be the key to the forthcoming analysis.

As λ decreases to -1, the error $e(\lambda) = \|f - g_\lambda\|^2_{L^2(I)}$ of the interpolation problem decreases to zero like in every instance of (BEP). However, the decay will turn out here to be considerably faster than it was for the approximation problem studied in Section 4. In addition, as pointed out in the introduction already, peculiar to the interpolation problem is the fact that $\|f - g_\lambda\|_{L^2(\mathbb{T})}$ itself goes to zero when $\lambda \to -1$, and we will estimate the corresponding error rate when f lies in a Sobolev class before giving, as corollaries, pointwise convergence results.

6.1 Estimates of the L^2 decay rates

The following estimate shows that the convergence of $e(\lambda)$ to 0 when $\lambda \to -1$ is much faster if $f \in H^2_{|_I}$ than the error rate (38), although we know the latter is sharp with respect to the approximation problem in a Sobolev class by Remark 4.4.

PROPOSITION 6.1 *If $f \in H^2_{|_I}$ then, as $\lambda \searrow -1$,*

$$e(\lambda) = O(1+\lambda).$$

PROOF.

$$e(\lambda) = (\lambda+1)^2 \, \|(1+\lambda\,\phi)^{-1}\,\phi\, f\|^2_{L^2(I)} = (\lambda+1)^2 \, ((1+\lambda\,\phi)^{-2}\,(1-\phi)\,\phi^2\, f, f)_{L^2(\mathbb{T})} \, .$$

Using again (49), we get

$$e(\lambda) = (\lambda+1)^2 \, \left[((1+\lambda\,\phi)^{-1}\,\phi^2\, f, f)_{L^2(\mathbb{T})} - (\lambda+1)\,((1+\lambda\,\phi)^{-2}\,\phi^3\, f, f)_{L^2(\mathbb{T})}\right]$$

$$\leq (\lambda+1)\,\|f\|^2_{L^2(\mathbb{T})} \, ,$$

from (50). ■

In contrast to Proposition 6.1 that provides an easy majorization of $e(\lambda) = \|f - g_\lambda\|^2_{L^2(I)}$, the convergence of $\|f - g_\lambda\|_{L^2(J)}$ to zero cannot be quantified in general, unless f assumes more smoothness than just being in H^2. In a vein similar to that of Theorem 4.3, we now derive estimates for this quantity when f belongs to a Sobolev class on I. This hypothesis will also improve the convergence rate we just gave for $e(\lambda)$.

THEOREM 6.2 *If f is the restriction to I of an H^2 function (still denoted by f) such that*

$$(1-e^{-i\theta}e^{ia})^{-1/2}(1-e^{-i\theta}e^{-ia})^{-1/2}\, f(e^{i\theta}) \in L^1(\mathbb{T}) \, ,$$

and

$$(1-e^{-i\theta}e^{ia})^{1/2}(1-e^{-i\theta}e^{-ia})^{1/2}\, f(e^{i\theta}) \in \mathcal{W}^{1,1}(\mathbb{T}) \, ,$$

then as $\lambda \searrow -1$,

$$\|f - g_\lambda\|^2_{L^2(J)} = o(|\log(\lambda+1)|^{-1}) \ \ ,$$

while

$$e(\lambda) = o((\lambda+1)|\log(\lambda+1)|^{-1}) \, . \tag{68}$$

PROOF. From (11) and (67):

$$\|f - g_\lambda\|^2_{L^2(J)} = (\lambda+1)^2 \, (\phi^3 \, (1+\lambda\phi)^{-2} f, f)_{L^2(\mathbb{T})}$$

$$= C\,(\lambda+1)^2 \int_0^1 \frac{t^3}{(1+\lambda t)^2} |v_0(t)|^2 \, dt \, , \tag{69}$$

where, this time, $v_0 = V\,f$. Theorem 3.1 then gives:

$$v_0(x) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{f(e^{i\theta})\, d\theta}{\overline{\psi_x(e^{i\theta})}(1-e^{-i\theta}e^{ia})^{1/2}(1-e^{-i\theta}e^{-ia})^{1/2}}$$

$$= v(x) + \frac{1}{2\pi} \int_J \frac{f(e^{i\theta})\psi_x(e^{i\theta})\, d\theta}{(1-x)(1-e^{-i\theta}e^{ia})^{1/2}(1-e^{-i\theta}e^{-ia})^{1/2}}\,.$$

for the function v defined by (30). Following (25), it holds that

$$v_0(x) = v(x) + \frac{1}{2\pi} \int_J \frac{F(\theta)\exp[i\gamma(x)\omega_I(\theta)]\, d\theta}{\sqrt{1-x}}\,, \tag{70}$$

where F and ω_I are now defined on the whole $[0, 2\pi)$ by (31) and (18). Moreover (19) is still valid for ω_I on $a \le \theta < 2\pi - a$ and therefore ω_I is still monotone (but this time decreasing) $(a, 2\pi - a) \to (-\infty, \infty)$. The remainder of this proof now goes as that of Proposition 4.1 by expressing $v_0 - v$ in terms of the Fourier transform of some $\mathcal{W}^{1,1}(\mathbb{R})$ function. We thus get

$$|v_0(x)| = o\left(\frac{1}{\sqrt{1-x}\,|\log(1-x)|}\right) \qquad \text{as } x \to 1. \tag{71}$$

Putting this in (69) and since we are mainly interested by the behaviour of this quantity for λ near to -1 where the behaviour of the integrand near $t = 1$ still dominates, there exists

$$\text{a positive increasing function } \varepsilon\,, \text{ with } \lim_{x\to 0^+} \varepsilon(x) = 0\,, \tag{72}$$

such that:

$$\|f - g_\lambda\|^2_{L^2(J)} \le (\lambda+1)^2 \left[\int_{1/2}^1 \frac{t^3\,\varepsilon(1-t)\, dt}{(1+\lambda t)^2(1-t)\log^2(1-t)} + O(1)\right]. \tag{73}$$

Hence, from the computations above (48), we get

$$\|f - g_\lambda\|^2_{L^2(J)} = o(|\log(\lambda+1)|^{-1})\,. \tag{74}$$

Also, from (12), we get that

$$\|f\|^2_{L^2(J)} - M^2(\lambda) = C\,(\lambda+1) \int_0^1 \frac{t^2\,(2+(\lambda-1)t)}{(1+\lambda t)^2}\,|v_0(t)|^2\, dt$$

$$\le C\,(\lambda+1) \int_{1/2}^1 \frac{t^2\,\varepsilon(1-t)\,(2+(\lambda-1)t)\, dt}{(1+\lambda t)^2\,(1-t)\,\log^2(1-t)} + O(\lambda+1)\,.$$

Moreover, writing $2 + (\lambda-1)t = 2(1-t) + (\lambda+1)t$,

$$(\lambda+1) \int_{1/2}^1 \frac{t^2\,\varepsilon(1-t)\,(2+(\lambda-1)t)\, dt}{(1+\lambda t)^2\,(1-t)\,\log^2(1-t)}$$

$$= (\lambda+1) \int_{1/2}^1 \frac{t^2\,\varepsilon(1-t)\, dt}{(1+\lambda t)^2\,\log^2(1-t)} + (\lambda+1)^2 \int_{1/2}^1 \frac{t^3\,\varepsilon(1-t)\, dt}{(1+\lambda t)^2\,(1-t)\,\log^2(1-t)}\,.$$

The first integral above is bounded by the one appearing in (42), which is itself, as in (43), dominated by $o((\lambda+1)^{-1}\log^{-2}(\lambda+1))$; the second integral coincides with the one involved in (73) whence, from (74),

$$\|f\|^2_{L^2(J)} - M^2(\lambda) = o(|\log(\lambda+1)|^{-1})\,.$$

Now, concerning $e(\lambda)$, we get from (13) that

$$e(\lambda) = -[(\tau+1)M^2(\tau)]_{-1}^{\lambda} - \int_{-1}^{\lambda} M^2(\tau)d\tau\,,$$

and, in view of the above bound,

$$e(\lambda) = o((\lambda+1)\,|\log(\lambda+1)|^{-1}) + \int_{-1}^{\lambda} \frac{d\tau}{\log(\tau+1)} = o((\lambda+1)\,|\log(\lambda+1)|^{-1})\,.$$

■

REMARK 6.3 Note that some further links can be derived between $v_0 = V f$ and v when $f \in H^2$. Indeed, $v = V P_{H^2}\tilde{f} = V(1-\phi)f$ and, using property (14) of the isometry V, $v(t) = (1-t)v_0(t)$, in this case. Recalling (71), this improves Proposition 4.1 whenever $f \in H^2$:

$$|v(x)| = o\left(\frac{\sqrt{1-x}}{|\log(1-x)|}\right) \qquad \text{as } x \to 1.$$

As a consequence of the estimate (68) in Theorem 6.2, we get a lower-bound for $\|f - g_\lambda\|_{L^2(J)}$ showing that, for the interpolation problem with smooth data, the error on J has to be significantly bigger than the squared error on I.

COROLLARY 6.4 *If f satisfies the assumptions of Theorem* 6.2*, then*

$$\frac{e(\lambda)}{\|f - g_\lambda\|_{L^2(J)}} = o(|\log(\lambda+1)|^{-1}) \qquad as \lambda \searrow -1.$$

PROOF. We have that

$$\|f - g_\lambda\|_{L^2(J)} \geq \|\phi(f - g_\lambda)\|_{L^2(\mathbb{T})} = (\lambda+1)\,\|(1+\lambda\,\phi)^{-1}\,\phi^2\,f\|_{L^2(\mathbb{T})}\,,$$

from (67), whence, for $-1 < \lambda \leq 0$,

$$\|f - g_\lambda\|_{L^2(J)} \geq \frac{\lambda+1}{\|1+\lambda\,\phi\|}\|\phi^2\,f\|_{L^2(\mathbb{T})} \geq (\lambda+1)\|\phi^2\,f\|_{L^2(\mathbb{T})}\,.$$

Moreover, we get from (68) that

$$\frac{e(\lambda)}{\lambda+1} = o(|\log(\lambda+1)|^{-1}) \qquad \text{as} \quad \lambda \searrow -1,$$

and combining the above two inequalities completes the proof. ■

6.2 Pointwise convergence

Concerning pointwise convergence of sequences of H^2 interpolants, it is of great interest to make use of Proposition 6.1 in order to get such results on the boundary $\mathbb{T}$, at least almost everywhere. Indeed, to our knowledge, it was only known up to now that pointwise convergence for such sequences holds *locally uniformly in* $\mathbb{D}$, from Goluzin and Krylov's Theorem, see [20, 21].

The result about convergence on I is simple, and we begin with this. Recall that, by convention, I is an open arc.

THEOREM 6.5 *If $f \in H^2_{|_I}$, then $g_\lambda(e^{i\theta}) \to f(e^{i\theta})$ uniformly on compact subsets of I, as $\lambda \to -1$.*

PROOF. It is simplest to use the equivalent expression of g_λ in terms of a Carleman-type integral formula, that links our family of approximants to the sequences of interpolants given in [21]. Indeed, we get from [6] that:

$$g_\lambda(z) = \hat{g}_\alpha(z) = \frac{1}{2\,i\,\pi} \int_{\mathbb{T}} \left(\frac{\varphi(\xi)}{\varphi(z)}\right)^\alpha (\chi_I f)(\xi)\,\frac{d\xi}{\xi - z}\,,\ \forall z \in \mathbb{D}\,, \tag{75}$$

or equivalently that:

$$g_\lambda = \hat{g}_\alpha = \frac{1}{\varphi^\alpha} P_{H^2}\left(\varphi^\alpha\left(\chi_I f\right)\right)\,,$$

where φ is the outer "quenching" function of modulus equal to $\varrho > 1$ on I and to 1 on J:

$$\varphi(z) = \exp\left\{\frac{\log\varrho}{2\,\pi} \int_I \frac{e^{it}+z}{e^{it}-z}\,dt\right\}\,,\ z \in \mathbb{D}\,,$$

and where

$$\alpha = -\frac{\log(\lambda+1)}{2\,\log\varrho}\,,\ \text{or } \lambda = -1 + \varrho^{-2\alpha}\,. \tag{76}$$

Thus

$$f(z) - g_\lambda(z) = \frac{1}{2\,i\,\pi} \int_J \left(\frac{\varphi(\xi)}{\varphi(z)}\right)^\alpha f(\xi)\,\frac{d\xi}{\xi - z}\,,\ \forall z \in \mathbb{D}\,,$$

and by continuity this extends to all z in I. Uniform convergence to zero on compact subsets K of I follows immediately given that $|\varphi(\xi)/\varphi(z)| = 1/\varrho < 1$ for all $\xi \in J$ and $z \in K$, and that $|\xi - z|$ is uniformly bounded away from zero. ∎

Pointwise convergence to f of the sequence $(\hat{g}_n)$, as defined in (75) with $\alpha = n$, also holds almost everywhere on J, and thus on $\mathbb{T}$, under some smoothness assumptions:

THEOREM 6.6 *If f is the restriction to I of an H^2 function whose derivative is absolutely continuous on $\mathbb{T}$, then the sequence $(\hat{g}_n)$ of Goluzin–Krylov approximants to f converges to f almost everywhere on J.*

Thus, combining this with Theorem 6.5, we see that if f is the trace on some subarc $I \subset \mathbb{T}$ of positive measure of an H^2 function whose derivative is absolutely continuous on $\mathbb{T}$ then, almost everywhere on $\mathbb{T}$, f is the pointwise limit of its sequence of H^2 approximants $(\hat{g}_n)$.

The proof of this result requires the following improvement of Theorem 6.2.

PROPOSITION 6.7 *If f is the restriction to I of an H^2 function whose derivative is absolutely continuous on $\mathbb{T}$, then, as $\lambda \searrow -1$,*

$$\|f - g_\lambda\|^2_{L^2(J)} = o(|\log^{-3}(\lambda+1)|)\,,$$

or, equivalently,

$$\|f - \hat{g}_\alpha\|^2_{L^2(J)} = o(\alpha^{-3})\,.$$

PROOF. Consider the expression (70) of v_0. Recalling (33), we get that

$$|\gamma^2(x)\sqrt{x}v(x)| = |\mathcal{F}(G'')(-\gamma(x))|\,,$$

while we analogously get from (70), by taking this time the variable $\sigma = \omega(\theta)$ with $a \le \theta < 2\pi - a$,

$$|\gamma^2(x)\sqrt{1-x}(v - v_0)(x)| = |\mathcal{F}(G'')(-\gamma(x))| \, .$$

Using the Riemann–Lebesgue lemma here also implies that, if $G'' \in L^1(\mathbb{R})$, then

$$\begin{aligned} |v(x)| &= o\left(\frac{1}{\log^2(1-x)}\right) \qquad \text{as } x \to 1, \\ |v_0(x)| &= o\left(\frac{1}{\sqrt{1-x}\log^2(1-x)}\right) \qquad \text{as } x \to 1. \end{aligned} \tag{77}$$

Now, the assumption that $f'' \in L^1(\mathbb{T})$ actually implies that $G'' \in L^1(\mathbb{R})$, as can be seen from (35).

Putting (77) into (69) implies:

$$\|f - g_\lambda\|^2_{L^2(J)} = (\lambda+1)^2 \int_0^1 \frac{t^3\,\varepsilon(1-t)\,dt}{(1+\lambda t)^2(1-t)\log^4(1-t)}$$

$$\le (\lambda+1)^2 \left[\int_{1/2}^1 \frac{t^3\,\varepsilon(1-t)\,dt}{(1+\lambda t)^2(1-t)\log^4(1-t)} + O(1) \right] .$$

for ε satisfying (72). Straightforward computations similar to the ones below (73) provide the desired estimate. ∎

PROOF OF THEOREM 6.6. For each $\delta > 0$, define

$$E_{n,\delta} = \{e^{it} \in J : |\hat{g}_n(e^{it}) - f(e^{it})| \ge \delta\}.$$

Let ℓ denote Lebesgue measure on $\mathbb{T}$. By Chebyshev's inequality, using Proposition 6.7, there is an absolute constant C, depending only on f, such that

$$\ell(E_{n,\delta}) \le C\delta^2/n^3.$$

Thus $\sum_{n=1}^\infty \ell(E_{n,\delta}) < \infty$, and so, by the Borel–Cantelli lemma (see e.g. [14, Lemma VIII.3.1]), for each $\delta > 0$ almost every $e^{it} \in J$ belongs to at most finitely many sets $E_{n,\delta}$. The result now follows on taking a countable sequence (δ_k) tending to zero. ∎

7 Numerical results

Figures 1, 2, and 3 illustrate the results of Theorem 4.3. In this example, the function f to be approximated has been built by classical interpolation procedure (splines) from pointwise experimental data provided by the French National Space Agency (CNES, Toulouse). These data correspond (through some conformal map) to reflection responses of a hyperfrequency filter, which will be part of on board devices (input /output multiplexors) for telecommunication satellites. From those data, the engineers want to robustly recover an H^2 (in fact, rational) function. The deep links between approximation by analytic functions and harmonic identification are discussed in [2, 8, 16, 20], among others, and the application to filter synthesis is more precisely handled in [4]. It is perhaps worth noticing that, although the function is to be approximated

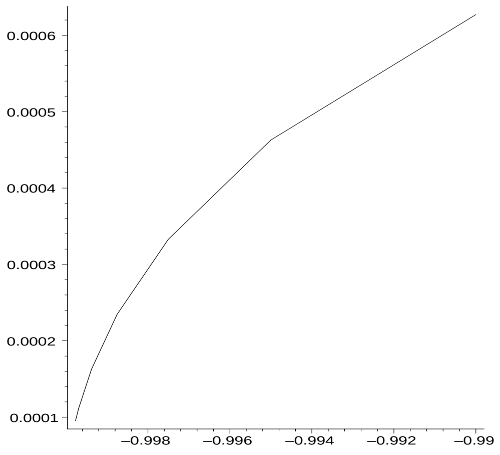

Figure 1: $-\log(1+\lambda)e(\lambda)$

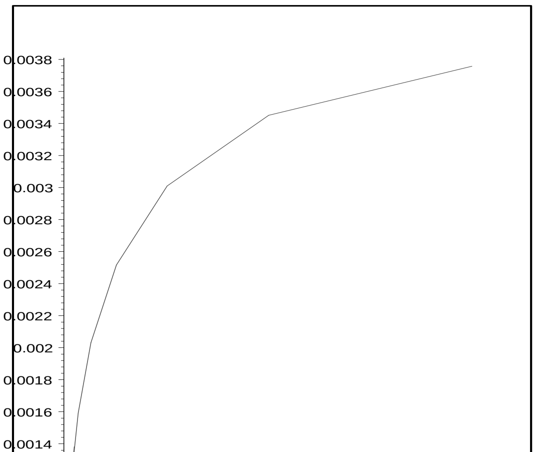

Figure 2: $(1+\lambda)\log^2(1+\lambda)M(\lambda)$

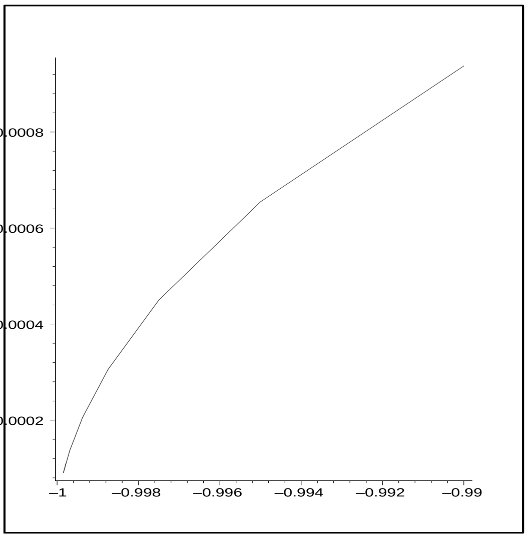

Figure 3: $e(\lambda)\log M(\lambda)$

in H^2, the given pointwise values do not coincide with those of an H^2 function in general, since they are provided by experimental devices and thus carry measurement errors.

We had here at our disposal 801 pointwise values in a high-frequency bandwidth, from which we computed 801 Fourier coefficients of some function $f \in L^2(I)$, with $I = (e^{-i\pi/2}, e^{i\pi/2})$. A number of approximants g_λ to f have been computed by a software package called *Hyperion*, developed at INRIA (Institut National de Recherche en Informatique et Automatique), for various values of λ near -1, together with the associated quantities $e(\lambda)$ and $M(\lambda)$.

The behaviour of $-\log(1+\lambda)e(\lambda)$, $(1+\lambda)\log^2(1+\lambda)M(\lambda)$, and $e(\lambda)\log M(\lambda)$ with respect to λ near -1 are plotted in Figures 1, 2, and 3.

8 Conclusion

The estimates of Theorem 4.3 considerably improve the ones that were established in [5]. We recall that, in this work, it was shown if $f \in \mathcal{W}^{1,2}(I)$ that, as λ approaches -1,

$$e(\lambda) \le O\left(\frac{\log(\log M(\lambda))}{\log M(\lambda)}\right).$$

This could in fact be improved, using unpublished results in [9] on the decay of the Hardy-Sobolev norm as a function of the L^2-norm on $I \subset \mathbb{T}$, to the effect that, in this case: $e(\lambda) \le O(1/\log M(\lambda))$. But if f belongs to $\mathcal{W}^{1,2}(I)$, then it is easy to see that it satisfies the hypotheses of Theorem 4.3, and we now see from that theorem, under even weaker assumptions on f, that the following stronger estimate holds:

$$e(\lambda) \le o\left(\frac{1}{\log M(\lambda)}\right).$$

This estimate sould be further held in contrast with Corollary 5.3 that shows a dramatic increase in the speed of approximation when f is meromorphic in $\mathbb{D}$.

Concerning the estimates of Theorem 6.2 for interpolating sequences, they imply that whenever $f \in H^2_{|_I}$, $f' \in L^1(\mathbb{T})$, if we set

$$e_J(\lambda) = \|f - g_\lambda\|^2_{L^2(J)} ,$$

then the convergence rates on I and J are linked by

$$e_J(\lambda) \leq o\left(\frac{1}{|\log e(\lambda)|}\right) ,$$

as λ approaches -1, although, as a consequence of [10], we obtained only the following pessimistic inequality[1]

$$e_J(\lambda) \leq O\left(\frac{\log(|\log e(\lambda)|)}{|\log e(\lambda)|}\right) .$$

A nice consequence of such estimates is that they seem to provide stability / instability properties for classes of 2D inverse problems arising in nondestructive control. This is already under study, while the basis of the strong and constructive links between 2D Laplace inverse problems and approximation in Hardy spaces from band-limited data is provided in [7, 12], whereas stability properties are discussed in [1], for example.

Let us finally mention another issue we have in mind that seems particularly relevant when using bounded extremal problems to express identification issues, either for transfer functions of linear dynamical systems or for solutions of inverse problems. It comes in cases where it is *a priori* known, for some physical reasons, that the function f to be approximated on I does "almost" belong to H^2, more precisely when $f = h + \delta$, say, with $h \in H^2_{|_I}$ and $\delta \in L^2(I) \setminus H^2_{|_I}$, $\|\delta\|_{L^2(I)}$ small. If we call, as usual, g_λ the solution in H^2 of the bounded extremal problems associated to $h + \delta$, we wonder if there exists a value of $\lambda > -1$ that minimizes $\|f - g_\lambda\|_{L^2(\mathbb{T})}$. It is easily seen that for $\delta = 0$, this quantity goes to an infimum, equal to 0, as $\lambda \to -1$, and that the same thing occurs for $f = 0$, as $\lambda \to \infty$. However, this remains unsolved for $\delta \neq 0$ and $f \neq 0$, although the L^2 representation of H^2 functions used in the present work may be of some use.

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AMS Subject Classification: 30D55, 30E10, 42A05, 47B35, 65E05.