Line-segment cracks recovery from incomplete boundary data

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Abstract

We are concerned with nondestructive control issues, namely detection and recovery of cracks in a planar (2D) isotropic conductor from partial boundary measurements of a solution to the Laplace-Neumann problem. We first build an extension of that solution to the whole boundary, using constructive approximation techniques in classes of analytic functions, and then use localisation algorithms based on boundary computations of the reciprocity gap.

Key-words: Geometrical inverse problems, 2D Laplace operator, cracks detection and recovery, analytic and meromorphic functions, uniform meromorphic approximation

1 Introduction

This paper is devoted to the inverse geometrical problem of detecting and locating cracks in a material by means of boundary measurements. These problems have attracted a lot of attention because of their various practical applications related to non destructive evaluation (medical imaging, tomography, ...). In such problems, the involved physical phenomena can be either thermal, electrical, electrostatic, acoustic, or elastostatic. We focus in this work on the Laplace equation in the bidimensional case and, for simplicity, we stick to the steady state thermal context.
The mathematical study of this inverse problem was initiated in [17] for a single buried crack in the 2D situation. The authors proved a uniqueness result for a single buried crack provided that two particular fluxes toguether with the corresponding measurements are available. This result has been proved for the case of a perfectly insulating or conducting crack, which are related one to the other by harmonic conjugation. For questions of uniqueness and stability, we refer to [3] and references there in.
There are, roughly speaking, two classes of inversion processes.
The first one consists in iterative processes which rely on multiple integration of the involved partial differential equation (PDE), hence are highly time consuming and usually sensitive to initial guesses. In [28], a very efficient computational algorithm to reconstruct a linear crack inside a homogeneous conductor has been performed while in [13], the method has been extended to the reconstruction of a collection of linear cracks.
The second class of inversion procedures contains "quasi-explicit" methods which do not require solving the backward problem at all and turn out to have very low computational costs. Up to our knowledge there are very few quasi-explicit methods and most of them give qualitative rather than quantitative results: they give information on the shape and location of the crack but do not allow to recover the crack itself, see [6] where tools from meromorphic approximations are used and a relation established between the poles of the meromorphic approximant and the geometric defect, the results being illustrated in the case of linear cracks. Another crack reconstruction

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method is proposed in [12], using the factorisation of the difference between the Neumann-Dirichlet operators of the safe body and of the damaged one. A further technique was introduced in [4] which turned out to be relevant for recovering 3D-planar cracks both for Laplace equation and elastostatic systems [5]. It constructively uses the "reciprocity gap" concept from which computation algorithms are deduced, see [11].
Notice that all the numerical works cited above are based on the essential assumption that overdetermined boundary conditions are *complete*, i.e. known on all the outer boundary $\partial\Omega$ of the body.

The present work is devoted to buried line segment cracks detection in 2D bodies from the knowledge of *partial boundary data* on $\partial\Omega$. We assume that the measured data u_K is given on a subset K of $\partial\Omega$ (u_K being the trace of the solution of the steady state Neumann heat conduction problem corresponding to the cracked body). This is a realistic assumption so far as only a part of the outer boundary of the body is accessible to the measurements. The aim of this work is to obtain a robust procedure to recover a line segment crack in this case.
We here approach this *incomplete data* issue by first extending them from K to the whole boundary $\partial\Omega$, in order then to run existing reconstruction algorithms from *complete extended data*. Observe however that some care has to be taken in this extension process. Indeed, such harmonic (or analytic) extrapolation issues from K to $\partial\Omega$ are ill–posed and lead to unstable schemes. More precisely, whenever the available data u_K does not coincide exactly with the trace on K of a harmonic function (which is likely to occur for u_K is given by corrupted numerical or experimental measurements), it will still admits a harmonic approximant on K, but whose norm on $\partial\Omega \setminus K$ will grow up as the error on K decreases, see e.g. [7, prop. 1, 2]. This phenomenon accounts for the necessity of adding a constraint on $\partial\Omega \setminus K$ for a best harmonic approximation problem to be well-posed on K, the choice of the constraint itself being another issue, see section 2.
Our recovery procedure is thus performed in two steps.
I./ The first one consists in finding an extension u to $\partial\Omega$ of the measured temperature u_K on $K \subset \partial\Omega$, the heat flux Φ being given on all $\partial\Omega$. We propose an algorithm related to best uniform approximation on K by analytic and meromorphic functions in Ω, whose behaviour is constrained on $\partial\Omega \setminus K$, as developed in [8].
II./ The reciprocity gap technique from [11] is then applied in order to recover the unknown geometry, when it is known to be a line-segment and once extended solution u is available on the whole boundary $\partial\Omega$ (from step I.). Here also the procedure splits into two steps:
i) the host line recovery, explicitly achieved,
ii) the endpoints identification.
In the sequel of this introduction, we set up the forward problem and give some indications about step II. of our recovery procedure issued from [11]. In section 2, we introduce some tools from complex analysis and approximation theory that are needed to solve for step I. of the above recovery procedure. Section 3 is devoted to the description of the algorithms as well as to a presentation of a discussion of some numerical trials.

1.1 The forward problem

One considers the case of buried cracks modelled by C^2 simple curves σ embedded in a smooth open set Ω of $\mathbb{R}^2$, Ω being simply connected with smooth boundary $\partial\Omega$, of class $C^{1,\beta}$, $0 < \beta \le 1$. We prescribe a heat flux $\Phi \in L^2(\partial\Omega)$, which does not identically vanish and satisfies:

$$\int_{\partial\Omega} \Phi ds = 0,$$

where s is the curvilinear abscissa on $\partial\Omega$. The forward problem is the following:

Given Ω, $\sigma \subset \Omega$ and Φ as described, find the solution u of :

$$
(1) \qquad \begin{cases} \Delta u = 0 & \text{in} \quad \Omega \backslash \sigma \\[2mm] \dfrac{\partial u}{\partial n} = 0 & \text{on} \quad \sigma \\[2mm] \dfrac{\partial u}{\partial n} = \Phi & \text{on} \quad \partial\Omega \end{cases}
$$

The second equation describes physically the case of an insulating crack. This problem has a unique solution u (temperature), up to an additive constant. We add the normalisation condition:

$$\int_{\partial\Omega} u ds = 0.$$

to insure uniqueness. Under the previous assumptions, we deduce from the classical regularity results [6, 17] that the trace of u belongs to the Hölder $C^{1/2}(\partial\Omega)$, that the heat jump across the crack: $[u] = u^+ - u^- \in C^{1/2}(\sigma)$, if $u^{\pm}$ denotes the trace of u on each side of σ.

1.2 The inverse problem

The inverse problem under consideration concerns the recovery of a linear crack $\sigma \subset \Omega$ by the knowledge of the heat flux Φ (input) and the temperature u_K measured on a subset K of the outer boundary of the body, assumed to have a positive measure.

Given Ω, $K \subset \partial\Omega$, Φ and u_K, find $\sigma \subset \Omega$ such that the solution u of (1) *satisfies:* $u_{|_K} = u_K$.

In 2D situations, the uniqueness (i.e. identifiability) result for a crack modelled by a C^2 curve has been proved in [17], provided that two partial measurements, corresponding to two particular fluxes, are available.
Assume for a while that step I. has been solved (see section 2) and the data u_K extended to $\mathbb{T}$, or that complete data is available ($K = \partial\Omega$). In the case of a linear crack or a collection of linear cracks supported by the same hosting line, one flux together with the corresponding temperature is enough to insure identifiability, provided that the mean of the induced jump of the solution across the crack does not vanish, see [4]. Even though the a priori selection of such a flux is an open question, notice that it is very easy to check if this condition is fulfilled a posteriori by using the over-specified data on $\partial\Omega$. The proof of this result is constructive and gives rise to two recovering algorithms developed in [10, 11] which we use here for solving step II.
These recovering procedures rely on the "reciprocity gap" (RG) as defined by the following boundary integral, which is completely accessible from the over-specified external boundary data:

$$
(2) \qquad RG(v) = \int_{\partial\Omega} (\Phi v - u \frac{\partial v}{\partial n}) ds \, ,
$$

evaluated for fields v harmonic in Ω. By Green's formula, this quantity is related to the unknown geometry σ by:

$$RG(v) = \int_{\sigma} [u] \frac{\partial v}{\partial n} \, d\tau \, ,$$

where τ designates the curvilinear abscissa on σ. The determination of the hosting line D is explicitely achieved via the inversion formula given in [4].
The line D being identified, the complete identification is done by the determination of the crack's endpoints. For this purpose two non iterative procedures (which do not require solving the forward problem) are proposed in [10, 11].
The first algorithm reconstructs the jump of the solution on the line D, by expanding $[u]$ on some Hilbertian basis of $L^2(D \cap \Omega)$ (for example, Legendre polynomials). The complete determination

of σ is given by that of the support of $[u]$. The second algorithm reconstructs the crack's endpoints by searching the minimum of the functional $\widetilde{\sigma} \longmapsto RG^2(w_{\widetilde{\sigma}})$, where $\widetilde{\sigma}$ is a fictitious crack included in $D \cap \Omega$ and $w_{\widetilde{\sigma}}$ is a harmonic field in $\Omega \backslash \widetilde{\sigma}$. We know by the reciprocity principle that $RG(w_\sigma) = 0$, see section 3.1.

1.3 Hypotheses, notations, definitions

In the following, $\Omega = \mathbb{D}$ will denote the unit disk of the complex plane $\mathbb{C}$ and $\partial\Omega = \mathbb{T}$ the unit circle. This is done without loss of generality, for the simply connected domain $\Omega \subset \mathbb{C}$ can be conformally mapped[1] onto $\mathbb{D}$.
When $E \subset \mathbb{T}$, we write $C(E)$ for the space of continuous complex-valued functions on E while $L^p(E)$ designates the familiar Lebesgue space for $1 \le p \le \infty$. We let $H^p \subset L^p(\mathbb{T})$ be the Hardy space with exponent p of $\mathbb{D}$ consisting of functions with vanishing Fourier coefficients of negative index. Alternatively, H^p is the set of non-tangential limits of functions analytic in $\mathbb{D}$ whose modulus has uniformly bounded L^p-mean over circles centered at 0 included in $\mathbb{D}$; when $p = \infty$, this simply says that the function is bounded in $\mathbb{D}$. We refer to [16, 18, 19, 20, 27] for definitions and properties of Hardy spaces.
We denote by d the distance (induced by norm) in $L^\infty(\mathbb{T})$. Define $R_N \subset L^\infty(\mathbb{T})$ to be the set of rational functions with no poles on $\mathbb{T}$ and at most N poles in $\mathbb{D}$. When $E \subset \mathbb{T}$, we write $C(E)$ for the space of continuous complex-valued functions on E, χ_E for the characteristic function of E, and

$$L^p(E) = \{f, \ \chi_E f \in L^p(\mathbb{T})\},$$

with norm $\|f\|_{L^p(E)} = \|\chi_E f\|_{L^p(\mathbb{T})}$. The concatenated function whose value is equal to f on E and to h on $\mathbb{T}\backslash E$ is expressed as

$$f \vee h = \chi_E f + \chi_{\mathbb{T}\backslash E} h .$$

The modulus of continuity of a function $f \in C(E)$ is defined by

$$\omega_f(\delta) := \sup_{\substack{x,y \in E \\ |x-y| \le \delta}} |f(x) - f(y)|.$$

For $0 < \alpha < 1$ fixed, introduce the Hölder–Zygmund[2] class:

$$C^\alpha(E) := \{f \in C(E), \ \omega_f(\delta) = O(\delta^\alpha)\},$$

with norm

$$\|f\|_\alpha := \|f\|_{L^\infty(E)} + \sup_{\delta > 0} \frac{\omega_f(\delta)}{\delta^\alpha}.$$

We define the *separable* Hölder–Zygmund class $\lambda_\alpha(E)$ to be the subspace of those functions f in $C^\alpha(E)$ such that

$$\frac{\omega_f(\delta)}{\delta^\alpha} \to 0 \qquad \text{as } \delta \to 0.$$

We will use the following subsets:

$$\begin{aligned} \lambda_\alpha^+(E) &:= \{f \in \lambda_\alpha(E), \ f \ge 0\}, \\ \lambda_\alpha^{+,0}(E) &:= \{f \in \lambda_\alpha^+(E), \ f > 0 \text{ excepted at boundary points of } E \text{ where } f = 0\}. \end{aligned}$$

In the sequel, K denotes a strict subset of $\mathbb{T}$ of positive Lebesgue measure, as well as its complementary set $\mathbb{T}\backslash K$. We finally introduce

$$\begin{aligned} \mathcal{M} &:= \{M \in L^\infty(\mathbb{T} \setminus K), \ M \ge 0 \text{ a.e. on } \mathbb{T} \setminus K\}, \\ \mathcal{M}_* &:= \{M \in \mathcal{M}, \ \inf_{\mathbb{T}\setminus K} M > 0\}. \end{aligned}$$

[1] Whenever the boundary $\partial\Omega$ is a $C^{1,\beta}$ Jordan curve, the conformal transformation admits a $C^{1,\beta}$ extension up to the boundary $\partial\Omega$, [14].

[2] Also commonly called Lipschitz class.

2 Data extension

In the framework of the unit disk, system (1) can be written as:

$$\text{(3)} \qquad \begin{cases} \Delta u &= 0 \quad \text{in} \quad \mathbb{D}\backslash\sigma\,, \\[2mm] \dfrac{\partial u}{\partial n} &= 0 \quad \text{on} \quad \sigma\,, \\[2mm] \dfrac{\partial u}{\partial n} &= \Phi \quad \text{on} \quad \mathbb{T}\,, \end{cases}$$

for a flux $\Phi \in L^2(\mathbb{T})$ which satisfies

$$\int_{\mathbb{T}} \Phi \, d\theta = 0,$$

and with the normalisation condition

$$\int_{\mathbb{T}} u \, d\theta = 0\,.$$

Given a function u_K (built from measurements) on $K \subset \mathbb{T}$, the inverse problems consists in determining $\sigma \subset \mathbb{D}$ such that the solution u to (3) verifies:

$$u_{|K} = u_K\,.$$

In order to solve our first identification step I. and to extend the available data u_K to the whole outer boundary $\mathbb{T}$, harmonic conjugation allows to express this issue in the framework of analytic function theory and Hardy spaces, where appropriate approximation techniques will be used.

2.1 Harmonic conjugation and analytic functions

The solution u to the direct problem being harmonic in $\mathbb{D}\backslash\sigma$, there exists a function U analytic in $\mathbb{D}\backslash\sigma$ such that $u = \operatorname{Re} U$. Up to an additive imaginary constant, it is defined by

$$U = u + i\widetilde{u}\,,$$

the function $\widetilde{u}$ being the harmonic conjugate of u, see e.g. [2, section 4.6.1]. As usual, u and $\widetilde{u}$ are linked on the boundary $\mathbb{T}$ by the Cauchy-Riemann equation:

$$\frac{\partial \widetilde{u}}{\partial \theta} = \frac{\partial u}{\partial n}.$$

It follows from (3) that this defines a harmonic single-valued function $\widetilde{u}$ in $\mathbb{D}\backslash\sigma$, up to an additive constant which can be fixed by adding the condition $\widetilde{u}(a) = 0$, at some point $a = e^{i\theta_a}$ of $\mathbb{T}$. Hence, for $\xi = e^{i\theta} \in \mathbb{T}$,

$$\widetilde{u}(e^{i\theta}) = \int_{\theta_a}^{\theta} \Phi(e^{i\tau})\, d\tau\,.$$

Here, the function

$$\text{(4)} \qquad U(\xi) = u_K(\xi) + i \int_{\theta_a}^{\theta} \Phi(e^{i\tau})\, d\tau\,,$$

is known on K because u is measured there while $\operatorname{Im} U = \widetilde{u}$ is available (by computation) on the whole of $\mathbb{T}$. Regarding smoothness on the boundary, Privalov or Carleson-Jacobs theorems [9, 18] ensure that harmonic conjugation preserves Hölder type regularity. By hypothesis, $\widetilde{u}$, thus also u and U, belong to $C^{1/2}(\mathbb{T}) \subset \lambda_\alpha(\mathbb{T})$, $\alpha < 1/2$. Also, the function U is Hölder smooth in $\mathbb{D} \setminus \sigma$ and belongs to Hardy classes of functions analytic and bounded in $\mathbb{D} \setminus \sigma$, while the temperature jump $[u] \in C^{1/2}(\sigma)$. In fact, it holds that:

$$\text{(5)} \qquad U(z) = \frac{1}{2\,i\,\pi} \int_{\sigma} \frac{[u](\tau)}{z - \tau}\, d\tau + g(z)\,,\ \forall\, z \in \mathbb{D} \setminus \sigma\,,$$

for some function $g \in H^\infty = H^\infty(\mathbb{D})$, see [6]. The inverse problem can then be formulated as follows:

being given $\mathrm{Re}\, U$ *on* K *and* $\mathrm{Im}\, U$ *on* $\mathbb{T}$, *locate the singularities* σ *of the function* U *in* $\mathbb{D}$.

As indicated above, we propose a two-stage resolution method. The first step consists in *finding an extension of* U (or $\mathrm{Re}\, U$), which is *known on* K, *to the whole boundary* $\mathbb{T}$. From exact data on K, such an extension is unique in Hardy classes of $\mathbb{D} \setminus \sigma$, the subset $K \subset \mathbb{T}$ being of positive measure. However, truly available data correspond to experimental or numerical measurements and may be corrupted by noise or errors. In order to ensure both well-posedness and robustness of the recovery process, we handle it as a constrained approximation problem in Hardy classes, rather than as an extrapolation one.

We describe in the following a computation scheme for some approximant U_N to U on K, which is robust in Hölder classes, and gives the extension we are looking for. It relies on the uniform approximation of a given function on K by functions that are meromorphic in $\mathbb{D}$ (whose singularities are poles) and bounded on $\mathbb{T}$. This is detailed in sections 2.2, for $K = \mathbb{T}$, and 2.3, for $K \subset \mathbb{T}$.

2.2 Adamjan-Arov-Krein problems

It is the following extremal problem of best uniform meromorphic approximation on $\mathbb{T}$. Being given some function $f \in L^\infty(\mathbb{T})$, we want to find a best approximant g_N to f in $L^\infty(\mathbb{T})$-norm among the set of functions meromorphic with at most N poles in $\mathbb{D}$ and bounded near $\mathbb{T}$, which coincides with $H^\infty + R_N$, see [8].
Recall that a Blaschke product of degree m is a rational function of the form

$$b(z) = e^{i\varphi} \prod_{j=1}^{m} \frac{z - a_j}{1 - \bar{a}_j\, z},$$

where φ is real and the a_j are complex numbers of modulus less than 1. Such functions belong to H^∞, are in fact analytic on the closed disk and have modulus 1 on $\mathbb{T}$, see [18]. For any integer $N \geq 0$,

$$H^\infty + R_N = \{b^{-1}\, g \,:\, b\,,\ g \in H^\infty\,,\ b \text{ a Blaschke product of degree } \leq N\}\,.$$

Problem 1 (AAK) *Let* $f \in L^\infty(\mathbb{T})$. *Find* $g_N \in H^\infty + R_N$ *such that*

$$\|f - g_N\|_{L^\infty(\mathbb{T})} = \min_{g \in H^\infty + R_N} \|f - g\|_{L^\infty(\mathbb{T})}.$$

This issue, originally handled in [1], has also been considered in [18, 24, 25, 26, 29], among others. It is known that problem (AAK) admits a unique solution if $f \in H^\infty + C(\mathbb{T})$. In this case, the solution is given by:

$$g_N = f - \frac{\Gamma_f\, \nu_N}{\nu_N}; \tag{6}$$

where Γ_f is the Hankel operator with symbol f defined by:

$$\Gamma_f : H^2 \to \overline{H}_0^2\,,\ \ \Gamma_f\, v = P_{\overline{H}_0^2}(f\, v),$$

if $\overline{H}_0^2$ denotes the subspace of $L^2(\mathbb{T})$ orthogonal to H^2 (consisting in functions analytic outside $\overline{\mathbb{D}}$), $P_{\overline{H}_0^2}$ the orthogonal projection from $L^2(\mathbb{T})$ onto $\overline{H}_0^2$, and if ν_N is the $(N+1)$th singular vector of Γ_f. Moreover, the error is circular: $|f - g_N| = \sigma_N$ a.e. on $\mathbb{T}$, σ_N being the $(N+1)$th singular value of Γ_f. Whenever $f \in H^\infty + C(\mathbb{T})$, the operator Γ_f is compact and the so-called Schmidt pair $(\sigma_N\,,\ \nu_N)$ is equivalently determined by the eigenvalues and eigenfunctions of $\Gamma_f^* \Gamma_f$:

$$\Gamma_f^* \,\Gamma_f\, \nu_N = \sigma_N^2\, \nu_N\,.$$

For rational symbols $f \in H^\infty + C(\mathbb{T})$, Γ_f has finite rank and the above procedure is constructive. The following continuity property show that it remains generically effective when f can be approximated by rationals in Hölder classes. Let us define the associated best approximation operator:

$$\mathcal{A}_N : H^\infty + C(\mathbb{T}) \to H^\infty + R_N \ \ : \ \mathcal{A}_N(h) = g_N.$$

Theorem 1 ([25]) *Let $f \in \lambda_\alpha(\mathbb{T})$. The restriction to separable Hölder classes $\lambda_\alpha(\mathbb{T})$ of the best approximation operator $\mathcal{A}_N$ is continuous for the Hölder norm at f whenever the $(N+1)$th singular value $\sigma_N(\Gamma_f)$ has multiplicity 1. In other words, if $(h_n)_{n\geq 0}$ is a sequence of $\lambda_\alpha(\mathbb{T})$ functions such that $\lim_{n\to\infty} \|h_n - f\|_{\alpha,\mathbb{T}} = 0$, then*

$$\lim_{n\to\infty} \|\mathcal{A}_N(h_n) - \mathcal{A}_N(f)\|_{\alpha,\mathbb{T}} = 0.$$

This continuity result is generic in Hölder classes, for the subset of those $f \in \lambda_\alpha(\mathbb{T})$ whose associated Hankel operator has simple singular values is dense in $\lambda_\alpha(T)$, as established in [8, cor. 2]. Observe however that $\mathcal{A}_N\ :\ H^\infty + C(\mathbb{T}) \to H^\infty + R_N$ is discontinuous, excepted at functions that already belong to $H^\infty + R_N$, see [23].

Back to the inverse problem, in case of *complete* boundary data $K = \mathbb{T}$, the function $f = U$ is available from (4) on the whole boundary and belongs to $\lambda_\alpha(\mathbb{T})$, $\alpha < 1/2$. The poles of the best meromorphic approximant $g_N = \mathcal{A}_N(f)$ to f provide a discrete approximation of the crack σ in some weak sense. Links between U, σ, and the localisation of the poles of best meromorphic or rational approximants are deeply studied in [6, 22].

2.3 Bounded extremal problems

In the present situation, we are provided with *incomplete* data on a strict part $K \subset \mathbb{T}$ of the boundary, accessible to measurements, as in section 2.1. *Assume that both K and $\mathbb{T} \setminus K$ have positive Lebesgue measure.* Given

$$f = U = u_K + i \int_a^\xi \Phi(e^{i\theta})\, d\theta\,,$$

on K, the issue of finding an extension to $\mathbb{T}$ can be constructively approached as a bounded extremal problem, if *some information is also available on* $\mathbb{T}\backslash K$. We now turn to the study of these extremal problems before being back to the inverse problem in section 2.4.

Let $\psi \in L^\infty(\mathbb{T}\backslash K)$ and $M \in \mathcal{M}$, and

$$D_{M,\psi} = \{h \in L^\infty(\mathbb{T}\backslash K),\ |h - \psi| \leq M \text{ a.e. on } \mathbb{T}\backslash K\}.$$

Problem 2 (BCP) *Given $f \in L^\infty(K)$, $\psi \in L^\infty(\mathbb{T}\backslash K)$, $M \in \mathcal{M}$,and $N \geq 0$, find $h_N \in D_{M,\psi}$ such that:*

$$d(f \vee h_N,\, H^\infty + R_N) = \min_{h\in D_{M,\psi}} d(f \vee h,\, H^\infty + R_N) = \gamma_N(f \vee \psi,\, M).$$

The issue here is to find an extension $h_N \in D_{M,\psi}$ of f which ensures that the extended function $f \vee h_N$ is as close as possible to the set $H^\infty + R_N$ of meromorphic functions bounded on $\mathbb{T}$. This is the main subject of [8] where the following result is established. Introduce the set:

$$B^N_{M,\psi} = \{g \in H^\infty + R_N,\ |g - \psi| \leq M \text{ a.e. on } \mathbb{T}\backslash K\},$$

and the set of its traces on K: $C^N_{M,\psi} = \{g_{|K},\ g \in B^N_{M,\psi}\}$.

Theorem 2 ([8]) *If $f \vee \psi \in H^\infty + C(\mathbb{T})$ and $M \in \mathcal{M}$, it holds that:*
(i) (BCP) *admits a unique solution h_N.*
(ii) When $f \notin C^N_{M,\psi}$, $|\psi - h_N| = M$ a.e. on $\mathbb{T} \backslash K$.

Moreover, an algorithm for the computation of the extension h_N is available which involves the solution of a bounded approximation problem (BEP). It consists in finding the uniformly closest meromorphic approximation to f on K among the functions in $C^N_{M,\psi}$.

Problem 3 (BEP) *Given $f \in L^\infty(K)$, $\psi \in L^\infty(\mathbb{T}\backslash K)$, and $M' \in \mathcal{M}$, find $g_N \in B^N_{M',\psi}$ such that:*

$$\|f - g_N\|_{L^\infty(K)} = \min_{g \in B^N_{M',\psi}} \|f - g\|_{L^\infty(K)} = \beta_N(f \vee \psi, M'). \tag{7}$$

When $f \vee \psi \in H^\infty + C(\mathbb{T})$ and $M' \in \mathcal{M}_*$, existence and uniqueness of the solution g_N are established in [8, thms 2, 5], as well as the circularity properties if $f \notin C^N_{M',\psi}$: $|f - g_N| = \beta_N$ and $|\psi - g_N| = M'$ a.e. on K and $\mathbb{T}\backslash K$, respectively. When approximating in K, the addition of a constraint on $\mathbb{T} \setminus K$ is necessary in order to avoid unstable behaviour of the approximants and to ensure well-posedness of (BCP) and (BEP), see [7, prop. 2].

Problems (BCP) and (BEP) can be implicitly formulated as (AAK) and this provides a way to solve them. For $\rho \in L^\infty(\mathbb{T})$ with positive values such that $\log \rho \in L^1(\mathbb{T})$, define the outer H^∞ function w_ρ of modulus equal to ρ a.e. on $\mathbb{T}$ [16, 18, 20, 27]:

$$w_\rho(z) = \exp\{\frac{1}{2\pi}\int_{\mathbb{T}} \frac{e^{it} + z}{e^{it} - z} \log \rho(e^{it}) dt\}.$$

If $\inf_{\mathbb{T}} \rho > 0$, then w_ρ is invertible in H^∞ and $w_\rho^{-1} = w_{1/\rho}$. When $f \vee \psi \in H^\infty + C(\mathbb{T})$, $M \in \mathcal{M}$, and $f \notin C^N_{M,\psi}$, we can compute the solutions h_N and g_N to (BCP) and (BEP) as follows from [8].

- Get $\gamma_N = \gamma_N(M)$ by solving (AAK) type problems: $\gamma_N(M)$ coincide with the unique $\gamma \in \mathbb{R}^*_+$ such that
$$\min_{v \in H^\infty + R_N} \|(f \vee \psi)\, w^{-1}_{\gamma \vee (M+\gamma)} - v\|_{L^\infty(\mathbb{T})} = 1\,.$$
This gives $\beta_N(M + \gamma_N(M)) = \gamma_N(M)$. Let then v_N be the unique solution to
$$\min_{v \in H^\infty + R_N} \|(f \vee \psi)\, w^{-1}_{\gamma_N \vee (M+\gamma_N)} - v\|_{L^\infty(\mathbb{T})} = \|(f \vee \psi)\, w^{-1}_{\gamma_N \vee (M+\gamma_N)} - v_N\|_{L^\infty(\mathbb{T})}\,,$$
- The solutions h_N and g_N to (BCP) and (BEP) are then given by:
$$h_N = \frac{M}{M + \gamma_N}\, v_N\, w_{\gamma_N \vee (M+\gamma_N)} + \left(1 - \frac{M}{M + \gamma_N}\right) \psi\ ,$$
$$g_N = v_N\, w_{\gamma_N \vee (M+\gamma_N)}\,.$$

It is also shown that if $f \vee \psi \in \lambda_\alpha(\mathbb{T})$ and $M \in \lambda^{+,0}_\alpha(\mathbb{T} \setminus K)$, then $h_N \in \lambda_\alpha(\mathbb{T} \setminus K)$ and $g_N \in \lambda_\alpha(\mathbb{T})$: both the data and the approximants belong to Hölder classes where the procedure is effective and generically robust.

2.4 Back to data extension

As indicated above, the addition of a constraint on $\mathbb{T} \setminus K$ is necessary to ensure well-posedness of (BCP) and stability of the extension U_N of U to $\mathbb{T}$. In the framework of the present inverse problem, the flux Φ is known on the whole boundary $\mathbb{T}$ where it determines $\operatorname{Im} U$. It would thus be natural to ask $\operatorname{Im} U_N$ to be close to $\operatorname{Im} U$ on $\mathbb{T} \setminus K$. However, this is not enough for the extension

/ approximation problems to be well-posed, in uniform norm (although such a constraint can be handled in H^2, [21]). Nevertheless, one can consider the function

$$F = \exp(i\,U),$$

which is still analytic in $\mathbb{D}\backslash\sigma$ and known on K, while it verifies on $\mathbb{T}$:

$$|F| = |\exp(iU)| = \exp(-\mathrm{Im}\,U)\,.$$

Hence, the constraint

$$\mathrm{Im}\,U = \int \Phi\,,$$

can be expressed on the modulus of F:

$$|F| = \exp\,(-\int \Phi)\,.$$

The problem now becomes that of determining an extension of F to the whole $\mathbb{T}$ whose modulus is constrained on $\mathbb{T}\setminus K$. This can now be done by looking for a solution h_N to problem 2 associated to $f = F$ on K, $\psi = 0$ and $M = \exp\,(-\int\Phi)$ on $\mathbb{T}\setminus K$, for $N \geq 0$. Finally, $U_N = -\,i\,\log h_N$, with an appropriate determination, gives an extension of U to $\mathbb{T}$.

3 An algorithm for cracks identification

3.1 Description of the scheme

Assume that K is an arc of $\mathbb{T}$ and let $K^\varepsilon \subsetneq K$ be a strict subset of K, with $K \setminus K^\varepsilon$ of small Lebesgue measure, say $\varepsilon > 0$. Let also α, $0 < \alpha < 1/2$. We propose the following identification algorithm for an line segment crack $\sigma \subset \mathbb{D}$.

0. From the data Φ on $\mathbb{T}$, $\Phi \in L^2(\mathbb{T})$, and u_K on K, get

$$\left\{\begin{array}{lcll} F & = & \exp\,(i\,U) = \exp i\,(u_K + i\int\Phi) & \text{in} \quad \lambda_\alpha(K^\varepsilon), \\ M & = & |F| = \exp\,(-\int\Phi) & \text{in} \quad \lambda_\alpha^+(\mathbb{T}\backslash K^\varepsilon). \end{array}\right.$$

Build a polynomial P_0 that vanishes at order one at the endpoints $z_\pm^\varepsilon$ of K^ε but not on $\mathbb{T}\backslash K^\varepsilon$.
1. Fix $N > 0$. Compute the solution h_N to problem (BCP) associated to $F_0 = P_0\,F \in \lambda_\alpha(K^\varepsilon)$, $M_0 = |P_0|\,M \in \lambda_\alpha^{+,0}(\mathbb{T}\backslash K^\varepsilon)$, and $\psi = 0$ (whence $F_0 \vee \psi \in \lambda_\alpha(\mathbb{T})$), using the procedure developed in [8] and briefly recalled in section 2.3:
i) Build a rational approximant r to $F_0 \vee 0$ in $\lambda_\alpha(\mathbb{T})$.
ii) Find the unique $\gamma > 0$ such that the $(N+1)$th singular value σ_N of the Hankel operator Γ_γ with symbol $r\,w^{-1}_{\gamma\vee(\gamma+M_0)}$ verifies:

$$\sigma_N(\Gamma_\gamma) = 1.$$

iii) Compute then the AAK approximant v_N to $rw^{-1}_{\gamma\vee(\gamma+M_0)}$ in $H^\infty + R_N$ given by (6) :

$$v_N = rw^{-1}_{\gamma\vee(\gamma+M_0)} - \frac{\Gamma_\gamma\,\nu_N}{\nu_N}\,,$$

where ν_N denotes the singular vector of Γ_γ associated to σ_N. For points *ii)* and *iii)*, one can use classically available singular value decomposition procedures.
iv) On $\mathbb{T}\backslash K^\varepsilon$, compute

$$h_N = \frac{M_0}{\gamma + M_0}\,v_N\,w_{\gamma\vee(\gamma+M_0)} + \frac{\gamma}{\gamma+M_0}\,r = r - \frac{M_0}{\gamma+M_0}\,w_{\gamma\vee(\gamma+M_0)}\,\frac{\Gamma_\gamma\,\nu_N}{\nu_N}\,.$$

2. Because $|h_N| = M_0 = |P_0|\, M > 0$ on $\mathbb{T}\backslash K^\varepsilon$, the function h_N/P_0 remains bounded (by M) and smooth on the closure of $\mathbb{T}\backslash K^\varepsilon$. Moreover, it does not vanish on $\mathbb{T}\backslash K \subset \mathbb{T}\backslash K^\varepsilon$ where its argument is thus also smooth. On $\mathbb{T}\backslash K$, we then take

$$U_N = -i \, \log \left[\frac{h_N}{P_0}\right] ,$$

and

$$u_N = \mathrm{Re}\, U_N = -i \, \log \left[M^{-1}\, \frac{h_N}{P_0}\right] = \arg \left[M^{-1}\, \frac{h_N}{P_0}\right] .$$

This defines u_N up to $2\,k\pi$, and the integer k can now be chosen so that u_N and u_K are as close as possible at the endpoints of K. The function $\mathcal{U}_N = u_K \vee u_N$ furnishes an extension of the data to the whole $\mathbb{T}$ and an approximate solution to (3). This solves for step I.

3. Apply the (RG) algorithm introduced in section 1.2, see [11], for the crack's complete determination with the extension $\mathcal{U}_N$ issued from the previous step **2.**. that is:

i) Recover the hosting line D equation by the determination of its normal unit vector n_σ and the position with respect to n_σ. Let us designate by $(v_k)_{k=1,2}$ the harmonic fields $v_k(x_1, x_2) = x_k$, where x_1, x_2 designate the Cartesian coordinate in $\mathbb{R}^2$, denoting $L_k = RG(v_k)$, we evaluate L_k via the boundary integral (2) using $u = \mathcal{U}_N$. The components of n_σ are therefore given by:

$$\frac{L_k}{\sqrt{L_1^2 + L_2^2}} \, , \quad k = 1, 2 \, .$$

Orthogonally to n_σ, the location of D is given by the constant :

$$\frac{RG(p)}{\sqrt{L_1^2 + L_2^2}} \, , \quad \text{where } p(x_1, x_2) = \frac{1}{2}\,(x_2^2 - x_1^2) \, .$$

ii) We define the function:

$$J : (x_a, x_b) \mapsto RG^2(w_{\tilde{\sigma}}) \, ,$$

x_a and x_b being the abscissa in D of the endpoints a and b of the "fictitious" crack $\tilde{\sigma}$, with

$$w_{\tilde{\sigma}}(z) = \mathrm{Im}\{(z-a)^{1/2}\,(z-b)^{1/2}\} \, , \; z \in \Omega \setminus \tilde{\sigma} \, .$$

The location of σ on the hosting line D being achieved as follows:

1. Give an initial guess (x_a^0, x_b^0) in D.

2. Find x_a^{k+1} such that:
$$J(x_a^{k+1}, x_b^k) = \min_{c_1 \le \tau \le c_2} J(\tau, x_b^k)$$

3. Find x_b^{k+1} such that:
$$J(x_a^{k+1}, x_b^{k+1}) = \min_{c_1 \le \xi \le c_2} J(x_a^{k+1}, \xi).$$

   Above, the constants c_1 and c_2 are adjusted in order to ensure that the line segments (τ, x_b^k) and (x_a^{k+1}, ξ) are contained in $D \cap \mathbb{D}$; we use the "golden number" method to minimise the one variable functions.

4. Evaluate the distance between (x_a^{k+1}, x_b^{k+1}) and (x_a^k, x_b^k) in D. If this distance is small enough, then $\tilde{\sigma} \approx \sigma$ and therefore (x_a^{k+1}, x_b^{k+1}) are (close to) the endpoints of σ on D; else, go back to step 2. with $k \leftarrow k+1$.

This solves for step II.

3.2 Numerical results

In this section, we present the numerical tests of the above algorithms, for data completion and crack identification.

The input data are numerically simulated. On the unit disk $\mathbb{D}$ deprived of an interior segment σ that represents the crack, one solves Laplace equation $\Delta u = 0$ using a finite elements method. As in (3), the crack σ is assumed to be perfectly insulating and the heat flux Φ is imposed on $\mathbb{T}$ by a Neumann type condition on the boundary. One chooses here:

$$\Phi = \begin{cases} -1 & \text{on} \quad [0, \pi) \\ \;\;\,1 & \text{on} \quad [\pi, 2\,\pi) \end{cases}$$

This makes it possible to obtain the measurement u_K of the solution on the part K of the boundary $\mathbb{T}$ assumed to be accessible to measurements. We take $K = [\frac{\pi}{2}, \frac{3\pi}{2}]$ excepted for some tests, for which we will explicitely choose a different subset K. One can then apply the procedure such as it is described in section 3.1. The approximation algorithm in $H^\infty + R_N$ (steps **0.** to **2.**) is developed with MATLAB, the algorithm (RG) – for the identification of σ from completed data (step **3.**) – with FORTRAN.

Figure 1 represents the various steps of the development of this procedure. The diagram (1.1) shows the temperature on $\mathbb{T}$ and the associated primitive of the flux. We represent in (1.2) the level-lines of the solution u. We emphasise on the fact that only the trace u_K on K of the temperature is assumed to be known for the algorithm to be run. Figure (1.3) shows Nyquist diagram of the function F_0 on K, which is the parametrised curve (Re $F_0(e^{i\theta})$, Im $F_0(e^{i\theta})$), for $\theta \in K$. Figure (1.4) shows the constraint M_0 on $\mathbb{T}\backslash K$. The approximation in $H^\infty + R_N$ then provides the function $F_0 \vee h_N$ on all the circle $\mathbb{T}$; its Nyquist diagram is plotted in figure (1.5). Here $N = 38$ and σ is the segment $[0, 0.4]$. The diagram (1.6) shows the function u_N reconstructed on $\mathbb{T} \setminus K$ with $\varepsilon = 0$; note the "bad" behaviour at the ends of K, due to the division by P_0 which vanishes in these points. On the other hand, the introduction in the algorithm of a parameter $\varepsilon > 0$ makes it possible to calculate u_N only sufficiently far from the zeroes of P_0 (located on ∂K^ε, at distance ε from $\mathbb{T} \setminus K$), and provides a better solution represented by the diagram (1.7). Here ε is equal to 5% of the Lebesgue measure of K, the error made on $\mathbb{T} \setminus K$ compared to the true function u is lower than 3% in uniform norm and to 2% in L^2 norm. Finally, the (RG) algorithm furnishes the location of the crack σ as indicated in the test (1.8), where we obtain a satisfactory result.

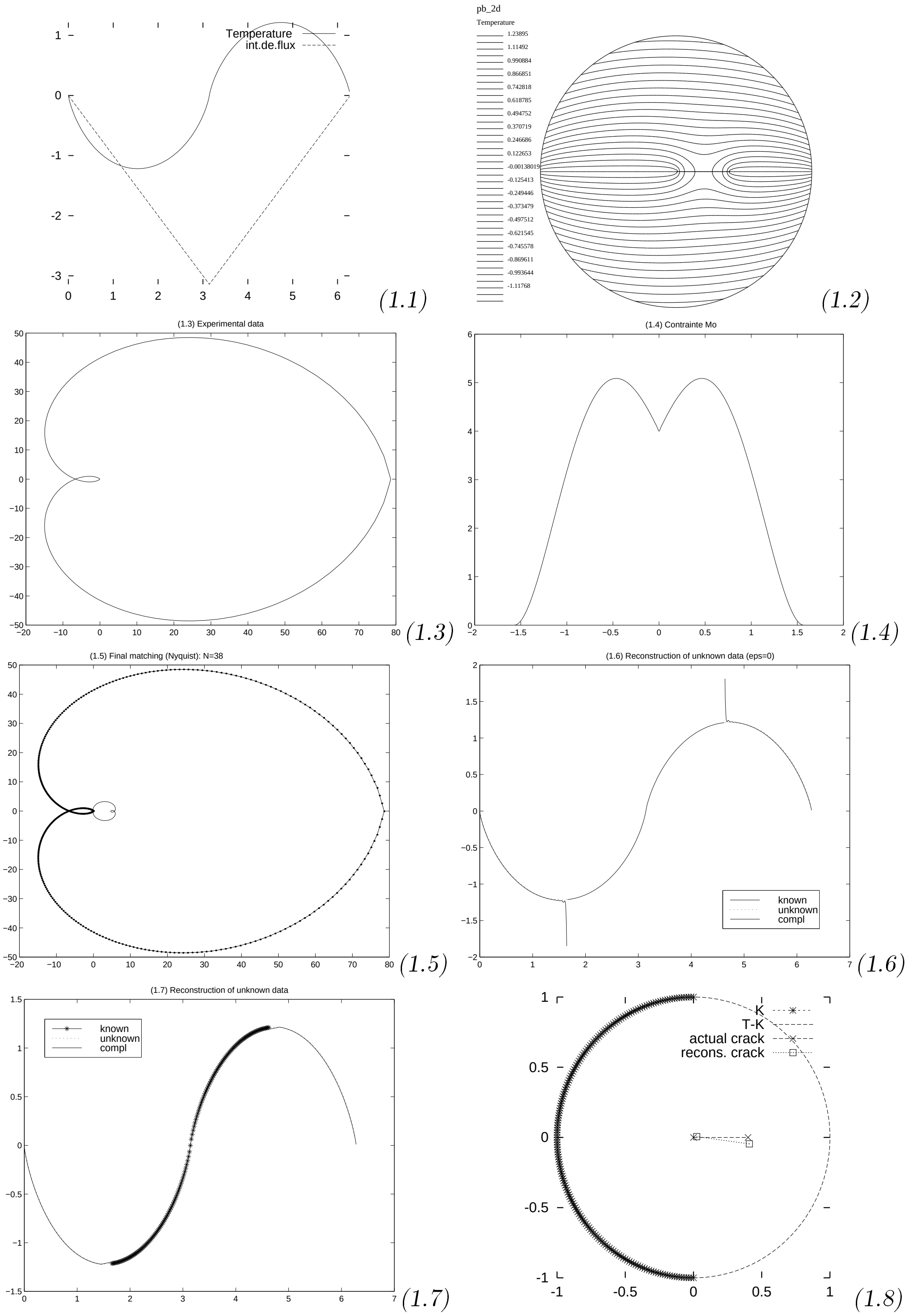


Figure 1: Data completion and crack identification.

Sensitivity to poles number N

Although the error γ_N in problem (BCP) decreases to zero as N increases, the proof of some convergence result of u_N to u on $\mathbb{T} \setminus K$ as $N \to \infty$ is not yet available and remains under study. However, numerical computations show that, as N increases, the error on $\mathbb{T} \setminus K$ between the extended function and the actual one decreases, in the present case. In figure 2, we show the behaviour of u_8 and u_{38} with the corresponding reconstructed crack. The value of γ_N with respect to N is plotted in the left picture of figure 3. For $N \geq 30$, $\gamma_N \leq 10^{-3}$, and we also get good approximations on $\mathbb{T} \setminus K$, see the relative errors there in the right plot of figure 3, for uniform and L^2 norms. Observe that, due to numerical precision on the computation of singular values, the ratio γ_{N+1}/γ_N becomes almost equal to 1, when N increases. This furnishes the largest value of N for which these computations are relevant.

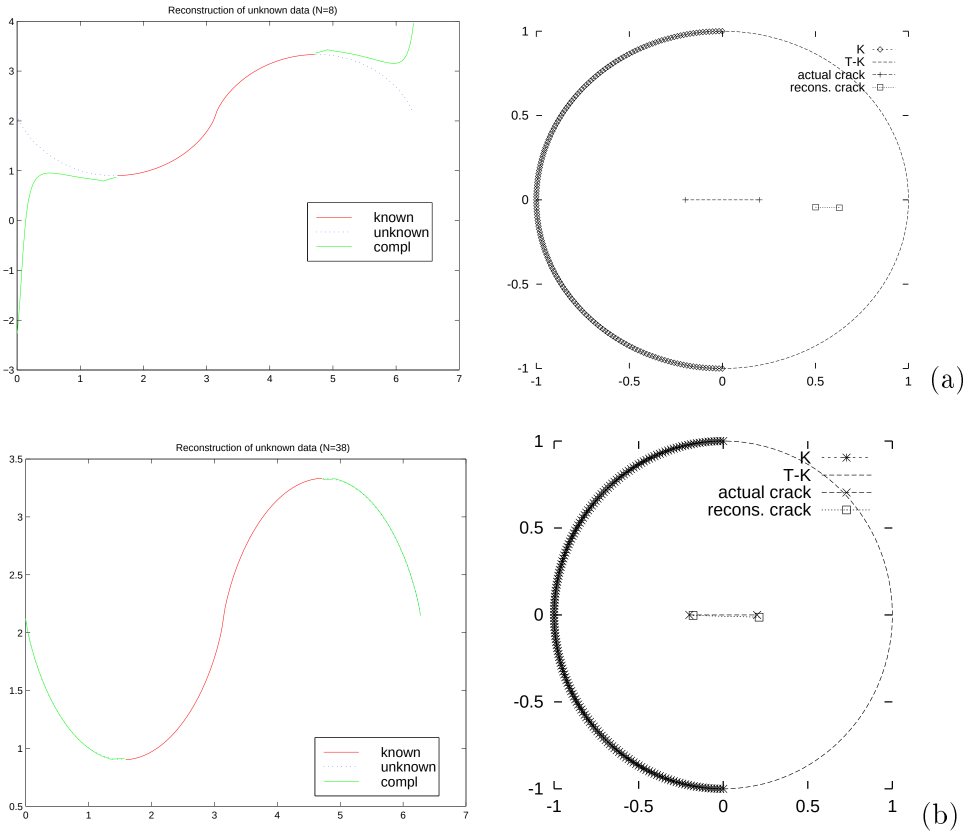


Figure 2: Reconstruction from $N = 8$ (a), and $N = 38$ (b).

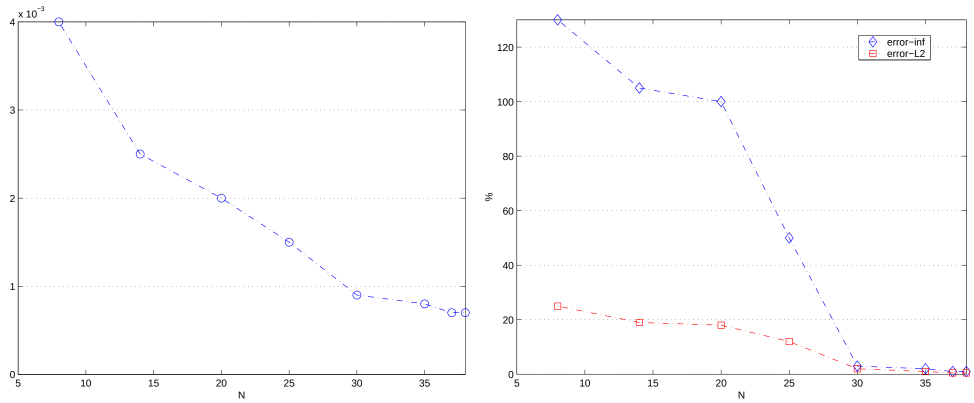


Figure 3: Completion error, with respect to N.

Behaviour with respect to σ

In the following, we keep $N = 38$. Figure 5 shows a case where the crack σ is not supported by the real line while, in figure 4, the crack is located near the boundary $\mathbb{T}$ of the domain, in two different situations for the relative positions of σ and K. In the first one, $K = [\frac{\pi}{2}, \frac{3\pi}{2}]$ as before and σ is close to the accessible part K; in the second one, $K = [-\frac{\pi}{2}, \frac{\pi}{2}]$ and σ is close to $\mathbb{T} \setminus K$ where the solution is to be extended. In both cases, the results are satisfactory.

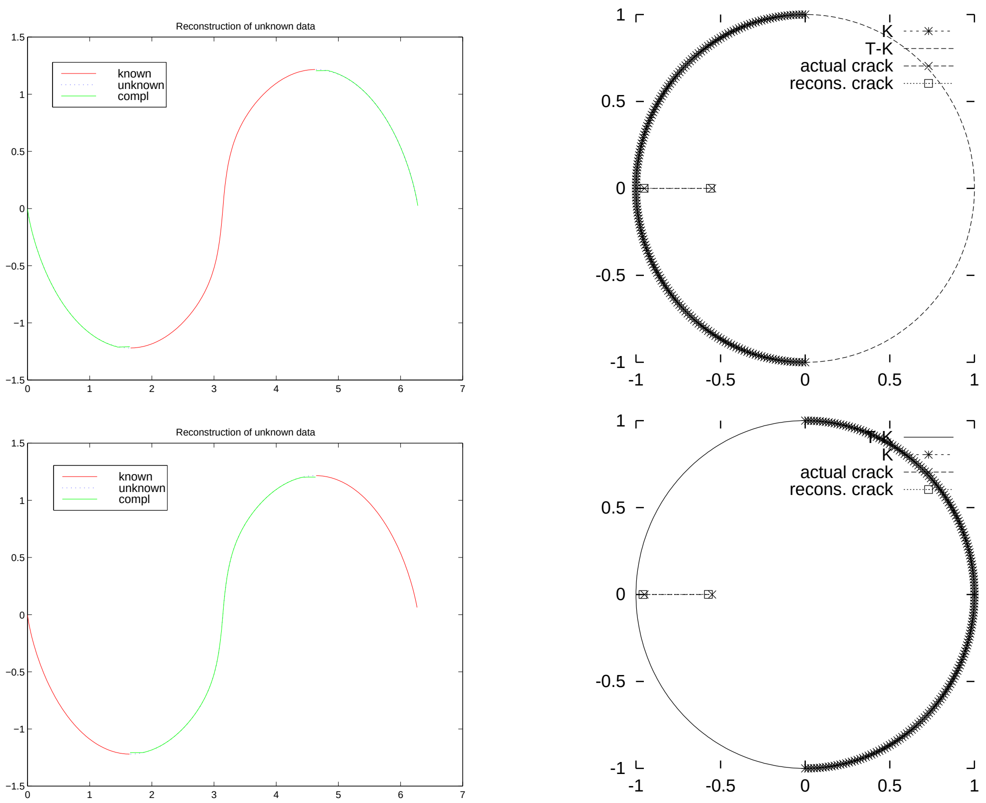


Figure 4: Crack σ close to K, to $\mathbb{T} \setminus K$.

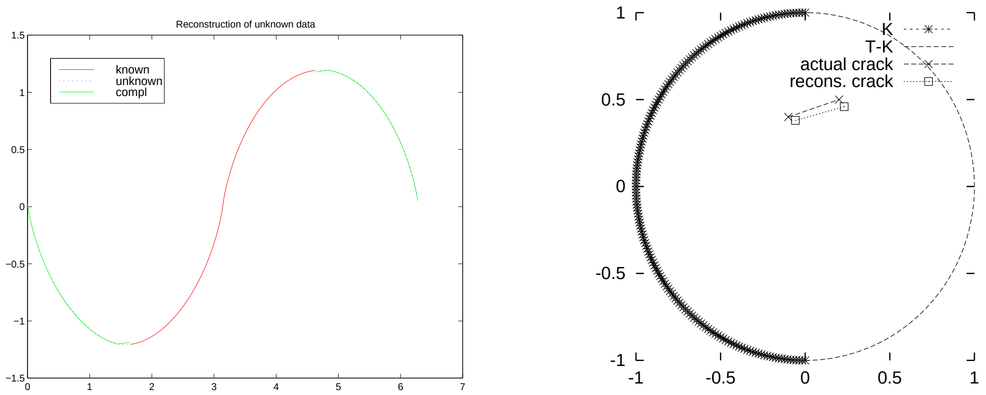


Figure 5: Crack σ located outside $\mathbb{R}$.

Behaviour with respect to K

In figure 6, two plots show the sensitivity with respect to the amount of available data. These error are plotted according to the ratio of provided data ($mes(K)/mes(\mathbb{T})$). In the left plot, the errors made on $\mathbb{T} \setminus K$ between u_N and the actual u are represented in uniform and L^2

norms. In the right plot, the error d is the distance between the true crack and the reconstructed one; given two straight cracks (AB) and $(A'B')$, their distance is given by: $d(AB, A'B') = \min\left\{\frac{1}{2}\left(d(A,A') + d(B,B')\right); \frac{1}{2}\left(d(A,B') + d(B,A')\right)\right\}$. Figure 7 gives the reconstruction from prescribed data on 1/3 the external boundary (a), and on 1/6 the external boundary (b).

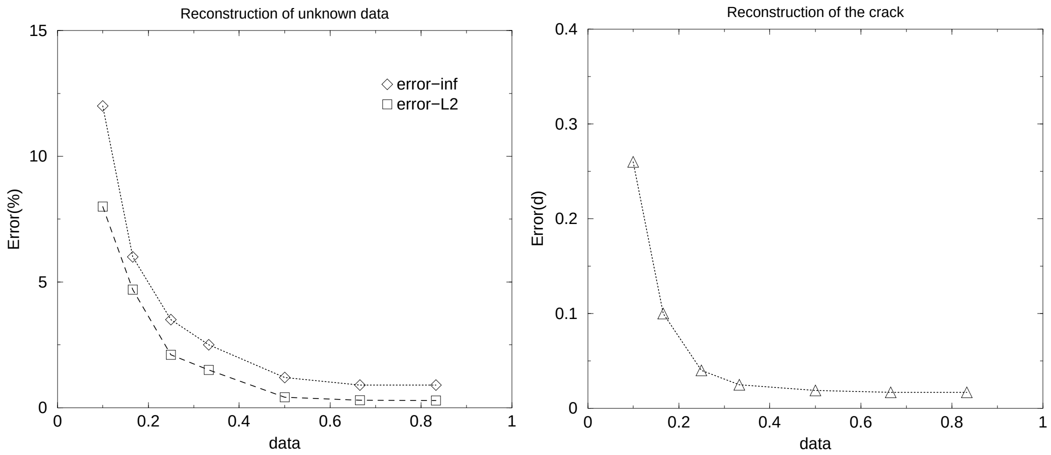
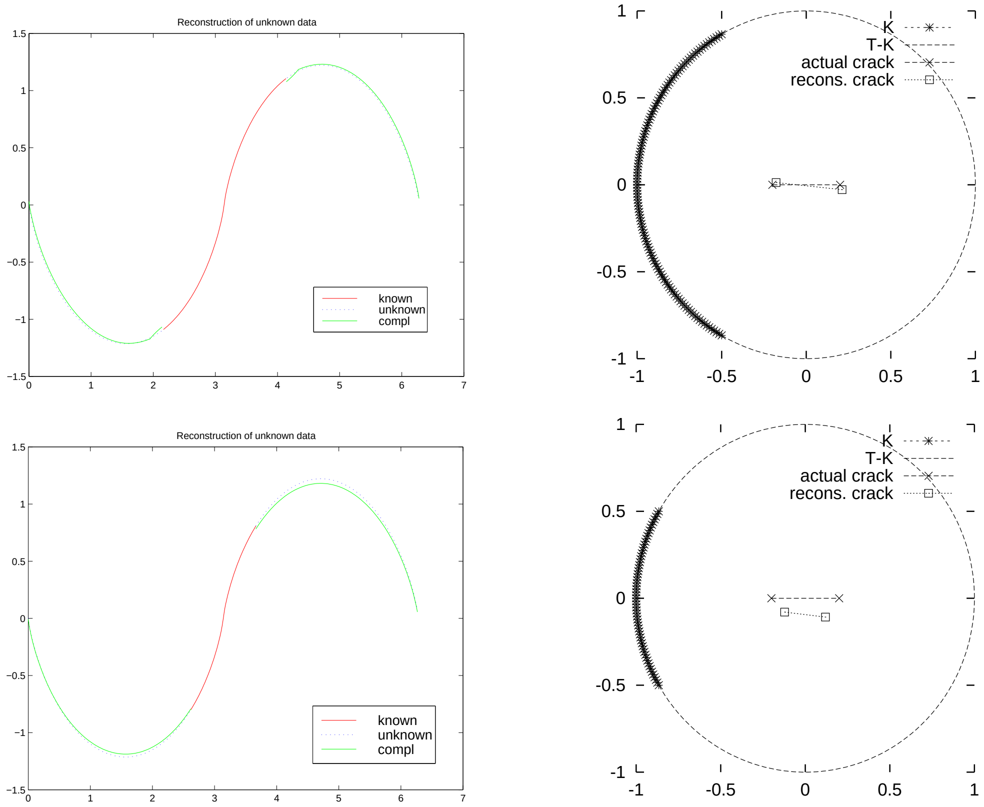


Figure 6: Errors w.r.t. the amount of prescribed boundary data.



(a)

(b)

Figure 7: Recovery from data on smaller parts K of the external boundary.

Figure 8 shows that the reconstruction is indeed sensitive to the prescribed data location K. The reconstruction is improved if the available data location is relevant to the crack, meaning that K may capture the field variations induced by σ. The sensitivity of this reconstruction scheme depends on the amount of available data and decreases as the length of K increases. The first numerical reconstruction has been run with the arc $K = [\frac{\pi}{3}, \frac{4\pi}{3}]$; for the others, K is rotated.

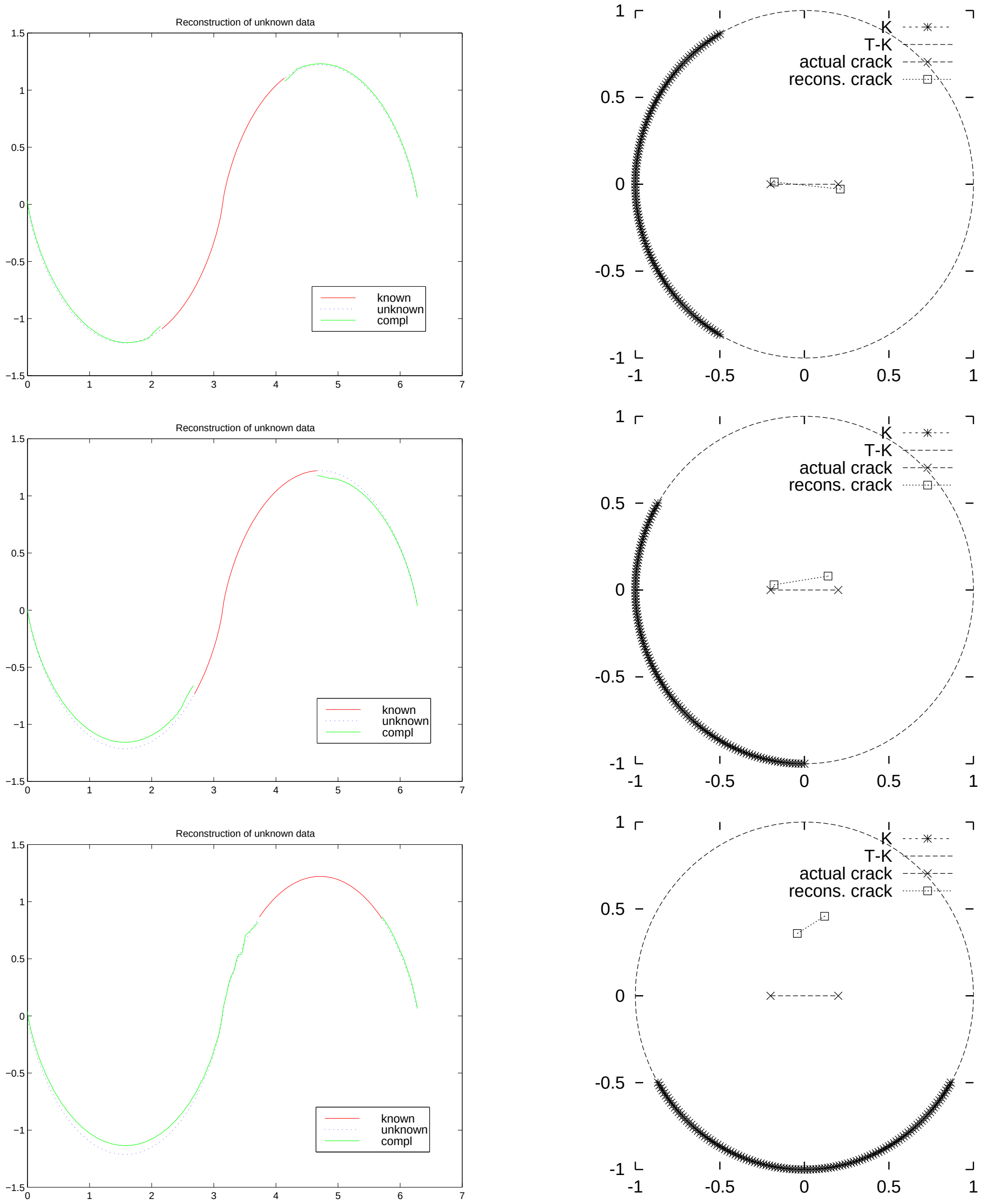


Figure 8: Sensitivity to the prescribed data location K.

Robustness

Let us recall that our experiments are all performed on synthetic data. It is our hope to eventually try our identification procedure on actual experimental data. To go towards a reconstruction based on real data, we focus in this last section on simulated noisy data.
Here, an additive noise (random from MATLAB) comes in to perturb the given data. Figures 9, 10 and 11 show the result for noises whose relative uniform norms on $K = [-\frac{\pi}{2}, \frac{\pi}{2}]$ (with respect to that of u) are equal to 2%, 5% and 10%, respectively. The tests indexed by (a) represent the results obtained from these noisy data (without smoothing). In (b), a preliminary smoothing step is added to the step **0.** of the algorithm, using cubic splines. The results are close enough to that obtained from data without noise (the crack is recovered in a satisfactory way for small noise levels). They show that the whole scheme carries some robustness with respect to perturbation.

Concerning the extension step I. of the procedure, this is in accordance with the (generic) continuity result in Hölder classes from [8].

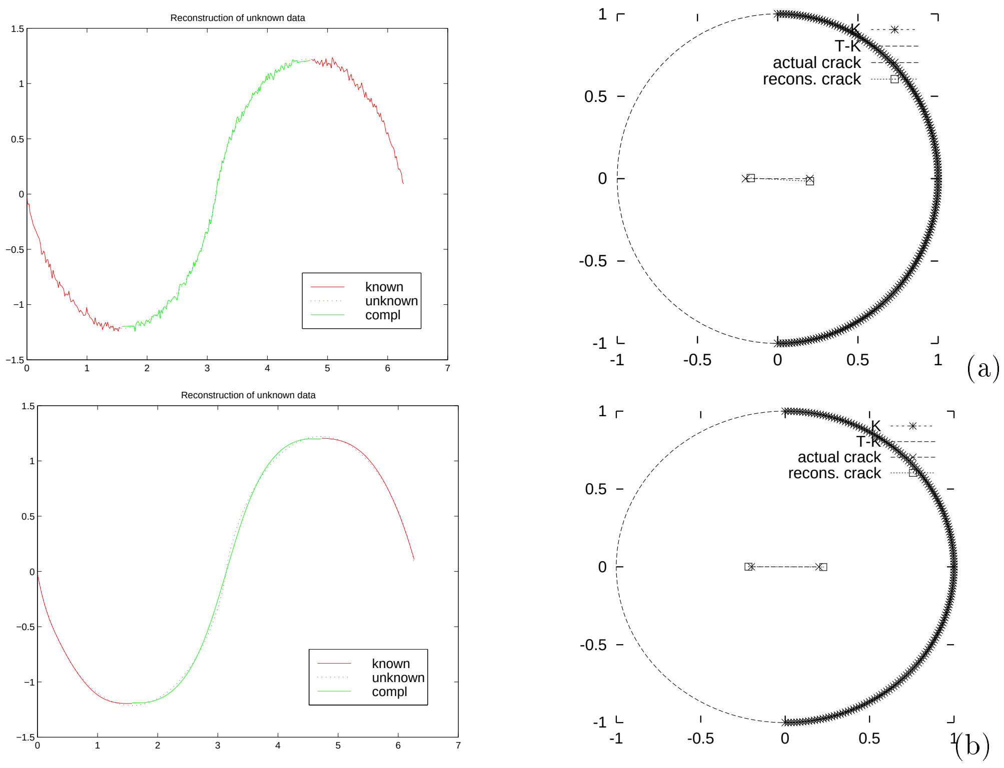
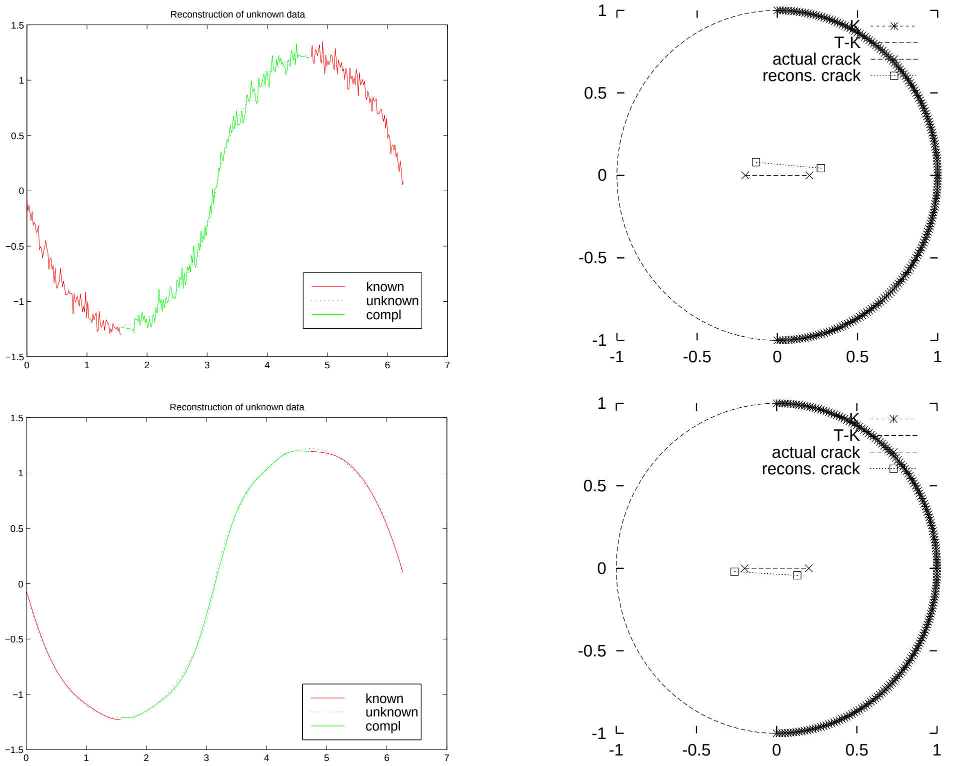


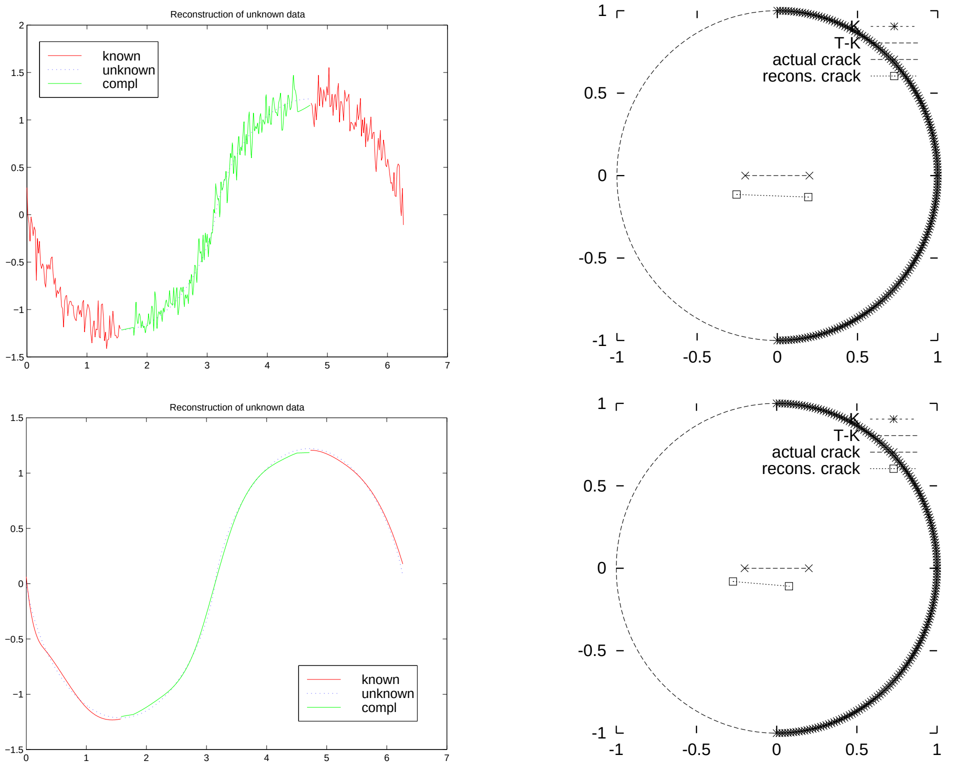
Figure 9: 2% noise



(a)

(b)

Figure 10: 5% noise



(a)

(b)

Figure 11: 10% noise

4 Conclusion

Observe first that the present class of approximation methods, that allow to extend lacking data, could be used for solving different kind of inverse problems defined by overdetermined data on a part of the boundary, such as identifying unknown geometries or parameters.
It is also worthful to explicit directly the (singular) integral obtained in (5) of the solution (1) to get information about the location of σ, using the poles of the uniform or L^2 meromorphic approximants of U on $\mathbb{T}$. Actually, the poles of the meromorphic approximant behave in relation with the endpoints of σ, as shown recently in [6, 22]. The underlying idea is to get a discretization of σ from the singularities in $\mathbb{D}$ of the approximant U_N to U. More generally, this can be viewed as a resolution scheme for elliptic two dimensional boundary values problems which no longer relates on the discretization of the operator but on the approximation of boundary conditions in a suitable class of functions (harmonic, for Laplace operator).
Furthermore, the results obtained in [21] allows the extension in the Hardy space H^2 with a constraint imposed on the imaginary part. This procedure, followed by a step of rational approximation, would allow an efficient resolution of the problem under consideration. Another possibility would be to substract from $f = U$ on K the analogous fonction associated to the domain without crack. One then deals with a real valued function on $\mathbb{T}$ and this raises another extremal problem, consisting in approximating a given function on K by the real part of an H^2 function that satisfies some constraint in $\mathbb{T} \setminus K$.
Note finally that available uniqueness, stability, and identification results usually require the domain Ω to be simply connected. However, the reciprocity gap identification process [11] does not require such an assumption. Moreover, in practice, relevant applications concern the detection and location of cracks in tubes. From data available on the outer boundary, we can hope to recover the trace of the solution on the inner boundary, using approximation in Hardy spaces of circular domains as in [15].

Acknowledgments

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