Sources recovery from boundary data: a model related to electroencephalography

Amel Ben Abda* Fahmi Ben Hassen*
Juliette Leblond† Moncef Mahjoub*

Abstract

We are concerned with an inverse problem related to sources detection from boundary data in a 2D medium with piecewise constant conductivity. It stands as a 2D version of the inverse problem of electroencephalography, where pointwise sources model epilepsy foci, with the so-called multi-layer spherical model of the head (scalp, skull, brain). When overdetermined electrical measurements (potential and current flux) are available on the scalp, one wants to recover the current sources (conductivity defaults) located in the brain (inner boundary). This recovery issue reduces to a number of inverse problems, where the sources identification process makes use of best rational approximation in the disk, whereas the preliminary cortical mapping step (Cauchy type issue) relies on best constrained harmonic or analytic approximation in an annulus (bounded extremal problems).

Classification numbers (AMS): 22E46, 53C35, 57S20

1 Introduction and motivation

We approach here the inverse problem of determining buried pointwise electrical sources from overspecified boundary data of the solution of heterogeneous electrical conduction equations.

Our main application is borrowed to the bioengineering communauty. It focuses on the identification of the location of epileptic centers in a human brain and of their corresponding electrical intensities. This inverse problem is known as the EEG (encephalography) one, where the data are values of the potential measured at a number of points on the scalp by electrodes. This method is the oldest one for the investigation of brain activity or deceases. It is the best known among the non invasive method and has been performed since 1929.

It allows to measure spontaneous electrical activity of the brain (electroencephalogram), either on the scalp, or directly (and invasively) in the brain (stereo-EEG). The identification of sources involved in epileptic spikes in such an electroencephalogram is clinically

*LAMSIN, ENIT, BP 37, 1002 Tunis Belvedere, TUNISIA, amel.benabda@enit.rnu.tn, fahmi.benhassen@enit.rnu.tn, moncef.mahjoub@lamsin.rnu.tn

†INRIA, BP 93, 06902 Sophia–Antipolis Cedex, FRANCE, leblond@sophia.inria.fr, phone: +33 4 92 38 78 76, fax: +33 4 92 38 78 58

crucial for the diagnosis and treatement. It is also used for the functional exploration of the brain.
The physical principle of such a method consists in considering the electrical activity generated by bioelectrical sources in the brain which corresponds to the dynamics of a large neuron population having the capacity to work in syncron. Notice that the fundamental feature of EEG relies on the fact that it follows the dynamics of the brain's activity up to a temporary scale of milliseconds order, which is the one of cognitive processes, contrarily to others well-known neuro-imaging modalities such that MRI or MST-DT.
We approach here the inverse problem of recovering bioelectric sources on the cerebral cortex, from the potential generated by these sources on the scalp. We refer to [15] for different statements. A classical one, known to be relevent in the case of epilepsy - at least at the first stage of the decease - is to take a pointwise dipolar model for the distribution of cortical current sources ([2], [17]).
We thus consider the inverse problem of Laplace operator which consists in recovering pointwise sources distributed in a 2D domain from available measurements of the solution on the boundary. This is as mensioned above, related to the inverse electroencephalography problem in spherical 3D domains ([17]), which gives simple models of the human head, assumed to be a ball made of (at least) three concentric spherical homogeneous layers Ω_i, $i = 0, 1, 2$, of constant conductivity (corresponding to the scalp, the skull, and the brain), see Figure 1. From the mathematical viewpoint, this inverse problem is known to be well-posed: uniqueness as well as weak stability results has been established ([6], [8], [11], [14]).
Of course, more realistic models could be obtained from anatomic MRI images, but the

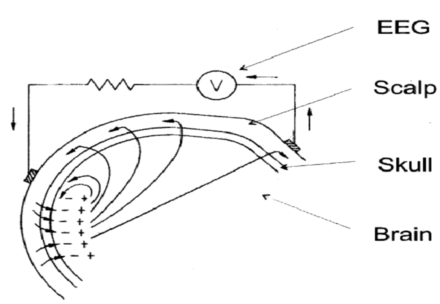

Figure 1: Realistic head model

actual spherical model is already suitable and allows explicit bases and analytical solutions for the potential problem.
In this setting, overdetermined electrical measurements (potential and current flux) are available on the scalp (external boundary), from which one wants to recover some current sources (conductivity defaults) located in the brain (inner layer). The situation where the data are already known on the boundary $\partial\Omega_0$ of the inner layer has been handled in [6], [8], in 2 and 3D. To test the capacity of the following proposed identification pocess we here restrict ourseves to a 2D situation, mainly in order to constructively solve the first cortical mapping step, since then, the problem may be solved using best analytic approximation. The propagation of outer Cauchy data to the interior interface is therefore handled using best approximation tools in Hardy classes of an annulus, as described in [19]. The sources are then located as the poles of a rational function, recovered from boundary data by best rational or meromorphic approximation algorithms ([6]).

The steady state electric potential is assumed to be the solution to

$$(NP) \qquad \begin{cases} -\nabla.(\sigma\nabla\, u) = F = \displaystyle\sum_{k=1}^{N} p_k.\nabla\, \delta_{c_k} \quad \text{in } \Omega, \\ \\ \sigma\partial_n u = \Phi \text{ on } \partial\Omega\,, \qquad\qquad \displaystyle\int_{\partial\Omega} \Phi = 0 \end{cases}$$

where n represents the outward unit normal vector to $\partial\Omega$ and σ is a piecewise constant conductivity which is equal to σ_i on both annuli $\Omega_i \setminus \bar{\Omega}_{i-1}$, $i = 1, 2$, and to σ_0 on the disk Ω_0. Being given measures (pointwise values, or a smooth enough interpolating function) u_b of u on the outer boundary $\partial\Omega$, the issue is to recover the location of N pointwise dipolar sources c_k located in the inner domain Ω_0, and their moment p_k, for $k = 1, .., N$. This involves harmonic data propagation from the outer boundary to the inner one as a preliminary step, after which the singularities are to be recovered. Indeed, on each circular layer:

$$\Delta\, u = 0\,, \text{ in } \Omega_i \setminus \Omega_{i-1}\,,\ i = 1, 2,\ -\Delta\, u = F/\sigma_0\,, \text{ in } \Omega_0,$$

and transmission condition are required on connected components on the boundaries $\Gamma_i \cap \Gamma_{i-1} = \partial\Omega_i$, $i = 1, 2$:

$$\begin{cases} u_{|\Gamma_{i-1}} = u_{|\Gamma_i}\,, \\ \\ \sigma_{i-1}\,\partial_n u_{|\Gamma_{i-1}} = \sigma_i\,\partial_n u_{|\Gamma_i}\,. \end{cases}$$

We handle the issue of harmonic data propogation in a family of annuli. We use a constraint analytic approximation scheme from [19], [25], by solving a bounded extremal problem there on both annulus $\Omega_2 \setminus \Omega_1 = R\mathbb{D} \setminus r\mathbb{D}$ and $\Omega_1 \setminus \Omega_0 = r\mathbb{D} \setminus \mathbb{D}$ to propogate the data onto the internal layer's boundary $\Gamma_0 = \mathbb{T}$, for $R > r > 1$ ($\lambda\mathbb{D}$ is the disk of radius $\lambda > 1$). Once data are available on the inner boundary Γ_0, best meromorphic or rational approximation schemes are used in order to approximately locate $\{c_k\}$, as in [6], [8].
This paper is outlined as follows: section 2 deals with the harmonic approximation which maps the Cauchy data from the scalp to the surface of the brain (cortical mapping step). In Section 3, we describe a best L^2 meromorphic approximation scheme as a tool for recovering the electrical sources as well as their intensities.
Section 4 is devoted to numerical results and we end up with some comments.

2 Harmonic approximation on an annulus: cortical mapping

The first step of the identification process consists in extending the available over-determined data on the exterior boundary $\partial\Omega$ of the domain Ω to the internal layer $\partial\Omega_0$. To this end, we choose to solve two consecutive bounded extremal problems on the annuli Ω_2 and Ω_1 (see also [2] where a boundary elements method is used). We describe explicit formulas allowing to extend the data on the innermost layer.
Let $G = r\mathbb{D} \setminus \overline{s\mathbb{D}} \subset \mathbb{C} \simeq \mathbb{R}^2$, $0 < s < r$, be an annulus, its boundary $\partial G = s\mathbb{T} \cup r\mathbb{T}$ being equipped with the Lebesgue measure normalized so that the circles $r\mathbb{T}$ and $s\mathbb{T}$ each have

unit measure.
We consider the intermediary problem: given two functions u_b and Φ defined on $r\mathbb{T}$, or a number of their pointwise measurements, with $\Phi \not\equiv 0$ such that

$$\begin{cases} \Delta u & = 0 \quad \text{in } G \\ u & = u_b \quad \text{on } r\mathbb{T} \\ \partial_n u & = \Phi \quad \text{on } r\mathbb{T} \end{cases} \tag{1}$$

where ∂_n stands for the partial derivative w.r.t. the outer normal unit vector to ∂G. We refer to [20] for the following properties of the solution.
Let $\Phi \in L^2(r\mathbb{T})$, then $u_{|\partial G} \in W^{1,2}(\partial G)$. Moreover, there exists a locally single-valued function v harmonic in G such that $\partial_\theta v = \partial_n u$ on ∂G, where ∂_θ stands for the tangential partial derivative on ∂G, from the Cauchy–Riemann equations.
Note that, from (1), the conjugate function v is given on $r\mathbb{T}$ up to a constant by:

$$v_{|r\mathbb{T}}(re^{i\theta}) = \int_{\theta_0}^{\theta} \Phi(re^{i\tau})\, d\tau\,,$$

for an arbitrary $re^{i\theta_0} \in r\mathbb{T}$, and that $v_{|\partial G} \in W^{1,2}(\partial G)$. Thus, $\tilde{f} = u + i\,v$ is analytic (and multiply-valued) in G; it is given on $r\mathbb{T}$ by

$$\tilde{f}(re^{i\theta}) = u_b(re^{i\theta}) + i \int_{\theta_0}^{\theta} \Phi(re^{i\tau})\, d\tau\,. \tag{2}$$

However, since the annulus is not simply-connected, it may not be possible to define $\tilde{f}$ globally in G as a single-valued function. In this case, there is a single-valued analytic function f defined on G such that

$$f(z) = \tilde{f}(z) - \frac{c}{2\pi} \log z\,, \tag{3}$$

hence $u(z) = \operatorname{Re} f(z) + \frac{c}{2\pi} \log|z|$, where

$$c = \int_0^{2\pi} \Phi(re^{i\theta})\, d\theta\,. \tag{4}$$

Introduce $H^2(\lambda\mathbb{D})$ the Hardy space of analytic functions in the disk $\lambda\mathbb{D}$ whose L^2 norms on circles centered at 0 of radius smaller than λ are bounded ([13], [16], [18], [24]). Let $\bar{H}_0^2(\lambda\mathbb{D})$ be its orthogonal complement in $L^2(\lambda\mathbb{T})$, which consists in functions analytic outside $\mathbb{D}$, vanishing at ∞, and whose L^2 norms on circles centered at 0 of radius greater than λ are bounded. From the above mentionned regularity properties, the function f is bounded in $L^2(r\mathbb{T})$, and we then find an extension of f in the so-called Hardy space denoted by $H^2(G) = H^2(r\mathbb{D}) \oplus \bar{H}_0^2(s\mathbb{D})$ defined in [24]. It is also possible to define the Hardy spaces $H^2(\partial G)$, as the closure in $L^2(\partial G)$ of the set R_G of rational functions whose poles lie in $\mathbb{C} \setminus \overline{G}$. The spaces $H^2(G)$ and $H^2(\partial G)$ are then isomorphic in a natural way, and so we identify the two spaces, see [4], [12].
For $m \geq 1$, introduce $H^{m,2}(G) = H^2(G) \cap W^{m,2}(\partial G)$, the Hardy-Sobolev space of the annulus G, with the $W^{m,2}(\partial G)$ norm:

$$\|g\|_{H^{m,2}(G)} = \|g\|_{W^{n,2}(\partial G)} = \sum_{p\in\mathbb{Z}} |g_p|^2 \left[w_{m,p} r^{2p} + \mu_{m,p} s^{2p}\right],$$

for functions $g \in H^{m,2}(G)$, $g(z) = \sum_{p\in\mathbb{Z}} g_p\, z^p$, $z \in G$, and

$$\begin{cases} w_{m,p} = 1 + p^2 r^{-2} + \cdots + p^2\,(p-1)^2 \cdots (p-m+1)^2 r^{-2p}\,, \\ \mu_{m,p} = 1 + p^2\, s^{-2} + \cdots + p^2\,(p-1)^2 \cdots (p-m+1)^2\, s^{-2p}\,. \end{cases} \tag{5}$$

For consistency of notation, we shall also write $H^{0,2}(G)$ for $H^2(G)$ and $W^{0,2}(\partial G)$ for $L^2(\partial G)$.
For fixed m, the family

$$e_{m,n} := \frac{z^n}{\sqrt{w_{m,n} r^{2n} + \mu_{m,n} s^{2n}}}, \quad z \in G, \quad n \in \mathbb{Z} \tag{6}$$

is an orthogonal basis of $H^{m,2}(G)$.
Let $m \geq 1$. We solve the minimization problem (bounded extremal problem):

$$(BEP) \begin{cases} \text{Given} \quad f \in W^{m,2}(r\mathbb{T}) \setminus H^{m,2}(G) \quad \text{and} \quad M > 0, \\ \text{find a function} \quad g \in H^{m,2}(G) \quad \text{such that} \quad \|g\|_{W^{m,2}(s\mathbb{T})} \leq M \quad \text{and} \\ \|f - g\|_{W^{m,2}(r\mathbb{T})} = \inf\{\|f - \psi\|_{W^{m,2}(r\mathbb{T})} :\ \psi \in H^{m,2}(G)\,,\ \|\psi\|_{W^{m,2}(s\mathbb{T})} \leq M\}. \end{cases}$$

From [12], the (BEP) problem admits a unique solution when $m = 0$ which can be obtained by solving a spectral equation for the Toeplitz operator $\mathcal{T}$ with symbol $\chi_{s\mathbb{T}}$, the characteristic function of the component $s\mathbb{T}$:

$$g = (Id + \lambda\,\mathcal{T})^{-1}\, P_{H^2(G)}\, f\,, \tag{7}$$

for the unique $\lambda > -1$ such that

$$\|g\|_{L^2(s\mathbb{T})} = M\,, \tag{8}$$

where

$$\begin{aligned} \mathcal{T} \quad : \quad & H^2(G) \to H^2(G) \\ & g \mapsto P_{H^2(G)}\, \chi_{s\mathbb{T}}\, g\,, \end{aligned}$$

and $P_{H^2(G)} : L^2(\partial G) \to H^2(G)$ is the orthogonal projection. Analogous results are still available for $m \geq 1$, that we establish now.

Proposition 1 *Let $m \geq 0$ and $f \in W^{m,2}(r\mathbb{T}) \setminus H^{m,2}(G)$:*

$$f(re^{i\theta}) = \sum_{n\in\mathbb{Z}} a_n r^n\, e^{in\theta} \quad a.\ e. \quad on \quad r\mathbb{T}\,,$$

then, problem (BEP) admits a unique solution:

$$g_m(z) = \sum_{n\in\mathbb{Z}} \frac{w_{m,n} r^{2n} a_n}{w_{m,n} r^{2n} + (1+\lambda)\mu_{m,n} s^{2n}}\ z^n, \tag{9}$$

for $z \in G$, where $\lambda > -1$ is the unique constant such that

$$\sum_{n\in\mathbb{Z}} \frac{w_{m,n}^2\, |a_n|^2\, s^{2n} r^{4n}}{\left(w_{m,n} r^{2n} + (1+\lambda)\mu_{m,n} s^{2n}\right)^2} = M^2\,.$$

For the proof, we need the following lemma.

Lemma 2 *Let $h \in W^{m,2}(\partial G)$ be given by*

$$h(re^{i\theta}) = \sum_{n\in\mathbb{Z}} a_n r^n \; e^{in\theta} \quad a.\ e. \quad on \quad r\mathbb{T}\,,$$

$$and \quad h(se^{i\theta}) = \sum_{n\in\mathbb{Z}} b_n s^n e^{in\theta} \quad a.\ e. \quad on \quad s\mathbb{T}\,.$$

Then

$$P_{H^{m,2}(G)}h(z) = \sum_{n\in\mathbb{Z}} \frac{w_{m,n}r^{2n}a_n + \mu_{m,n}s^{2n}b_n}{l_{m,n}} \; z^n, \quad z \in G, \tag{10}$$

where $l_{m,n} = w_{m,n}r^{2n} + \mu_{m,n}s^{2n}$.

Proof : Let $h \in W^{m,2}(\partial G)$:

$$P_{H^{m,2}(G)}h(z) = \sum_{n\in\mathbb{Z}} c_n \; z^n\,, \quad \{c_n\} \subset \mathbb{C}\,.$$

The family $\left\{ e_{m,n}(z) = \dfrac{z^m}{\sqrt{l_{m,n}}} \right\}$ being a total basis of the Hilbert space $H^{m,2}(G)$, therefore $h - P_{H^{m,2}(G)}h \perp e_{m,n}$, for all $n \in \mathbb{Z}$.
On the other hand,

$$h - P_{H^{m,2}(G)}h(re^{i\theta}) = \sum_{n\in\mathbb{Z}} (a_n - c_n)\, r^n \; e^{in\theta} \quad \text{and} \quad h - P_{H^{m,2}(G)}h(se^{i\theta}) = \sum_{n\in\mathbb{Z}} (b_n - c_n) \; s^n e^{in\theta}\,,$$

then

$$\begin{aligned} 0 &= \, <h - P_{H^{m,2}(G)}h, e_m(z)>_{W^{m,2}(\partial G)} \\ &= \, <h - P_{H^2(G)}h, e_{m,n}(z)>_{W^{m,2}(r\mathbb{T})} + <h - P_{H^2(G)}h, e_{m,n}(z)>_{W^{m,2}(s\mathbb{T})} \\ &= \frac{1}{\sqrt{l_{m,n}}} \left[w_{m,n}\,(a_n - c_n)\, r^{2n} + \mu_{m,n}\,(b_n - c_n)\, s^{2n} \right]. \end{aligned}$$

The result follows. ■

Proof of Proposition 1: Thanks to Ref. [12], the solution g_m is given by

$$(1 + \lambda\, \mathcal{T}) \; g = P_{H^{m,2}(G)}\, f.$$

Let $h = f$ on $r\mathbb{T}$ and $h = 0$ on $s\mathbb{T}$. From Lemma 2, for all $z \in G$

$$P_{H^{m,2}(G)}h(z) = \sum_{n\in\mathbb{Z}} \frac{w_{m,n}r^{2n}a_n}{l_{m,n}} \; z^n\,.$$

On the other hand, if $g_m(z) = \sum_{n\in\mathbb{Z}} c_n z^n$, $z \in G$, then from Lemma 2, we have

$$\mathcal{T} g(z) = \sum_{n\in\mathbb{Z}} \frac{\mu_{m,n}s^{2n}c_n}{l_{m,n}} \; z^n\,.$$

Since

$$c_n + \frac{\lambda \mu_{m,n} s^{2n} c_n}{l_{m,n}} = \frac{w_{m,n} r^{2n} a_n}{l_{m,n}} \, ,$$

we obtain

$$c_n = \frac{w_{m,n} r^{2n} a_n}{w_{m,n} r^{2n} + (1+\lambda)\mu_{m,n} s^{2n}} \, .$$

From [12], the parameter $\lambda > -1$ is determined by the condition

$$\| \, g_m \, \|_{W^{m,2}(s\mathbb{T})} = M \, .$$

Therefore,

$$M^2 = \left\| \sum_{n\in\mathbb{Z}} \frac{w_{m,n} r^{2n} \left(a_n - b_n\right)}{w_{m,n} r^{2n} + (1+\lambda)\mu_{m,n} s^{2n}} \; z^n \right\|^2_{W^{m,2}(s\mathbb{T})} = \sum_{n\in\mathbb{Z}} \frac{w^2_{m,n} r^{4n} \left|a_n\right|^2 s^{2n}}{\left(w_{m,n} r^{2n} + (1+\lambda)\mu_{m,n} s^{2n}\right)^2} .$$

■

Once $g = g_m$ is computed, one gets Cauchy data on $s\mathbb{T}$ by taking $u = \mathrm{Re}\, g$ and $\partial_n u = \partial_\theta \, \mathrm{Im}\, g$.

3 Pointwise sources identification process: best rational approximation

Once the overdetermined computed boundary values (electric potential and current flux), available on the outer boundary $\partial\Omega$, are propagated to the inner boundary $\mathbb{T} = \partial\Omega_0$ by the techniques from the previous section, we are ready to handle the problem of recovering the source term from the data

$$u_{|\partial\Omega_0} = u_0 \quad \text{ and } \partial_n u_{|\partial\Omega_0} = \Phi_0 \tag{11}$$

of a solution u to (assuming that $\sigma_0 = 1$)

$$-\Delta u = F \;\text{ in } \Omega_0.$$

There exists a function $\mathcal{A}$ analytic in the unit disk $\mathbb{D}$ such that, if we define:

$$f(z) = \mathcal{A}(z) - \sum_{k=1}^{N} \frac{p_k}{2\pi\left(z - c_k\right)} \, , \tag{12}$$

then

$$u(z) = \mathrm{Re}(f(z)) \, , \; z \in \mathbb{D} \setminus \{c_k\} \, .$$

Moreover, f is given on $\mathbb{T}$ (up to an additive constant) by

$$f(z) = u_0(z) + i \int_{\xi_0}^{z} \Phi_0(\xi) ds(\xi), \tag{13}$$

for every $z \in \mathbb{T}$, where $\xi_0 \in \mathbb{T}$ is fixed once and for all.
As in [6] The inverse source problem in $\mathbb{D}$ can then be formulated as that of locating the

singularities of the function f given by (12) from its values on the boundary $\mathbb{T}$, available by (13) from the Cauchy data u_0, Φ_0, see (11).

In the case where there are no sources in $\mathbb{D}$, i.e $m = 0$, the function u is the real part of the analytic function $\mathcal{A} = f$ which is known on $\mathbb{T}$. This provides the basis of a test to establish the presence of sources in the domain. Indeed, some distance on $\mathbb{T}$ between f and the set of boundary values of analytic functions in $\mathbb{D}$ can be constructively computed, at least for quadratic and uniform norms ([1], [7], [9], [23]).

If this distance is strictly positive (not too small, in practice) f does necessarily possess singularities in $\mathbb{D}$ and we are led to the determination of the number of sources, their positions and their moments. This issue can be approached through the study of the behaviour of the poles of rational or meromorphic approximants on the unit circle $\mathbb{T}$.

Best L^2 meromorphic approximation on the boundary is used for the recovery of the singularities in $\mathbb{D}$ from available data on $\mathbb{T}$, because equation (12) defines a function $f - \mathcal{A}$ that belongs to $\bar{H}^2_0$.

Indeed, let $R_n \subset \bar{H}^2_0$ be the set of strictly proper rational functions with at most n poles in $\mathbb{D}$, none on $\mathbb{T}$. For functions $h \in L^2(\mathbb{T})$, the issue of finding the best L^2 meromorphic approximation to h with less than n poles in $\mathbb{D}$ amounts to get $\psi_n \in H^2 + R_n$ such that

$$\|h - \psi_n\|_{L^2(\mathbb{T})} = \min_{\psi \in H^2 + R_n} \|h - \psi\|_{L^2(\mathbb{T})} .$$

It can be expressed in terms of the best rational approximation as follows. Let P_{H^2} denote the orthogonal projection from $L^2(\mathbb{T})$ onto H^2 and $P_{\bar{H}^2_0}$ denote the one from $L^2(\mathbb{T})$ onto $\bar{H}^2_0$. Put $h_- = P_{\bar{H}^2_0} h$.

Now, being given a function $h_- \in \bar{H}^2_0$, one can look for its best L^2 rational approximation of degree less than n in $\bar{H}^2_0$: find polynomials π_n, q_n, with $deg(q_n) \leq n$, such that $\pi_n / q_n \in \bar{H}^2_0$ (this forces $deg(\pi_n) < deg(q_n)$), that minimizes $\|h_- - p/q\|_{L^2(\mathbb{T})}$ among such functions, see Ref. [9]:

$$\left\| h_- - \frac{\pi_n}{q_n} \right\|_{L^2(\mathbb{T})} = \min_{deg(p) < deg(q) = n} \left\| h_- - \frac{p}{q} \right\|_{L^2(\mathbb{T})} .$$

It then holds that $\psi_n = P_{H^2} h + \pi_n / q_n$.

The existence of such a minimum is established in [5]. In fact, q_n will have degree n, except if h_- is already a rational $\bar{H}^2_0$ function of degree strictly less than n (normality property, [9]).

As to uniqueness of the best rational approximant, it is known to be true for an open and dense subset of $\bar{H}^2_0$. Whenever h_- is already a rational function of degree n in $\bar{H}^2$, the unique minimum at order n is h itself, a consequence of the consistency property from [10] (which is to the effect that h_- is the unique critical point of the criterion).

Concerning constructive aspects, algorithms to generate local minima can be obtained using Schur parametrization which induces a map on the manifold consisting in rational $\bar{H}^2_0$ functions of given degree and of uniform norm equal to 1 on $\mathbb{T}$. Computing the gradient and the Hessian of the criterion with this parametrization produces an efficient resolution scheme ([22]).

When the function h to be approximated already possesses N poles as singularities in $\mathbb{D}$, which is the case in the present situation, that is when the anti-analytic part $h_- = P_{\bar{H}^2_0} h$ belongs to R_N, the best rational approximant π_n / q_n of degree n will coincide with h_- and provide 0 as error value, as soon as $n \geq N$. This occurs in the case of dipolar sources since then $f - \mathcal{A} \in R_N$. This allows us to recover the number N of poles, together with

their locations and residues. Indeed, if $n > N$, π_n/q_n possesses n poles, among which $n - N$ have a residue equal to zero.
Observe that for more general cases of functions with branched singularities, some results concerning the convergence of the poles of the rational approximants to these singularities are available, that are discussed in [8].

4 Numerical trials

We consider the $2D$ configuration of the so-called spherical (here circular) model of the head, where the domain is a disk Ω made up of 3 disjoint homogeneous connected circular/annular layers Ω_i, $i = 0, 1, 2$, (corresponding to brain, skull and scalp) of radii $s = 1$, $r = 1.0574$ and $R = 1.1494$, delimiting surfaces with conductivities $\sigma_0 = 1$, $\sigma_1 = 0.0125$ and $\sigma_2 = 1$, from inside towards outside, see [2].
The steady state electric potential is a solution to problem (NP), where σ is constant in each layer (in particular $\sigma = \sigma_2$ on $\partial\Omega = R\mathbb{T} \subset \partial\Omega_2$) and $c_k \in \mathbb{D}$, $p_k \in \mathbb{R}^2$.
We first solve the direct Neumann problem (NP), for a situation where $N = 4$ pointwise dipoles are modelled. The domain Ω is meshed using P1 finite elements. The boundary $\partial\Omega$ is discretized with 512 points at which we compute the potential u associated with the boundary current $\Phi = 0$. This example is illustrated in figure 2, where the domain is shown with level curves for the numerical solution u to the direct problem.
Now, we extend the function f, given on $\partial\Omega = R\mathbb{T}$ by (2), (3), (4), to each inner layer,

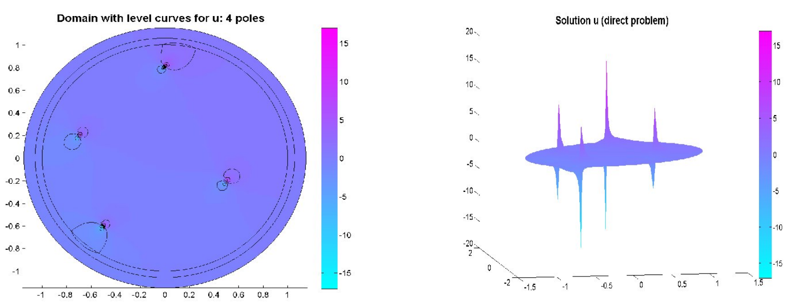

Figure 2: Domain $\Omega = R\mathbb{D}$ with level curves for u; solution to Neumann direct problem.

first $r\mathbb{T}$, then $\mathbb{T}$, by using the technics of harmonic approximation described in section 2 and Proposition 1, with $m = 2$. Let us note that λ plays the role of a Lagrange multiplier which makes implicit the dependence of the solution on the bound M; this can be adjusted by dichotomy. Indeed, the error $e(\lambda) := \|f - g(\lambda)\|_{L^2(s\mathbb{T})}$ smoothly decreases to 0 as $\lambda \to -1$, and $\lambda \to M(\lambda)$ is C^1, bijective and decreasing on $]-1, +\infty[\to]0, +\infty[$, see Ref. [26]. The determination of the actual bound M is achieved by functional minimization (Ref. [19]). The numerical tests are produced by the software $AEAD$[1] which runs the procedure described in section 2 in order to compute the best harmonic $H^{2,2}$ approximants to the function f. Figure 3 shows the reconstruction data on the innermost layer. The gap, in L^2-norm between the reconstructed data and the actual one is equal to $2, 23 \times 10^{-2}$. Notice that, due to the small value of the specific conductivity σ_1, information is lost on the innermost layer.

[1] Analytic Extension on Annular Domain: developed at LAMSIN and INRIA (APICS team) using Matlab 7.

As a second step, once the Cauchy data $u = \operatorname{Re} g$ and $\partial_n u = \partial_\theta \operatorname{Im} g$ are available on the

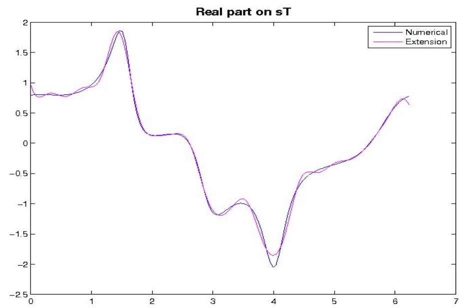

Figure 3: Real part of analytic extension g of boundary solution (actual/computed u) on the inner layer $\partial\Omega_0 = \mathbb{T}$.

inner boundary $\mathbb{T}$, meromorphic or rational approximation schemes, described in section 3, are used in order to approximately locate $\{c_k\}$, as in Ref. [6]. They are run by the software *Rarl2*[2] which uses the procedure described in section 3, relying on Schur parameters, in order to compute the best rational $\bar{H}_0^2$ approximants π_n/q_n of given degree n to the function $h_- = P_{\bar{H}_0^2} g = g - \mathcal{A} \in \bar{H}_0^2$, see (12).
In the following experiments, the original sources are represented by (small) $*$, the recovered poles by the $\circ$ (which are often superposed on the $*$, which makes them look like single black circles), while the lines are the moments for both functions (symbolized by a dot when equal to 0 or sufficiently small). Figure 4 shows how the relative error on the moments, the absolute error of the sources location, and the relative error on the extended data on $\partial\Omega_0 = \mathbb{T}$ increase as the $N = 1$ dipolar current source gets closer to the boundary $\mathbb{T}$. The source is a unitary current dipole with moment $p_1 = (1+i)/\sqrt{2}$ and located at a distance from the origine equals to 49%, 78%, 88%, 93%, or 97% of the radius of the innermost layer ($s = 1$, at the moment).
Figure 5 shows the evolution of the errors as the ratio σ_1/σ_0 increases, for the conductiv-

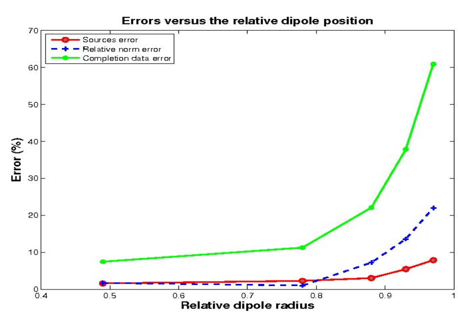

Figure 4: Errors w.r.t. dipole position.

ity σ_1 of the intermediate layer of our three layer model, and the two others that satisfy $\sigma_0 = \sigma_2$. Note that in the above case, we had $\sigma_1/\sigma_0 = 80$.
Figure 6 illustrates how the computation of the poles of best L^2 rational approximants

[2]Developed jointly at INRIA (Miaou team) and Ecole des Mines de Paris (CMA) using Matlab 6. This software also performs AAK (best uniform meromorphic) approximation, using singular value decompositions together with state space representations. This is used in order to initialize the present L^2 scheme.

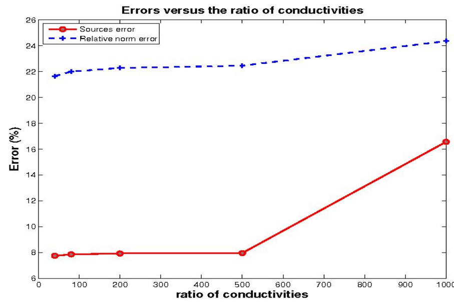

Figure 5: Errors w.r.t. ratio of conductivities between neighboring layers, for $N = 1$ dipole at radius .97.

of $g - \mathcal{A}$ on $\mathbb{T}$ allows to recover the dipoles centers.

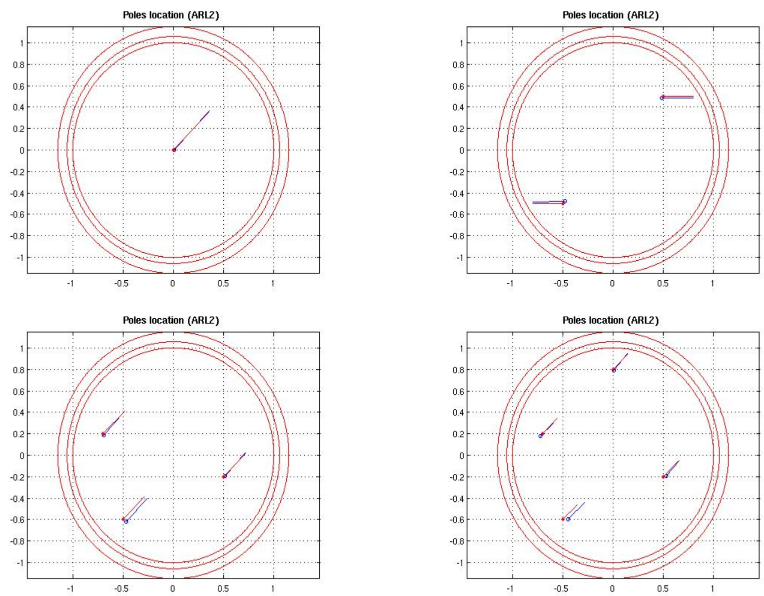

Figure 6: $N = 1, 2, 3, 4$.

We now consider the robustness properties of our identification processes. Noise is generated by a random variable whose uniform norm ranges from 1% to 20% of $\|f\|_{L^\infty(\partial\Omega)}$. Figure 8 shows the evaluation of the absolute error on the sources and the relative error for the moments.

5 Comments

This work shows that analytic and meromorphic approximation tools from complex and functional analysis provide a relevent and costless method in order to solve some 2D non-destructive control issues, and in particular dipolar sources recovery from boundary data, a problem arising in medical engineering (inverse EEG problem). See [6] where the detection of small inclusions is also handled by these techniques.

Generalization to 3D situations for the three sphere model as well as for more realistic

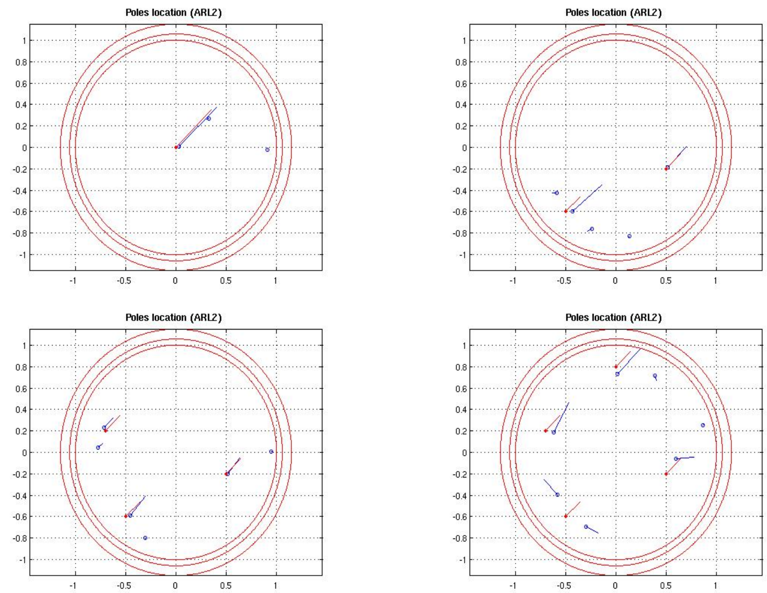

Figure 7: $N = 1, 2, 3, 4$ and $n = 3, 5, 6, 7$.

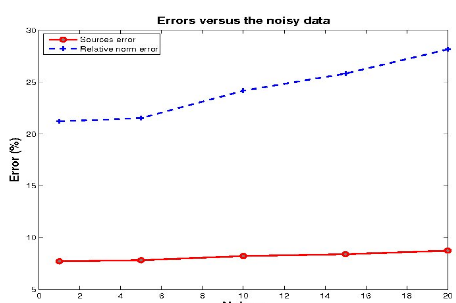

Figure 8: Errors with respect to the noise level for $N = 1$.

geometries is under consideration. Note that the main difficulty is to constructively solve the first cortical mapping step: even if gradients of harmonic functions are still analytic functions in 3D, one needs to compute the solution on a basis of spherical harmonic. This leads to a number of numerical difficulties, mainly in realistic medical situations where data are missing on a part of $\partial\Omega$ (see [3] and [21] where ellipsoidal geometries are considered). However, the second step of sources recovery is already performing well enough in 3D, by looking for meromorphic approximants on 2D sections of the inner domain Ω_0, see [8].

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