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Parameter identification for Laplace equation and approximation in Hardy classes

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Abstract — We consider the inverse problem of identifying a Robin coefficient on some part of the boundary of a smooth 2D domain from overdetermined data available on the other part of the boundary, for Laplace equation in the domain. Using tools from complex analysis and analytic functions theory, we provide a constructive and convergent identification scheme for this inverse problem, together with numerical experiments.

1. INTRODUCTION

Several inverse problems may need to be solved that some lacking data on a part of the boundary be recovered from overdetermined data on another part of the boundary. In this paper, the issue we are interested in is the recovery of a Robin coefficient from measurements performed on some part of the boundary. Such an issue arises for example in corrosion detection by electrical impedance tomgraphy. An effective non linear boundary condition that reduces the knowledge of corrosion effects to that of a function defined on the corroded boundary has been derived by using a multiscale analysis expansion by Santosa et al. [25]. In the simplest linear case, the parameter characterizing the damage caused by corrosion is a Robin exchange coefficient, the direct problem to be solved being a Laplace equation. Sticking to this simple model, our purpose here is to set up a numerical algorithm based on constructive approximation, using analytic functions tools. Alternative algorithms have been investigated by several authors, using quasi-reversibility or least squares approaches see [13, 15, 18, 20, 22].

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We first recall an identifiability result proved in [13], ensuring that the unknown Robin coefficient is uniquely determined in a proper class from the knowledge of the Neumann data (prescribed current flux) and of additional Dirichlet data (measured voltage potential), on a suitable proper part of the boundary.

We then approach constructively the issue of determining the unknown coefficient from the available boundary data. The point here is that the above problem amounts to that of recovering an analytic function from its trace on a proper subset of the boundary of its analyticity domain. In order to ensure robustness properties of the recovery procedure, we are led to turn this interpolation issue into an approximation one in Hardy classes. After conformally mapping the involved domain into the unit disk $\mathbb{D}$ of the complex plane, we handle these problems using complex analysis tools and analytic approximation results [2, 4, 11]. The main difficulty one has to face in processing such an approach is that related to instabilities characterizing data completion problems of this kind, i. e. the solution of Cauchy problems for elliptic operators, which are well known — since Hadamard around 1920 — to be severely ill-posed. As a matter of fact, any given data may be fit as closely as desired on the prescribed part of the boundary, provided that hectic behaviours are tolerated on the remaining part of the boundary. Setting a bound on the data to be recovered, as proposed in the bounded extremal extension approach [4], can avoid the extended solution from blowing up away from the prescription part of the boundary. However, doing so provides us with nothing but an approximate extension that saturates the constraint, and is therefore arbitrary unless this constraint is close enough to be the proper bound. The reconstruction algorithm thus needs to tackle in a single movement both issues of determining the extended function *and* the right bound. To this end, a cross validation procedure is set up, some part of the available data being devoted to it. On the other hand, extending the function would be hardly enough since, our purpose being to recover a Robin coefficient from extended data, accuracy is not only required on the function itself, but on its normal derivative as well. This compels us to consider higher order methods, based on the same extension process applied to the derivatives of the function to be extended.

The kernel of the whole numerical process makes use of the approximation software *Hyperion* — developped at INRIA — in order to compute the desired coefficient. This approach extends for the reconstruction of lacking data in cracks recovery: in such a case, the data to recover are not harmonic, and meromorphic extension is therefore used instead of the analytic one [5]. The issues remain essentially the same, especially that related to the cross validation stabilzation procedure.

The outline of the paper is the following. In Section 2, notations are set and identifiability results are recalled; further regularity properties are also checked. Section 3 is devoted to the presentation of the bounded extremal problems in Hardy spaces, and a thorough study of their solutions continuity with respect to the data and bounds is then conducted in Section 4. Section 5 is devoted to the numerical methods and results. Appropriate recovery algorithms are first described, and their robustness is established according to the continuity properties of Section 4. A bunch of numerical experiments, highlighting accuracy and robustness of the methods, eventually closes the issue.

2. STATEMENT OF THE PROBLEM, IDENTIFIABILITY, AND SMOOTHNESS PROPERTIES

2.1. Notations

Let D be a bounded domain of $\mathbb{R}^2$ (or $\mathbb{C}$) with Jordan closed boundary T.

The k^{th} derivative with respect to the ambient real or complex variable of some function ϕ will be written as $\phi^{(k)}$, $k \geq 0$, with the usual convention $\phi' = \phi^{(1)}$.

For $n \geq 0$ and $0 \leq \beta \leq 1$, we note $\mathcal{C}^{n,\beta}(\overline{D})$ for the space of functions f on $\overline{D}$ whose derivatives $f^{(k)}$ are of Hölder class with order β for $0 \leq k \leq n$. We put $\mathcal{C}^{n,0} = \mathcal{C}^n$.

Also, D is said to have a $\mathcal{C}^{n,\beta}$ boundary if T admits a $\mathcal{C}^{n,\beta}$ parametrization [23].

The Lebesgue measure on T will be noted μ in general. However, for $T = \mathbb{T}$, the unit circle, we shall write $d\mu = d\theta$.

For any connected open subset $E \subset T$, let χ_E be the characteristic function of E; traces on E of both functions and spaces will be indicated by $_{|E}$.

The Hilbert space $L^2(E)$ of square summable functions with respect to μ on E is equipped with the classical norm and inner product, that we simply write $\| \ \|_E$ and $(\ , \ )_E$, respectively.

For $s > 0$, the Sobolev Hilbert space $W^{s,2}(D)$ equipped with $\| \ \|_{s,D}$ is classically defined, see e.g. [14]. Whenever $n \in \mathbb{N}$, the norm on $W^{n,2}(E)$ is the usual one:

$$\|f\|^2_{n,E} = \sum_{k=0}^{n} \|f^{(k)}\|^2_E,$$

(here, $f^{(k)}$ is the k^{th} derivative of h with respect to the complex variable z). Note, as usual, that $W^{0,2}(E) = L^2(E)$ while $\| \ \|_E = \| \ \|_{0,E}$.

Whenever $E \subsetneq T$, $\mathcal{C}^n_0(E)$ is the subset of $\mathcal{C}^n(E)$ consisting of functions f that vanishes at ∂E together with their derivatives $f^{(k)}$, $k = 0, \cdots, n-1$. The set $W^{n,2}_0(E)$ stands for the $W^{n,2}(E)$ closure of $\mathcal{C}^n_0(E)$. We also use the Sobolev Banach space $W^{n,\infty}(E)$ of functions belonging to $L^\infty(E)$ together with their derivatives up to order n.

As to Hardy spaces of the unit disk $\mathbb{D}$, $H^2 = H^2(\mathbb{D})$ can be viewed as the space of functions analytic in $\mathbb{D}$ that are square-summable on circles of radius less than 1 centered at 0. It is a consequence of this definition that traces on the unit circle $\mathbb{T}$ of H^2 functions belong to $L^2(\mathbb{T})$, and in this sense $H^2 \subset L^2(\mathbb{T})$ inherits the inner product and can be described as the subspace of $L^2(\mathbb{T})$ consisting in functions whose Fourier coefficients of negative order do vanish:

$$H^2 = \Big\{ g(z) = \sum_{p \geq 0} a_p z^p \mid \sum_{p \geq 0} |a_p|^2 < \infty \Big\}, \tag{1}$$

see e.g. [16, 17, 24] for definitions and properties of Hardy spaces. A further equivalent definition of H^2 is that it is the space of complex valued functions whose real and imaginary parts are both harmonic in $\mathbb{D}$ and such that their L^2 norm on circles of radius $r < 1$ remains bounded as $r \to 1$.

Note that H^2 strictly contains the functions that are analytic and uniformly bounded in $\mathbb{D}$.

We finally let $\mathcal{H}^{n,2}$ be the Hardy—Sobolev Hilbert space:

$$\mathcal{H}^{n,2} = \{g \in H^2 \mid g^{(k)} \in H^2,\ 0 \le k \le n\} = H^2 \cap W^{n,2}(\mathbb{T}), \tag{2}$$

(here, $g^{(k)}$ is the k^{th} derivative of g with respect to z in $\mathbb{D}$) equipped with the norm $\| \ \|_{n,\mathbb{T}}$; of course $\mathcal{H}^{0,2} = H^2$.

2.2. The 2D Robin inverse problem

Let D be a simply connected bounded domain of $\mathbb{R}^2$ with boundary T, a $\mathcal{C}^{1,\beta}$ Jordan curve, for $\beta \in (0,1)$. Let then γ, K be two nonempty open subsets of T, satisfying $T = \overline{\gamma} \cup \overline{K}$.

We address the following inverse problem:

> Being given a prescribed flux $\phi \not\equiv 0$ together with measurements u_m on K, find a function φ on γ such that the solution u of
>
> $$\begin{gathered} \Delta u = 0 \quad \text{in} \quad D, \\ \frac{\partial u}{\partial n} = \phi \quad \text{on } K, \qquad \frac{\partial u}{\partial n} + \varphi u = 0 \quad \text{on } \gamma, \end{gathered} \tag{PR}$$
>
> also satisfies $u_{|K} = u_m$.

In the sequel, we assume that *both the measurement part $K \subset T$ and the "Robin" part $\gamma = T \setminus K$ have positive Lebesgue measure and finitely many components*, the simplest case being the one where K is an arc of T.

Remark. The present work is still valid if measurements u_m are available on some subset of K only, provided it also has positive Lebesgue measure, or (at least partially) if the Neumann boundary condition was replaced there by a Dirichlet one. However, for sake of simplicity, we shall stick here to the above case.

2.3. Identifiability

That the above inverse problem is well-posed (meaning that its solution is unique) as soon as the additional measurements are available on any set K of positive measure, is a result proved in [13] for classes of continuous Robin coefficients, and recalled herafter in Theorem 1. Intending to work out higher order methods, we shall need however that Robin coefficients hold more regularity, which leads us to restrict somewhat the class of admissible Robin coefficients used in [13]. Let $c, c' > 0$, and $\mathcal{K}$ be a nonempty connected subset of γ the boundary of which does not intersect that of γ. Define then:

$$\Phi^n_{ad} = \{\varphi \in \mathcal{C}^n_0(\overline{\gamma}) \mid |\varphi^{(k)}| \le c',\ 0 \le k \le n,\ \text{and}\ \varphi \ge c\chi_{\mathcal{K}}\}. \tag{3}$$

Recall that $\mathcal{C}^n_0$ is the set of n-times differentiable functions, that vanish on the boundary as well as their $(n-1)$ first derivatives. The identifiability result can then read as follows:

Theorem 1 [13, Theorem 1, Lemma 1]. *Let $\phi \in L^2(K)$ and for $i = 1, 2$, let $\varphi_i \in \Phi^0_{ad}$. Let $u_i \in \mathcal{C}^0(\overline{D})$ be the unique solution of* (PR) *associated to $\varphi = \varphi_i$. It holds that, if $u_{1|_K} = u_{2|_K}$, then $\varphi_1 = \varphi_2$ on $\overline{\gamma}$.*

2.4. More regularity

More regularity on the solution of the Robin problem (PR), provided the coefficient φ holds itself enough regularity, is needed in view of working out higher order methods, which means methods based on analytic extensions of the derivatives, not only of the prescribed data function. The following result then holds:

Theorem 2. *If $\phi \in W^{n,2}_0(K)$ and $\varphi \in \Phi^n_{ad}$, then the solution u_φ to* (PR) *belongs to $W^{n+3/2,2}(D) \subset \mathcal{C}^{n,1/2}(\overline{D})$ and there exist some constant $\gamma_n > 0$ such that:*

$$\forall \varphi \in \Phi^n_{ad}, \quad |u^{(k)}_\varphi(x) - u^{(k)}_\varphi(y)| \le \gamma_n |x-y|^{1/2} \quad \forall x, y \in \overline{D} \quad \forall k = 0, \ldots, n. \tag{4}$$

The proof makes use of a bootstrap technique, supported by the precise knowledge of the behaviour of the solution u_φ of the Robin problem (PR) with respect to φ provided by the following lemma:

Lemma 1 [13]. *Let ϕ be a non negative flux, and u_φ be the solution of problem* (PR) *associated to φ. Therefore:*

i) Positivity. $\forall \varphi \in \Phi^0_{ad}$, there exists a constant $\varrho > 0$ such that $u_\varphi > \varrho$ in $\overline{D}$.

ii) Monotonicity. Let $\varphi_1, \varphi_2 \in \Phi^0_{ad}$, such that $\varphi_1 \ge \varphi_2$ in $\overline{\gamma}$. Then $u_{\varphi_1} \le u_{\varphi_2}$ in $\overline{D}$.

iii) Control. Let c and c' be the two positive constants of Φ^0_{ad}. Let us denote by u_c the solution of (PR) *for $\varphi = c$ on $\mathcal{K}$ and $\varphi = 0$ on $\gamma \setminus \mathcal{K}$ and $u_{c'}$ the one for $\varphi = c'$ on γ. Then, $\forall\ \varphi \in \Phi^0_{ad}$, $0 < u_{c'} \le u_\varphi \le u_c$.*

Proof of Theorem 2. Let u_φ solve to (PR), and introduce:

$$\phi_2 = \begin{cases} \phi & \text{on } K, \\ -\varphi u_\varphi & \text{on } \gamma. \end{cases}$$

Let now w_φ solve the problem:

$$\begin{gathered} \Delta w_\varphi = 0 \qquad \text{in} \quad D, \\ \frac{\partial w_\varphi}{\partial n} = \phi_2 \quad \text{on } T, \qquad \int_T w_\varphi = 0. \end{gathered} \tag{5}$$

Setting $c_\varphi = \int_T u_\varphi > 0$, we derive immediately:

$$u_\varphi = w_\varphi + c_\varphi. \tag{6}$$

Let us first notice that, as a constant, c_φ can be controlled independently of φ by using the monotonicity result from Lemma 1: from $\varphi \in \Phi^0_{ad}$, we derive that $u_\varphi \le u_c$ a.e. Hence, $c_\varphi = \int_T |u_\varphi| \le \int_T |u_c| := \beta$, which puts an end to the issue.

Theorem 2 will now be proved as soon as we establish that w_φ belongs to $W^{n+3/2,2}(D)$, and that its norm in that space is controlled by some constant $C_n > 0$ not depending on φ, namely:

$$\exists C_n > 0; \quad \varphi \in \Phi^n_{ad} \text{ and } \phi \in W^{n,2}_0(K) \implies$$
$$w_\varphi \in W^{n+3/2,2}(D) \text{ and } \|w_\varphi\|_{n+3/2,D} \le C_n. \tag{7}$$

• Let us prove (7) for $n = 0$. From Theorem 1, we get $\varphi u_\varphi \in C^0(\gamma) \subset L^2(\gamma)$, whence $\phi_2 \in L^2(T)$. Thanks to the shift theorem [14], the solution w_φ of (5) belongs to $W^{3/2,2}(D)$ and there exists some constant $c_0 > 0$ such that:

$$\|w_\varphi\|_{3/2,D} \le c_0 \, \|\phi_2\|_{2,T} = c_0(\|\phi\|^2_K + \|u_\varphi \varphi\|^2_\gamma)^{1/2}.$$

Since $\varphi \in \Phi^0_{ad}$, it follows from Lemma 1 that: $\|u_\varphi \varphi\|_\gamma \le c' \|u_c\|_\gamma$, which yields:

$$\|w_\varphi\|_{3/2,D} \le C_0 = c_0(\|\phi\|^2_K + c'^2 \|u_c\|^2_\gamma)^{1/2}.$$

• Assume now that (7) holds for some $n \ge 0$, and let us prove it to hold for $(n+1)$.

Let $\phi \in W^{n+1,2}_0(K)$ and $\varphi \in \Phi^{n+1}_{ad} \subset \mathcal{C}^{n+1}_0(\overline{\gamma})$. We are claiming that $\phi_2 \in W^{n+1,2}(T)$. Indeed:

$$\|\phi_2\|^2_{n+1,T} = \|\phi\|^2_{n+1,K} + \|\varphi u_\varphi\|^2_{n+1,\gamma}. \tag{8}$$

Because $\Phi^{n+1}_{ad} \subset \Phi^n_{ad}$, we get from (7) that w_φ is bounded in $W^{n+3/2,2}(D)$ by a constant independant on φ. It comes out that so is its trace in $W^{n+1,2}(T)$, which means that some positive constant α_n exists, such that:

$$\|w_\varphi\|_{n+1,T} \le \alpha_n. \tag{9}$$

which, together with (6), yields:

$$\|u_\varphi\|_{n+1,T} \le \alpha_n + \beta. \tag{10}$$

From (8) and (10), we derive the existence of a constant $\beta_{n+1} > 0$ such that:

$$\|\phi_2\|^2_{n+1,T} \le \|\phi\|^2_{n+1,K} + \beta_{n+1},$$

which proves the claim. Hence, thanks to the shift theorem [14], w_φ belongs to $W^{n+5/2,2}(D)$ and some constant $c_{n+1} > 0$ exists, such that:

$$\forall \varphi \in \Phi^{n+1}_{ad}, \quad \|w_\varphi\|_{n+5/2,D} \le c_{n+1} \|\phi_2\|_{n+1,T}.$$

Thus:

$$\forall \varphi \in \Phi_{ad}^{n+1}, \quad \|w_\varphi\|_{n+5/2,D} \leq c_{n+1}(\|\phi\|_{n+1,K}^2 + \beta_{n+1})^{1/2}.$$

Property (7) is then established at order $n+1$, with $C_{n+1} = c_{n+1}(\|\phi\|_{n+1,K}^2 + \beta_{n+1})^{1/2}$.

Using now (6), we derive the existence of some constant $\delta_n > 0$ such that $\forall \varphi \in \Phi_{ad}^n$, $u_\varphi \in W^{n+3/2,2}(D)$ and $\|u_\varphi\|_{n+3/2,D} \leq \delta_n$. The Sobolev imbedding $W^{n+3/2,2}(D) \subset \mathcal{C}^{n,1/2}(\overline{D})$ with continuous injection [19], is the last argument needed to derive (4). $\square$

3. PARAMETER IDENTIFICATION AND APPROXIMATION IN HARDY CLASSES

Up to a conformal mapping, problem (PR) can be expressed in the unit disk $\mathbb{D}$, in order to work with Hardy classes in one of their classical framework [11]. Indeed, whenever D possesses a $C^{n,\beta}$ boundary for some $\beta \in (0,1)$, the Kellogg—Warschawski theorem [23, Theorems 3.5, 3.6] implies that there exists a conformal mapping from the unit disk $\mathbb{D}$ into D having a C^n extension to $\overline{\mathbb{D}}$.

In the remainder of this section, we shall thus assume that $D = \mathbb{D}$ *and* $T = \mathbb{T} = \overline{T \setminus K} \cup \overline{K}$.

The inverse problem (PR) we are concerned with now takes place in the unit disk where it can be constructively solved using interpolation / approximation results in the Hardy space H^2.

3.1. From harmonic functions to Hardy classes

Back to problem (PR), we assume now that $\phi \in L^2(K)$ and $\varphi \in \Phi_{ad}^0 \subset \mathcal{C}^0(\overline{T \setminus K})$. It then follows from Theorem 2 that $u \in \mathcal{C}^{0,1/2}(T)$.

3.1.1. Harmonic conjugation

From the knowledge of the flux $\phi \in L^2(K)$ and of the temperature $u_m \in \mathcal{C}^{0,1/2}(K)$ in system (PR), we can in principle build the trace on $K \subset \Gamma_N$ of a function analytic in D. This holds because u is harmonic in D and Cauchy—Riemann equations ensure that, if ω is a harmonic function in D satisfying

$$\frac{\partial \omega}{\partial \theta} = \frac{\partial u}{\partial n} \qquad \text{on} \quad T$$

(ω is the harmonic conjugate of u), then $f = u + i\,\omega$ is an analytic function in D. Thus, if we note $\int \phi d\theta$ for some primitive of ϕ on K, then the function:

$$f = u_m + i \int \phi \, d\theta \qquad \text{on} \quad K,$$

is actually the trace on K of a (unique) function g analytic in D: $f = g_{|K}$. Moreover, smoothness preserving properties of the harmonic conjugation operator, namely Privalov's theorem or the Carleson—Jacobs one, implies that g

also belongs to the Hölder class $C^{n,1/2}(T)$, see [1, 8, 17]. It thus belongs to Hardy classes and in particular to $\mathcal{H}^{n,2}$, see the definitions in Section 2.1.

Our aim is then to recover g in the whole of D, or at least on $\gamma = T \setminus \overline{K}$, from the knowledge of its trace f on K. Indeed, this function would solve for (PR) since

$$u = \operatorname{Re} g \qquad \text{in} \quad D,$$

and then

$$\varphi = -(\operatorname{Im} g)' / \operatorname{Re} g \qquad \text{on} \quad \gamma, \tag{11}$$

where the above equality should be properly understood (non tangential limits of the right hand side). Now, φ is the expected solution to (PR) on γ.

Moreover, we want the recovery procedure to be convergent in order to carry some stability and robustness properties. Our concern here is that, in practice, one may not know exactly u_m nor f on K: for example, pointwise values of u_m might only be available through experimental devices that necessarily induce noises and perturbations of the measurements. Also, the knowledge of f requires that of some "primitive of the flux", which is to be computed numerically.

A more realistic problem is thus to approximately and robustly recover g in the whole of D from the knowledge of a perturbation f_ε of its trace f on $K \subset T$, where ε is a small parameter that stands for a (deterministic) measure of the perturbation. However, classical analytic interpolation or extrapolation results from data on part of the boundary (Carleman integrals, for example) do not possess any stability properties on their own and are not suitable to ensure robustness with respect to perturbations, as shows the next proposition from [6]. This is the reason why we need a constrained approximation framework.

Now, despite these recovery / approximation questions should be approached in uniform norm (see e.g. [6, 7]), it is simpler to handle them in the Hardy space H^2 which possesses a Hilbertian structure. Also, for various reasons, robustness properties are easier to ensure there.

3.1.2. Basic properties of the Hardy space H^2

Recall first the following basic uniqueness result, in Hardy spaces.

Proposition 1 [16, 17, 24]. *Let K be an nonempty subset of $\mathbb{T}$ such that $\mu(K) > 0$ and let $g \in H^2$ verifying $g_{|_K} = 0$; then:*

$$g \equiv 0 \text{ on the whole unit disk } \mathbb{D}$$

Also, we have the density property:

Proposition 2 [6]. *Let $K \subset T$ such that K and $T \setminus K$ have positive Lebesgue measure.*

(i) $H^2_{|_K}$ is dense in $L^2(K)$.

(ii) Let $h \in L^2(K)$ and suppose (g_n) is a sequence of H^2 converging to h in $L^2(K)$. If h is not the trace of an H^2 function, then $\|g_n\|_{T\setminus K} \to \infty$.

In view of Proposition 2, we see that as soon as a perturbation is involved in the measurements on K and prevents the data from being truly the trace of an H^2 function by loose of its analyticity property, any H^2 interpolating procedure from K *only* will exhibit a wild behavior outside K.

A remedy for this unstable behavior is to state our recovery problem as a (best) constrained approximation issue which is a bounded generalization on subsets of T of classical (dual) extremal problems in H^2.

3.2. Best H^2 approximation

We now explain how to robustly recover an H^2 function on the whole $\overline{D}$ from approximate measures of its boundary values on K.

To give some more feeling about this approach, assume for a while that we want to solve the direct problem of finding the solution v to

$$\Delta v = 0 \qquad \text{in} \quad D,$$
$$v = u_m \quad on \quad T, \qquad \frac{\partial v}{\partial n} = \phi \quad \text{on } T.$$

Put $f = u_m + i \int \phi \, d\theta$ on T. The considerations of Section 3.1.1 are in this case also to the effect that f coincides with the trace on the whole boundary T of a (unique) function g_* analytic in D and bounded in L^2 norm (at least if u_m, $\phi \in L^2(T)$). Hence, $f = g_{*|_T}$, for the solution $g_* \in H^2$ of

$$\|f - g_*\|_T = \min_{g \in H^2} \|f - g\|_T,$$

which is given by the orthogonal (analytic) projection of f onto H^2. Classical (dual) extremal problems may thus be of constructive use to get v, since it holds that $v = \operatorname{Re} g_*$. Hence, best H^2 approximation on the whole boundary already provides a constructive way to solve for direct Dirichlet or Neumann problems. Our purpose now will be to show that this is still the case for inverse problems; moreover, partially overdetermined situations where u_m or ϕ are not available on the whole boundary may be handled as well, since H^2 extremal problems can be solved from data on part on the boundary only, if some rough information is given on the complementary part.

3.2.1. Bounded extremal problems (BEP)

These are as follows, in the case of the Hardy space H^2; observe that such issues do also make sense in general Hardy classes H^p, $1 \leq p \leq \infty$ were they have also been approached, see [4, 6, 7].

Given $h \in L^2(K)$ and $M > 0$,

$$\begin{gathered}\text{Find } g = g(h, M) \in H^2, \ \ \|g\|_{T\setminus K} \leq M \text{ such that} \\ \|h - g\|_K = \min_{\substack{\omega \in H^2 \\ \|\omega\|_{T\setminus K} \leq M}} \|h - \omega\|_K. \end{gathered} \tag{BEP}$$

Under the norm constraint, this problem becomes well-posed. Existence and uniqueness of solutions to (BEP) are established in [2, 4] as well as a constructive formula to compute g_0. Denote by $P_{H^2}\colon L^2(T) \to H^2$ the orthogonal (analytic) projection and by $\mathcal{T}$ the Toeplitz operator with symbol $\chi_{T \setminus K}$

$$\begin{aligned}\mathcal{T}\colon \quad H^2 &\to H^2\\ \omega &\mapsto P_{H^2}(\chi_{T\setminus K}\omega).\end{aligned}$$

On the Fourier basis, the operator $\mathcal{T}$ is a semi-infinite Toeplitz matrix: $\mathcal{T}_{k,l} = \mathcal{T}_{k-l}$, $k, l \geq 0$. Whenever $T \setminus K$ coincides with the arc $(e^{-i\theta_0}, e^{i\theta_0})$, $\mathcal{T}$ can be expressed as:

$$\mathcal{T}_{k,l} = \sin(k-l)\theta_0/\pi(k-l), \qquad \text{for} \quad k \neq l, \quad \mathcal{T}_{l,l} = \theta_0/\pi.$$

Hence,

$$g = g(\lambda) = (I + \lambda\mathcal{T})^{-1} P_{H^2}(\chi_K h), \tag{12}$$

for the unique value of the (Lagrange type) parameter $\lambda > -1$ such that $\|g\|_{T\setminus K} = M$, excepted if h already belongs to the approximant class ($h \in H^2_{|K}$, $\|h\|_{T\setminus K} \leq M$) in which case $g = h$ corresponds to $\lambda = -1$. Note that integral formulas of Carleman type are also available to represent $g(\lambda)$, see [4].

The behavior with respect to λ of the error $e(\lambda)$ and of the constraint $M(\lambda)$ defined by:

$$e(\lambda) = \|h - g(\lambda)\|_K^2, \qquad M(\lambda) = \|g(\lambda)\|_{T\setminus K},$$

is smooth and monotonous. In particular, as $\lambda \searrow -1$,

$$e(\lambda) \searrow 0, \quad M(\lambda) \nearrow \infty, \quad \text{if} \quad h \notin H^2_{|K}, \qquad M(\lambda) \nearrow \|h\|_{T\setminus K}, \quad \text{if} \quad h \in H^2_{|K}.$$

Although these formulae remain implicit (the parameter λ being involved in place of the norm constraint M), they give rise to a robust algorithm that allow to build $g(h, M)$, see Sections 4 and 5.

4. CONTINUITY OF THE (BEP) SOLUTIONS WITH RESPECT TO THE DATA AND BOUNDS

In this section, we shall be concerned with continuity properties of the solutions of (BEP) problems with respect to the data h and M. For $h \in H^2_{|K}$, we still denote by h its unique H^2 extension to the whole unit disk D (see Proposition 1). Let:

$$\begin{aligned}\mathcal{D} &= \{(h, M) \in H^2_{|K} \times \mathbb{R}_+^* \mid \|h\|_{T\setminus K} = M\},\\ \mathcal{D}_- &= \{(h, M) \in H^2_{|K} \times \mathbb{R}_+^* \mid \|h\|_{T\setminus K} < M\}.\end{aligned}$$

Let $\mathcal{E}$ the approximation or extension operator, which maps $h \in L^2(K)$ and $M \in \mathbb{R}_+^*$ onto the unique associated solution $g(h, M) = g$ to problem (BEP):

$$\begin{aligned}\mathcal{E}\colon \quad L^2(K) \times \mathbb{R}_+^* &\longmapsto H^2\\ (h, M) &\longmapsto g.\end{aligned}$$

Theorem 3. *Let $L^2(K) \times \mathbb{R}_+^*$ be equipped with its usual norm. The mapping $\mathcal{E}$ is continuous on the whole $(L^2(K) \times \mathbb{R}_+^*)$ with respect to the weak topology of H^2, whereas it is only continuous on $(L^2(K) \times \mathbb{R}_+^*) \setminus \mathcal{D}_-$, with respect to its strong topology.*

The proof requires the following lemma.

Lemma 2. *The map $\mathcal{E}_K$: $(h, M) \mapsto \mathcal{E}(h, M)|_K$ is continuous from $L^2(K) \times \mathbb{R}_+^*$ onto $L^2(K)$.*

Proof. Let (h_n, M_n) be a sequence of $L^2(K) \times \mathbb{R}_+^*$ strongly convergent to (h, M), and let g_n and g solve the (BEP) problems related to (h_n, M_n) and (h, M).

Define now:

$$\mathcal{B}_M := \{g \in H^2 \mid \|g\|_{T\setminus K} \le M\}, \qquad \mathcal{C}_M := \{g_{|K} \mid g \in \mathcal{B}_{\mathcal{M}}\},$$
$$\mathcal{B}_n := \mathcal{B}_{M_n}, \qquad \mathcal{C}_n := \mathcal{C}_{M_n}.$$

Being the best approximation of the data h in the closed convex subset $\mathcal{B}_M$ of the Hilbert space H^2, the following classical characterization holds for g [10]:

$$\mathrm{Re}\,((w - g), (g - h))_K \ge 0, \qquad \forall w \in \mathcal{B}_M. \tag{13}$$

We have:

$$\|g_n - g\|_K \le \|g_n - h_n\|_K + \|h_n - g\|_K \le 2\|h_n\|_K + \|g\|_K$$

and therefore $\|g_n - g\|_K$ is bounded.

By the characteristic equation (13), we get for all $n \in \mathbb{N}$:

$$\begin{cases} \mathrm{Re}\,(w_n - g_n, g_n - h_n)_K \ge 0, & \forall w_n \in B_n \\ \mathrm{Re}\,(w - g, g - h)_K \ge 0, & \forall w \in B_M \end{cases} \tag{14}$$

Let us choose some $\varepsilon \in]0,1[$. Therefore, there exists some integer $N(\varepsilon) \in \mathbb{N}$, such that $\forall n \ge N(\varepsilon)$:

$$g_\varepsilon := (1-\varepsilon)g \in B_n \qquad \text{and} \qquad g_{n,\varepsilon} := (1-\varepsilon)g_n \in B_M.$$

Using then equation (13) with g_ε and $g_{n,\varepsilon}$ as test functions, we get:

$$\begin{cases} \mathrm{Re}\,(g - g_n - \varepsilon g, g_n - h_n)_K \ge 0 & \forall n \ge N(\varepsilon) \\ \mathrm{Re}\,(g_n - g - \varepsilon g_n, g - h)_K \ge 0 & \forall n \ge N(\varepsilon) \end{cases}$$

and:

$$\begin{cases} \mathrm{Re}\,(g - g_n, g_n - h_n)_K \ge \varepsilon\,\mathrm{Re}\,(g, g_n - h_n)_K, \\ \mathrm{Re}\,(g_n - g, g - h)_K \ge \varepsilon\,\mathrm{Re}\,(g_n, g - h)_K. \end{cases}$$

From both the above inequations, we get:

$$\mathrm{Re}\,(g - g_n, (g_n - g) + (h - h_n))_K \ge \varepsilon[\mathrm{Re}\,(g, g_n - h_n)_K + \mathrm{Re}\,(g_n, g - h)_K]$$

which yields:

$$\operatorname{Re}(g-g_n, g-g_n)_K \le \operatorname{Re}(g-g_n, h-h_n)_K \\ - \varepsilon[\operatorname{Re}(g, g_n-h_n)_K + \operatorname{Re}(g_n, g-h)_K]. \quad (15)$$

But $\|g_n\|_K$ and $\|h_n\|_K$ are both real bounded sequences. There exists two positive real numbers α, β such that:

$$\|g_n - g\|_K^2 \le \alpha\|h_n - h\|_K + \beta\varepsilon, \qquad \forall n \ge N(\varepsilon).$$

Now, λ being any accumulation point of the bounded real sequence $\|g_n - g\|_K$, we derive by making $n \to \infty$:

$$\lambda \le \beta\varepsilon.$$

Since this holds for any value of ε, the real sequence $\|g_n - g\|_K$ has only 0 as an accumulation point, and is thus convergent to 0, which proves g_n to strongly converge to g in $L^2(K)$. $\square$

Proof of Theorem 3. Let (h_n, M_n) be a sequence of $L^2(K) \times \mathbb{R}_+^*$ such that:

$$h_n \longrightarrow h_0 \quad \text{in } L^2(K), \qquad M_n \longrightarrow M > 0 \quad \text{in } \mathbb{R}_+.$$

Let g_n and g be the solutions of the (BEP) problems related to the pairs (h_n, M_n) and (h, M). From Lemma 2, we derive that $g_n \to g$ in $L^2(K)$, and $\|g_n\|_K$ is hence bounded. Since $\|g_n\|_{T\setminus K} \le M_n$ and $\lim_{n\to\infty} M_n = M$, we get that $\|g_n\|_T$ is bounded. H^2 being a Hilbert space, there exists some $w \in H^2$, and a subsequence of (g_n), still denoted by (g_n), such that $g_n \rightharpoonup w$ weakly in H^2. Lemma 2 thus implies that $w = g$ on K and, by Proposition 1, $w = g$ on the whole $\overline{D}$. The subsequence g_n then weakly converges to g in H^2, which does not depend on the chosen subsequence. Hence, g_n weakly converges to g in H^2. This establishes the first part of Theorem 3.

We now claim that, if $h \notin \mathcal{C}_M$ or if $(h, M) \in \mathcal{D}$ (actually meaning $(h, M) \in (L^2(K) \times \mathbb{R}_+^*) \setminus \mathcal{D}_-$), then $\lim_{n\to\infty} g_n = g$ in H^2. Indeed, according to lemma 2 and the already proved first part of the present theorem, we shall be done by proving that $\lim_{n\to\infty} \|g_n\|_{T\setminus K} = \|g\|_{T\setminus K}$. To this end, let $(\|g_\nu\|_{T\setminus K})$ be any convergent subsequence of $\|g_n\|_{T\setminus K}$, and let $l \ge 0$ be its limit. But, whenever $h \notin \mathcal{C}_M$, or $(h, M) \in \mathcal{D}$, the constraint of the related (BEP) problem is saturated and we thus get:

$$M = \|g\|_{T\setminus K} \le \liminf_{\nu\to\infty} \|g_\nu\|_{T\setminus K} = l \le \lim_{\nu\to\infty} M_\nu = M.$$

It follows that $l = M = \|g\|_{T\setminus K}$, which means the sequence $(\|g_n\|_{T\setminus K})$ has a single accumulation point. In addition to being bounded, this makes it convergent and proves the claim.

Finally, let us prove that, if $(h, M) \in \mathcal{D}_-$, and $\lim_{n\to\infty} g_n = g$ in H^2, then there exists some integer N such that $\forall n \ge N$, $h_n \in C_n$. Indeed, if $(h, M) \in \mathcal{D}_-$, then $g = h$, and moreover $\|g|_K\|_{T\setminus K} < M$. Let (h_ν) be some subsequence of (h_n), such that $h_\nu \notin C_\nu$. In such a case, $\|g_\nu\|_{T\setminus K} = M_\nu$. But $\lim_{\nu\to\infty} M_\nu = M$, so that we get:

$$\lim_{\nu\to\infty} \|g_\nu\|_{T\setminus K} > \|g\|_{T\setminus K},$$

which cannot hold, because g_ν strongly converges to g in $L^2(T)$. $\square$

In order to achieve convergence of the reconstruction scheme, continuity ensured by Theorem 3 is hardly sufficient. Provided the available data are smooth enough, higher order schemes, holding better continuity properties, can be worked out in a similar way. To this end, $\mathcal{H}^{n,2}$ being the Hardy—Sobolev space of $\mathbb{D}$ as defined in Section 2.1, let $\mathcal{E}_n$ be the mapping:

$$\mathcal{E}_n\colon \quad W^{n,2}(K) \times \mathbb{R}_+^* \longmapsto \mathcal{H}^{n,2}$$

defined by:

$$[\mathcal{E}_n(h,M)]^{(n)} = \mathcal{E}(h^{(n)}, M), \qquad \mathcal{E}_n^{(k)}(h,M) = h^{(k)}(x_0), \quad 0 \le k \le n-1 \quad (16)$$

for some fixed $x_0 \in K$ (all the derivatives are taken with respect to the arclength θ on $\mathbb{T}$). Note that $\mathcal{E}_0 = \mathcal{E}$. Solving the so-called (BEP)$_n$ problem thus amounts to differentiate n times the given data, solve the (BEP) problem for the so obtained n-th derivative with bound M, and then integrate n times to get $\mathcal{E}_n(h,M)$ as a function of $\mathcal{H}^{n,2}$. As expected, continuity on the derivatives, up to the $n-$th one, is gained this way.

Theorem 4. *Let $W^{n,2}(K) \times \mathbb{R}_+^*$ be equipped with its usual norm. $\mathcal{E}_n$ is therefore continuous as a mapping from $W^{n,2}(K) \times \mathbb{R}_+^*$ onto $\mathcal{H}^{n,2}$, with respect to its weak topology for all $n \ge 0$, and it is continuous as a mapping from $W^{n,2}(K) \times \mathbb{R}_+^*$ onto $\mathcal{H}^{n-1,2}$ with respect to its strong topology.*

Theorem 4 is a straightforward consequence of Theorem 3 and the definition of $\mathcal{E}_n$, by using the compactness of the imbedding of $W^{n,2}(\mathbb{T})$ into $W^{n-1,2}(\mathbb{T})$ for $n \ge 1$ [10, Theorem IX.16].

5. COMPUTATIONAL ALGORITHMS AND NUMERICAL RESULTS

In view of the results of Section 3, we are able to provide a constructive scheme to solve for (PR). Under some smoothness assumptions, this procedure is effective and robust with respect to measurement noises. This will provide us with an original and efficient method that permits the Robin coefficient recovery.

Assume that we are given some nonnegative flux ϕ such that $\phi \not\equiv 0$ and let f be the measurements performed on $K \subset T$. When $D = \mathbb{D}$, the successive steps of a reconstruction algorithm for φ on γ are the following. If $D \neq \mathbb{D}$, a conformal transformation is required as preliminary and final steps. In the sequel, we shall describe and study the method in the unit disk $\mathbb{D}$.

1. Compute from the available data the restriction to K of the analytic function $h = u_m + i\int \phi\, d\theta$;

2. Solve the (BEP) problem related to the data h on K, and a suitable constraint $M > 0$ on $\mathbb{T} \setminus K$. This gives $g_M = g(h|_K, M)$ on $\overline{\mathbb{D}}$;

3. Compute
$$\varphi_M = -(\operatorname{Im} g_M)' / \operatorname{Re} g_M \qquad \text{on} \quad \mathbb{T} \setminus K. \tag{17}$$

This process is expected to end up with the actual solution to problem (PR), provided the constraint M is properly chosen. In that case, expression (17) makes sense thanks to Lemma 1, since the positivity of ϕ yields $\operatorname{Re}(g_M) > 0$ on $\overline{\mathbb{D}}$.

Assuming h is indeed the restriction to K of some analytic function also denoted by h, g_M exactly fits h whenever the bound M is larger than $\|h\|_{\mathbb{T} \setminus K}$. However, this value is unknown and thus needs to be found out. A cross validation process may be laid out to fulfill the task, some part of the available data needing to be devoted to it.

5.1. The zero-order method

Let us briefly describe the procedure, designed to replace the second step of the above algorithm, in order to provide it with the "right" value of the bound to be used, and eventually with the proper bounded extension:

(a) Split the measurement set into two parts $K = K_1 \cup K_2$.

(b) Given $\mu > 0$, solve the (BEP) problem with respect to $(h|_{K_1}, \mu)$ and get g_μ.

(c) Define $M := \operatorname{Argmin}_{\mu > 0} \|g_\mu - h\|_{K_2}$.

The full zero-order algorithm reads now as follows:

Algorithm A_0:

1. Compute from the available data the restriction to K of the analytic function $h = u_m + i \int \phi \, d\theta$.

2. (a) Split the measurement set into two parts $K = K_1 \cup K_2$.
   (b) Given $\mu > 0$, solve the (BEP) problem with respect to (h_{K_1}, μ) and get g_μ.
   (c) Define $M := \operatorname{Argmin}_{\mu > 0} \|g_\mu - h\|_{K_2}$ and compute $g = g(h|_{K_1}, M)$ on $\overline{\mathbb{D}}$.

3. Compute
$$\varphi = -(\operatorname{Im} g)' / \operatorname{Re} g \qquad \text{on} \quad \mathbb{T} \setminus K.$$

In case of analytic data, the so-computed g is as expected the desired analytic extension of the data to the whole of $\overline{\mathbb{D}}$, which is established in the following proposition.

Proposition 3. *Assume $h \in H^2_{|K}$, and let $M := \|h\|_{T\setminus K_1}$. Then, M is the smallest constant minimizing the mapping τ defined by:*

$$\tau\colon\ \mathbb{R}^+_* \longmapsto \mathbb{R}^+$$
$$\mu \longmapsto \|g(h|_{K_1}, \mu) - h\|_{K_2}$$

and moreover $\tau(\mu) = 0$, $\forall \mu \in [M, \infty]$.

Proof. Since $g(h, M) \equiv h$, we get $\tau(M) = 0$ and M is a minimum of τ. The remaining follows immediately from Proposition 1. $\square$

Because of noise, the data are however unlikely to be analytic, which compels us to check the robustness of the whole process. Namely, the issue is: *given slightly perturbated (i. e. close to analytic) data, would the extended ones, using the above algorithm (A_0), be close to the analytic ones ?*

Let $h \in H^2$, and $M := \|h\|_{\mathbb{T}\setminus K_1}$. Given two positive real numbers A, $B > 0$, $A < B$, and a non analytic perturbation ε ($\varepsilon \in L^2(K)$ and $\varepsilon \notin H^2_{|K}$), let us consider $h_\varepsilon = h + \varepsilon$, and define the following mapping:

$$\tau_\varepsilon\colon\ [A, B] \longmapsto \mathbb{R}^+$$
$$\mu \longmapsto \|g(h_\varepsilon|_{K_1}, \mu) - h_\varepsilon\|_{K_2}.$$

where $g(h_\varepsilon|_{K_1}, \mu)$ solves the (BEP) problem related to $(h_\varepsilon|_{K_1}, \mu)$. The quantity $\tau_\varepsilon(\mu)$ stands for the misfit value on K_2 of the prescribed data h_ε on K_1 to its bounded "extension" $g(h_\varepsilon|_{K_1}, \mu)$ on $\mathbb{T} \setminus K_1$, with bound μ. Let δ_ε be the lower bound, with respect to μ, of that misfit value

$$\delta_\varepsilon = \inf_{\mu \in [A,B]} \tau_\varepsilon(\mu).$$

The following result then holds:

Lemma 3. *For all ε, the set $\mathcal{I}_\varepsilon = \{\mu_\varepsilon \in [A, B] \mid \delta_\varepsilon = \tau_\varepsilon(\mu_\varepsilon)\}$ is a non empty and closed one. Moreover, $\lim_{\|\varepsilon\|_K \to 0} \delta_\varepsilon = 0$.*

Proof. Given $\varepsilon \notin H^2|_K$, the mapping τ_ε is continuous on $[A, B]$ since by Theorem 3, $\lim_{k\to\infty} \|g(h_\varepsilon, \mu_k) - g(h_\varepsilon, \mu)\|_K = 0$ whenever $\mu_k \to \mu$. The minimum value δ_ε of τ_ε is thus reached on the compact set $[A, B]$ and the set $\mathcal{I}_\varepsilon$ is therefore not empty. Moreover, it is closed since $\mathcal{I}_\varepsilon = \tau_\varepsilon^{-1}(\delta_\varepsilon)$. For $M \in [A, B]$, we have:

$$0 \le \delta_\varepsilon \le \|g(h_\varepsilon, M) - h_\varepsilon\|_{K_2}.$$

Let ε_n be any sequence such that $\lim_{n\to\infty} \varepsilon_n = 0$. Therefore, (h_{ε_n}, M) strongly converges to (h, M) in $L^2(K) \times \mathbb{R}^+$ and moreover $(h, M) \notin \mathcal{D}_-$ (in fact, $(h, M) \in \mathcal{D}$, by hypothesis). According to Theorem 3, it comes out that $g(h_{\varepsilon_n}, M)$ strongly converges to $g(h, M) = h$ in H^2. Both $g(h_{\varepsilon_n}, M)$ and h_{ε_n} hence strongly converge to h in $L^2(K_2)$, which yields $\lim_{n\to\infty} \delta_{\varepsilon_n} = 0$ and therefore $\lim_{\|\varepsilon\|_K \to 0} \delta_\varepsilon = 0$. $\square$

According to Lemma 3, we can now define $M_\varepsilon \in [A, B]$ and $g_\varepsilon \in \mathcal{H}^2$ as follows:

$$M_\varepsilon = \min\{\mu_\varepsilon \in \mathcal{I}_\varepsilon\}, \qquad g_\varepsilon = g(h_\varepsilon|_{K_1}, M_\varepsilon). \tag{18}$$

Lemma 4. *Any accumulation point M_1 of the family $(M_\varepsilon)_\varepsilon$ is such that $M_1 \geq M$.*

Proof. Assume that there exists some subsequence $(M_{\varepsilon_n})_n$ of $(M_\varepsilon)_\varepsilon$, such that $\lim_{n\to\infty} M_{\varepsilon_n} = M_1 < M$. In such a case, $(h, M_1) \notin D_-$, and hence $g(h_{\varepsilon_n}, M_{\varepsilon_n})$ strongly converges to $g(h, M_1)$ in H^2. By the above lemma, we get

$$0 = \lim_{n\to\infty} \delta_{\varepsilon_n} = \|g(h, M_1) - h\|_{K_2},$$

whence $g(h, M_1)$ coincides with h on K_2. Because they both belong to H^2, they coincide on the whole of $\bar{\mathbb{D}}$ and hence:

$$M_1 = \|g(h, M_1)\|_{\mathbb{T}\setminus K_1} = \|h\|_{\mathbb{T}\setminus K_1} = M,$$

a contradiction. □

The convergence result for the zero-order algorithm (A_0) is the following.

Theorem 5 Robustness of the zero-order method. *The family $(g_\varepsilon)_\varepsilon$ strongly converges to h on K, whereas only weak convergence is achieved on $\mathbb{T} \setminus K$.*

Proof. First, the convergence property on K_1 is a direct consequence of Lemma 2 applied to $\mathcal{E}_{K_1}$. Now, according to Lemma 4, it may happen that $(h, M_1) \in \mathcal{D}_-$. The weak convergence on $\mathbb{T} \setminus K$ directly follows from Theorem 3, while the strong one on K_2 is given by Lemma 3. □

5.2. Higher order methods

The above result (strong convergence on the prescription part of the boundary, weak convergence elsewhere) is very close to that obtained in [15], using an alternative extension method. Aiming to make use of the extended data in order to recover the Robin coefficient, the weak convergence on $\mathbb{T}\setminus$ K is hardly sufficient. Provided the data and the boundary are smooth enough, using a $(\text{BEP})_n$ extension instead of the (BEP) one brings additional continuity properties for the derivatives, as emphasized in Theorem 4, and hopefully on the Robin coefficient itself.

Let $n \geq 1$, and define the n-th order algorithm as follows:

Algorithm A_n:

1. Compute from the available data the restriction to K of the analytic function $h = u_m + i\int \phi\, d\theta$.
2. (a) Split the measurement set into two parts $K = K_1 \cup K_2$.
   (b) Given $\mu > 0$, solve the (BEP) problem with respect to $(h|_{K_1}^{(n)}, \mu)$ and get $g_\mu^{(n)}$.

(c) Define $M_n := \operatorname{Argmin}_{\mu>0} \|g_\mu^{(n)} - h^{(n)}\|_{K_2}$ and compute $g_n^{(n)} = \mathcal{E}(h|_{K_1}^{(n)}, M_n) \in H^2$ on $\overline{\mathbb{D}}$.

(d) Integrate $g_n^{(n)}$ n times, using proper initial conditions, in order to get $g_n \in \mathcal{H}^{n,2}$.

3. Compute

$$\varphi_n = -(\operatorname{Im} g_n)' / \operatorname{Re} g_n \qquad \text{on} \quad \mathbb{T} \setminus K.$$

Theorem 6 [*.Robustness of the n-th order method*] *Let $h \in \mathcal{H}^{n,2}$, and ε be some smooth non analytic perturbation ($\varepsilon \in W^{n,2}|_K$, $\varepsilon \notin \mathcal{H}^{n,2}|_K$). Let $h_\varepsilon = h + \varepsilon$ and $g_{n,\varepsilon} = \mathcal{E}_n(h_\varepsilon, M_\varepsilon)$. The following properties then hold as $\|\varepsilon\|_{n,K} \to 0$:*

1. *$g_{n,\varepsilon} \rightharpoonup h$ weakly in $\mathcal{H}^{n,2}$.*
2. *$g_{n,\varepsilon} \to h$ strongly in $\mathcal{H}^{n-1,2}$.*
3. *$\varphi_{n,\varepsilon} \rightharpoonup \varphi$ weakly in $W^{n-1,2}(\mathbb{T} \setminus K)$ for $n \geq 1$ (and strongly in $W^{n-2,2}(\mathbb{T} \setminus K)$ for $n \geq 2$).*

Proof. By Theorem 5, $g_{n,\varepsilon}^{(n)}$ weakly converges to $h^{(n)}$ in $\mathcal{H}^2$. Integrating n times with proper initial conditions, we derive thus strong convergence of the $(n-1)$ first derivatives of $g_{n,\varepsilon}$ to the corresponding ones of h, which proves the first and second point.

To establish point 3, first observe that, if ϕ is non negative and $\varphi \in \Phi_{ad}^n$ then Lemma 1 ensures that the analytic function h associated to the measured data and prescribed flux ($h = u_m + i \int \phi$) verifies:

$$\operatorname{Re} h \geq \alpha > 0 \qquad \text{on} \quad \overline{\mathbb{D}}. \tag{19}$$

For ε small enough, we then get from

$$\operatorname{Re} g_{n,\varepsilon} \geq \alpha/2 \qquad \text{on} \quad \overline{\mathbb{D}}. \tag{20}$$

From formula

$$\varphi_{n,\varepsilon} - \varphi = \frac{[\operatorname{Im}(g_{n,\varepsilon} - h)]'}{\operatorname{Re} g_{n,\varepsilon}} + [\operatorname{Im} h]' \frac{\operatorname{Re}(h - g_{n,\varepsilon})}{\operatorname{Re} g_{n,\varepsilon} \operatorname{Re} h}, \tag{21}$$

one can check that, as distributions supported on $\mathbb{T} \setminus K$, $\varphi_{n,\varepsilon}$ weakly converges to φ.

Making now use of (19), (20), and of formula (21) together with its first $n-1$ derivatives, ensure that there exist positive constants β_1 and β_2 such that for small enough ε and $n \geq 1$:

$$\|\varphi_{n,\varepsilon} - \varphi\|_{n-1,2} \leq \beta_1 \|g_{n,\varepsilon} - h\|_{n,2} + \beta_2 \, \|h\|_{n,2} \, \|g_{n,\varepsilon} - h\|_{n-1,\infty}.$$

This proves the weak convergence of $\varphi_{n,\varepsilon}$ to φ in $W^{n-1,2}(\mathbb{T} \setminus K)$. Strong convergence in $W^{n-2,2}(\mathbb{T} \setminus K)$ is then a straightforward consequence of this together with the compactness of the imbedding $W^{n-1,2}(\mathbb{T} \setminus K) \subset W^{n-2,2}(\mathbb{T} \setminus K)$, for all $n \geq 2$. □

From Theorem 6, we can deduce that using the second order method (A_2) is necessary to achieve robustness on the recovered Robin coefficient. Numerical results will however show that the first order method (A_1) is usually satisfactory, at least for smooth enough data.

5.3. Numerical results

In this section, we are going to display some numerical trials proving that the above described algorithm (A_n) is indeed effective. The trials have been run on the unit disk $\mathbb{D}$, for $n = 0, 1, 2$, using the *Hyperion* software, developed in INRIA. The data to be reconstructed are those resulting from the function:

$$h(z) = (z - a)^{1/2} + cst,$$

where a is any complex number not belonging to the unit disk $\mathbb{D}$. Hence, the function h indeed belongs to H^2 and its boundary values are assumed to be available on the arc $K \subset \mathbb{T}$ that corresponds to $[\pi/2, 3\pi/2]$. The software *Hyperion* makes use of the expansion of H^2 functions on a truncated Fourier basis, of which we took 50 coefficients to run all the computations displayed in this section.

5.3.1. Recovery of smooth data

The first numerical test (see Figure 1) is devoted to the reconstruction of the Robin coefficient φ in a smooth case (for $a = 2(1 + i)$). A Nyquist diagram is obtained by plotting the imaginary parts with respect to the real part of a complex valued function. Those shown in Figure 1a) correspond to the parametrized curve $(\operatorname{Re} h(e^{i\theta}), \operatorname{Im} h(e^{i\theta}))$, for $\theta \in [0, 2\pi)$.

Figures 1c and 1d show that the zero-order method is uneffective for the Robin coefficient recovery, whatever smooth the prescribed data are, and we shall thus drop it for the forthcoming experiments.

5.3.2. Recovery of non smooth data

Let us now study the sensitivity of the reconstruction method to the data smoothness. By making a closer to the unit circle, the function h achieves harsher behaviours, though remaining *smooth* as stated in Theorem 6.

It should be noted that the most varying part of the data (i. e. the one on part of the boundary closer to a) is reconstructed from the smoother part. Hence, although K and $\mathbb{T} \setminus K$ remain of equal Lebesgue measure, the amount of data to be recovered, with respect to that of the prescribed, is larger when a is closer to $\mathbb{T} \setminus K$. The distance $\mathrm{d}(a, \mathbb{T})$ is thus a proper way to parametrize the numerical study. Actually, Figures 2 and 3 show that the Robin coefficient recovered from the extended data extension is no longer acceptable when a becomes close to the boundary: as a matter of fact, higher order methods cannot make up for the loss of smoothness, since they do need regularity in order to be effective.

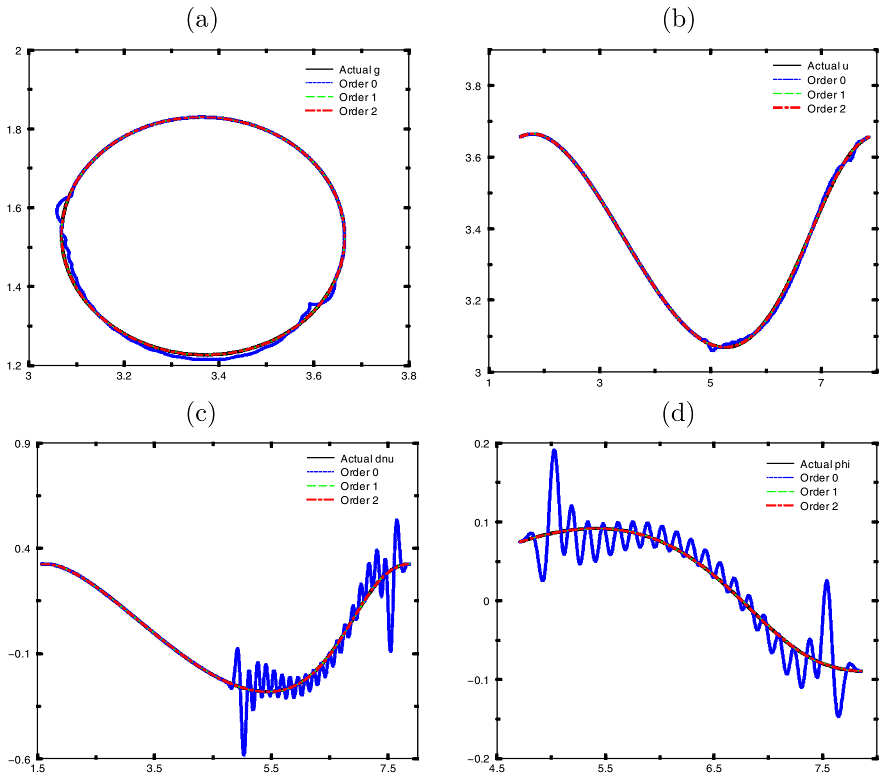

Figure 1. Nyquist plot of the bounded extension (a), and plots of its real part (b), of the normal derivative $\partial u/\partial n$ (c) and of the recovered Robin coefficient φ (d) at orders 0, 1 and 2

5.3.3. Noisy data

The study is run in a smooth data case ($a = 2(1+i)$). Noise is generated by a random variable whose uniform norm ranges from 1% to 15% of $\|h\|_\infty$. As expected from the robustness results of the above Subsections 5.1 and 5.2, the data extension process (Figure 4) resists to noise better than the Robin coefficient recovery one does (Figure 5), although this latter is pretty robust.

6. CONCLUSION

The method we have been presenting in this paper first reads as a data completion one, solving the Cauchy problem for the Laplace equation. The (BEP) framework, enriched with a cross validation procedure to control the instabilities inherent to such problems, turns out to provide with an effective and robust method to built up the data extension. By designing higher order methods based on the same framework, additional robustness is gained on the derivatives, thus permitting to derive the Robin coefficient from the extended data. Numerical results confirm the robustness of the higher order methods and prove moreover

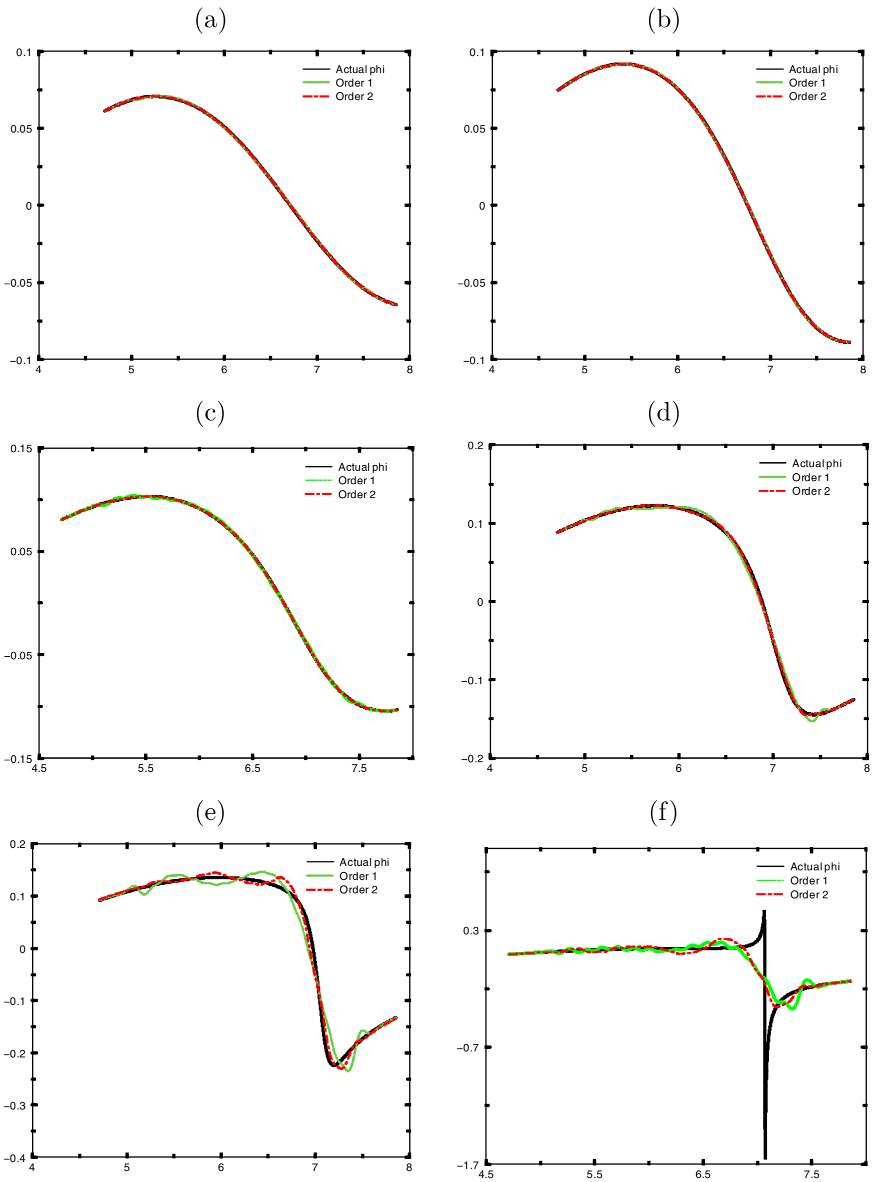

Figure 2. Recovered φ for different values of $\delta = \mathrm{d}(a, \mathbb{T})$: $\delta = 4$ (a), $\delta = 2$ (b), $\delta = 1.5$ (c), $\delta = 1$ (d), $\delta = 0.8$ (e), $\delta = 0.708$ (f) at orders 1 and 2

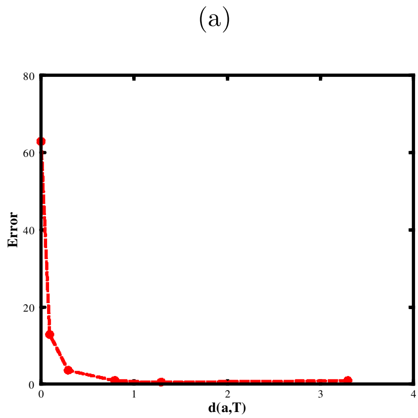

Figure 3. L^2 errors on φ with respect $d(a, \mathbb{T})$ (1st order method (a) and 2nd order (b))

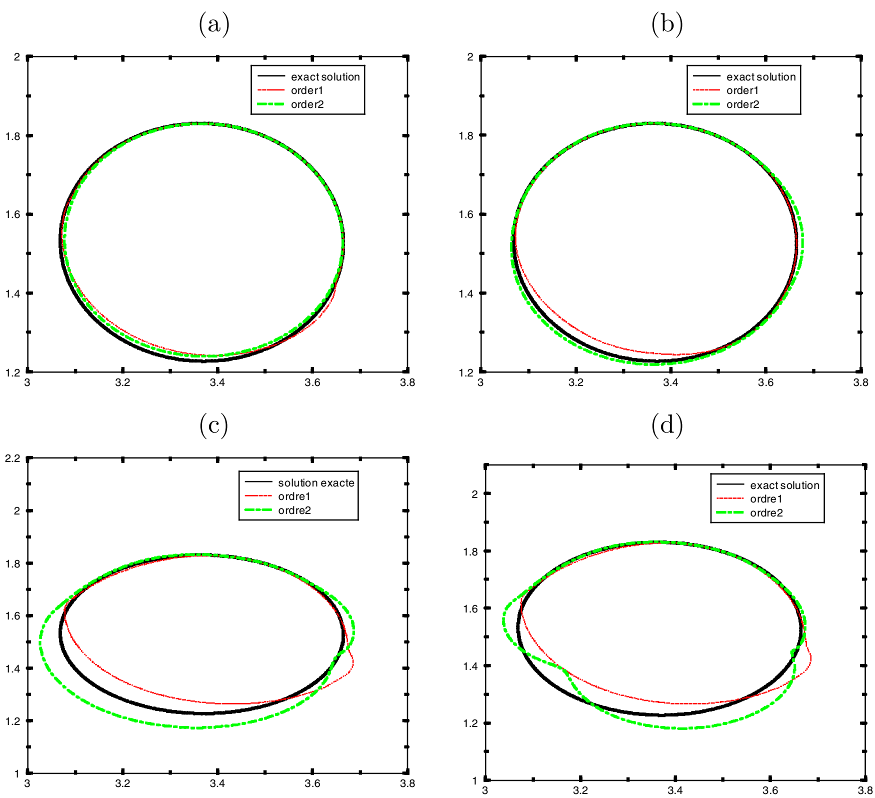

Figure 4. Nyquist plots of g for noise at levels 1% (a), 5% (b), 10% (c), and 15% (d)

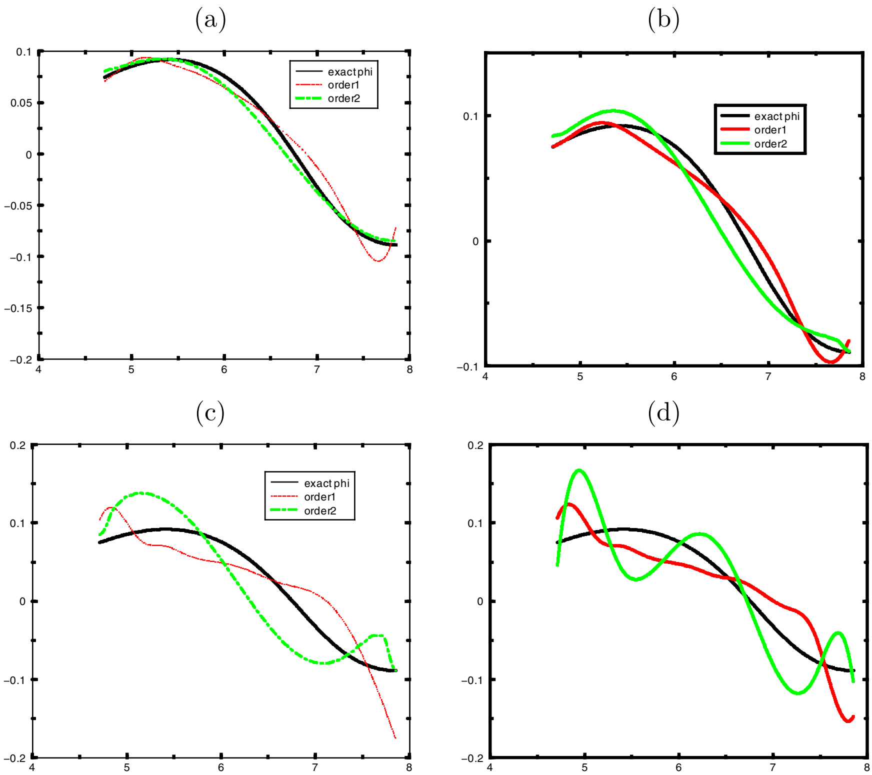

Figure 5. Reconstruction of φ from extended noisy data: 1% (a), 5% (b), 10% (c), and 15% (d)

the smoothness assumptions the convergence results are based to be necessary, if not sharp.

Limitations of the method follow from the features of the complex analysis tools used to work it out: it cannot directly extend to 3D, and to other operators but the Laplacian, although a variable conductivity may perhaps be handled. However, possible extensions are of several kinds:

- Cracks: Assuming the body to be cracked, is there still possible to make use of the method in order to recover the Robin coefficient and/or the geometrical defect ? Alternative methods have proved to be effective in such cases: in [9], a meromorphic extension is used instead of the analytic one, analyticity being lost because of the defect. Analytic extension might however be used, but to some annular domain obtained by removing from the actual one some part expected to host the flaws, which has be done sucessfully using the alternative data extension method presented in [15]. This raises the issue of solving (BEP) problems in non simply connected domains.

- Setting the bound on the imaginary part of the analytic function to be recovered, instead of its whole norm, would be in that case of great interest, see [21]. Indeed, the imaginary part is nothing but some integral of the prescribed flux, the bound of which is hence prior information needing no recovery.

- Stability: Stability properties, together with estimates, for (PR) and the above resolution algorithm can also be deduced from links between the error e on K and the constraint M on $T \setminus K$, when solving (BEP), see [3].

A similar study may now be run in H^∞ (uniform norm) instead of H^2.

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