Constrained approximation and interpolation in Hilbert function spaces

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1 Introduction

Many approximation and interpolation questions that have significant applications may be expressed in terms of constrained optimization problems in Hilbert function spaces. For example, the Hardy spaces H^2 are of importance in systems and control and certain identification problems are naturally solved by optimization [1, 17, 25]. Then again, the study of inverse 2D Dirichlet-Neumann problems leads naturally to questions of harmonic approximation [4]. A third example is to be found in signal processing [10, 17] where the Paley-Wiener spaces are of crucial importance and again certain sampling problems can be expressed in terms of Hilbert space optimization.

We consider here a family of bounded extremal problems and of interpolation problems expressed using two distinct orthogonal decompositions of some Hilbert space H; we establish existence results as well as a way of characterizing the solutions using Toeplitz-type operators.

Our primary aim is to provide a suitable Hilbertian framework in order to unify the formulation of problems of approximation that involve the restriction of Hardy class functions to subsets of the boundary of their domain. These problems have been studied before when $H = L^2$, both in the unit disk $\mathbb{D}$ (or in any conformally equivalent half-plane) and on multiply connected domains, among which is the annulus [1, 3, 8, 15, 16]. For decomposition into Hardy spaces on $\mathbb{D}$, these bounded extremal problems, as well as related best extension issues, have been studied in the general L^p case, $1 \leq p \leq \infty$, see also [5]; they generalize classical (dual) extremal problems [12].
However, other orthogonal decompositions in terms of Paley-Wiener spaces of band-limited functions may be studied using this approach. Extrapolation issues that gen-

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eralize [18] will be handled as well.

More precisely, we suppose that we are given a Hilbert space H that can be decomposed in two distinct ways into the orthogonal direct sum of subspaces,

$$H = H_+ \oplus H_- = K_+ \oplus K_- \,, \tag{1}$$

and we are interested in the following approximation and interpolation issues. Given a function in K_+, we want to approximate it as well as possible by the projection on K_+ of an element of H_+ with controlled norm (Section 3) or to extend it to the whole of H by a function in K_- such that the resulting extension is as close as possible to H_+ (Section 5). We also want to determine whether the given K_+ function is already the projection on K_+ of an element of H_+ with or without controlled norm and, in the latter case, to recover this H_+ function (Section 4).

One important example, motivated by previous work [1, 2, 3, 5, 15, 18], is the decomposition

$$L^2(\mathbb{T}) = H^2 \oplus \overline{H}_0^2 = L^2(K) \oplus L^2(\mathbb{T} \setminus K), \tag{2}$$

where K is a compact subset of $\mathbb{T}$, such that K and $\mathbb{T} \setminus K$ have positive measure, and where by a convenient abuse of notation we identify $L^2(K)$ with the subspace of functions in $L^2(\mathbb{T})$ that vanish almost everywhere on $\mathbb{T} \setminus K$; and similarly for $L^2(\mathbb{T} \setminus K)$. The Hardy space H^2 is the subspace of $L^2(\mathbb{T})$ functions whose Fourier coefficients of strictly negative indices vanish, while its orthogonal complement $\bar{H}_0^2$ consists naturally of those $L^2(\mathbb{T})$ functions whose Fourier coefficients of non-negative indices vanish. These spaces can be identified with those of functions analytic inside or outside $\mathbb{D}$ of bounded L^2 norm on circles, see for example [12, 13] for properties of Hardy spaces. Further, *weighted* approximation problems on such spaces have been treated in [16]; they fit into this context as well.
Constrained approximation for the decomposition (2) has wide applications in robust identification of transfer functions of a class of stable linear time-invariant causal dynamical systems. Indeed, such a transfer function is classically available by pointwise measurements on a bandwidth K; the identification issue then consists in finding an approximate H^2 interpolant for these data on K [1, 3, 5].

Another example is the decomposition

$$L^2(\mathbb{R}) = PW(\Omega) \oplus PW(\mathbb{R} \setminus \Omega) = L^2(J) \oplus L^2(\mathbb{R} \setminus J), \tag{3}$$

where J and Ω are compact subsets of the real line of positive measure, and $PW(\Omega)$ denotes the Paley–Wiener space [17, ch. 5], namely the subspace of $L^2(\mathbb{R})$ consisting of all functions f for which the Fourier transform $\hat{f}$ vanishes almost everywhere on the complement of Ω, that is

$$f(t) = \frac{1}{2\pi} \int_\Omega \hat{f}(x)\, e^{ixt}\, dx \,,$$

and $PW(\mathbb{R} \setminus \Omega)$ is its orthogonal complement in $L^2(\mathbb{R})$, the space of $L^2(\mathbb{R})$ functions whose Fourier transform is supported on $\mathbb{R} \setminus \Omega$. Again we regard $L^2(J)$ and $L^2(\mathbb{R} \setminus J)$ as subspaces of $L^2(\mathbb{R})$ in the natural way.
This second decomposition is itself useful in signal processing when one wants to obtain a suitable band-limited approximation to some time-limited signal [10, 22].

Yet another relevant example can be constructed by using the following decomposition, also studied in [8]

$$L^2(\partial G) = H^2(G) \oplus H^2(G)^\perp = L^2(K) \oplus L^2(\partial G \setminus K), \tag{4}$$

where G is now a complex domain whose boundary ∂G is the union of a finite number of disjoint rectifiable Jordan curves. Here $H^2(G)$ is the closure in $L^2(\partial G)$ of the set of rational functions with poles in $\mathbb{C} \setminus \overline{G}$, and $H^2(G)^\perp$ is its orthogonal complement; again K is a compact subset of ∂G such that K and $\partial G \setminus K$ have positive measure, and we regard $L^2(K)$ as a subspace of $L^2(\partial G)$.
This can be related to non-destructive control and 2D inverse problems for the Laplacian. These consist in finding the location of a possible singularity in a domain from partial boundary measurements of functions that are harmonic in the domain except at the singularity. This can be written as an analytic approximation problem (using conjugate harmonic functions) and the question then arises of finding the suitable domain G of analyticity [4].

2 Preliminaries

For E a subspace of a Hilbert space H we write P_E for the orthogonal projection from H onto E. The inner product on H and the associated norm will be written $\langle \, , \, \rangle$ and $\| \, \|$. The operator norm will also be noted $\| \, \|$.
In each of the three examples above, we are interested in approximating elements of K_+ by the projections (restrictions) of elements in H_+. We are aided in this by the fact, that we shall demonstrate, that in each case the conditions of the following lemma hold.

Lemma 2.1 *Assume that we have the decomposition* (1). *Then the following conditions are equivalent:*
(i) $P_{K_+}H_+$ *is dense in* K_+*;*
(ii) $K_+ \cap H_- = \{0\}$*;*
(iii) $P_{H_-}K_-$ *is dense in* H_-.

Proof. Condition (i) holds if and only if the only vector $k \in K_+$ satisfying $\langle P_{K_+}h, k\rangle = 0$ for all $h \in H_+$ is the zero vector.
But $\langle P_{K_+}h, k\rangle = \langle h, k\rangle$ if $k \in K_+$, and so condition (i) is equivalent to the condition that the only vector $k \in K_+$ orthogonal to all vectors in H_+ is the zero vector. This is clearly the same as condition (ii).

The equivalence of condition (iii) follows by interchanging the roles of K_+ and H_- in (ii).

■

Proposition 2.1 *The conditions of Lemma* 2.1 *hold in each of the decompositions* (2), (3) *and* (4).

Note that in each of the three examples we have given, the projection P_{K_+} can be naturally thought of as a restriction mapping.

Proof. In (2), condition (ii) asserts that there is no nonzero function in $\overline{H}_0^2$ for which the boundary values are supported on K, a well-known result due to Riesz [13, p. 52], $\mathbb{T} \setminus K$ being of positive measure. Here, point (i) is the $p = 2$ case of the density in $L^p(K)$ of traces on K of H^p function, whenever $\mathbb{T} \setminus K$ has positive measure and $1 \leq p < \infty$ [5, prop. 1].
In (3), the result (ii) holds because, if f has compact support, then $\hat{f}$ is the restriction of an entire function on $\mathbb{C}$ [17, p. 205], and hence cannot vanish on a non-discrete set of positive measure unless it is identically zero [21, thm 10.18].
In (4), we may easily obtain (i) from standard results in approximation theory such as Mergelyan's theorem [21, p. 264], see also [8, thm 4.1].

■

The next result, which has its counterpart in [5], shows that, although approximations to elements of K_+ by projections of elements of H_+ do exist, they show a behaviour that is undesirable from the point of view of well-posed approximation.

Proposition 2.2 *Suppose that the conditions of Lemma* 2.1 *hold. Let $k \in K_+$ and suppose that (h_n) is a sequence in H_+ such that $\|P_{K_+}h_n - k\| \to 0$. Then either $k = P_{K_+}h$ for some $h \in H_+$, or $\|h_n\| \to \infty$ as $n \to \infty$.*

Proof. Suppose that $h_n = f_n + g_n$, $f_n \in K_+$, $g_n \in K_-$, and $\|f_n - k\| \to 0$. Then if (h_n) has any bounded subsequence, we may without loss of generality relabel it and assume that (h_n) and hence (f_n) and (g_n) are bounded. By passing to a further subsequence we may assume that (g_n) has a weak limit point $g \in K_-$. Now $h = k + g$ is the weak limit of (h_n), and hence it lies in H_+. But now $k = P_{K_+}h$, as asserted.

■

3 Approximation problems

Proposition 2.2 motivates the following *Bounded Extremal problem* (BEP), which provides the best approximation to $k \in K_+$ by elements $P_{K_+}h$, $h \in H_+$ subject to a norm constraint on $P_{K_-}h$. Thus in each example given above we would approximate an L^2 function on a set by the restrictions of functions in Hardy classes or Paley–Wiener classes, subject to constraints on the complementary set.

Problem 3.1 *Given a decomposition* (1) *satisfying the conditions of Lemma* 2.1*, let* $M > 0$ *and let* $k_2 \in K_-$ *be such that*

$$S_M = S_M(k_2) = \{g \in H_+ : \|P_{K_-}g - k_2\| \leq M\} \neq \emptyset. \tag{5}$$

Fix $k_1 \in K_+$*, such that* $k_1 \notin P_{K_+}S_M$*. Then we seek* $h_0 \in S_M$ *such that*

$$\|P_{K_+}h_0 - k_1\| = \inf\{\|P_{K_+}g - k_1\| : g \in S_M\} = \beta, \text{ say}. \tag{6}$$

Theorem 3.1 *Problem* 3.1 *has a unique solution* h_0*, and the constrained is saturated, in that* $\|P_{K_-}h_0 - k_2\| = M$*.*

Proof. The existence of a solution to Problem 3.1 follows by a weak compactness argument—take a sequence $(g_n) \in S_M$ such that $\|P_{K_+}g_n - k_1\| \to \beta$; then it has a weak limit point $h_0 \in S_M$ and this is easily seen to be an extremal h_0 for the problem.
To see that the constraint is saturated, suppose to the contrary that $\|P_{K_-}h_0 - k_2\| < M$. Then, writing $k = k_1 - P_{K_+}h_0 \in K_+$, which is necessarily non-zero, we can use the density hypothesis on $P_{K_+}H_+$ (Lemma 2.1, (i)) to find $h_1 \in H_+$ such that $\|k - P_{K_+}h_1\| < \|k\|$, and hence, writing $x = P_{K_+}h_1$, we have

$$\begin{aligned} \|k_1 - P_{K_+}(h_0 + \lambda h_1)\| &= \|k - \lambda x\| \\ &\leq \|\lambda(k - x)\| + \|(1 - \lambda)k\| \\ &< \|k\| = \|k_1 - P_{K_+}h_0\| \end{aligned}$$

for any $\lambda \in (0, 1)$. Now, for λ sufficiently small, we have also that $h_0 + \lambda h_1 \in S_M$, which contradicts the extremal property of h_0. Hence the constraint is saturated.

To prove uniqueness, assume that g_1, $g_2 \in S_M$, with $g_1 \neq g_2$, and $\|P_{K_+}g_1 - k_1\| = \|P_{K_+}g_2 - k_1\| = \beta$, say. Then, $g = (g_1 + g_2)/2$ lies in S_M and at least one of the following two conditions holds:
(i) $P_{K_+}g_1 \neq P_{K_+}g_2$, in which case the strict convexity of the norm in H implies that $\|P_{K_+}g - k_1\| < \beta$, contradicting the optimality of g_1 and g_2;
(ii) $P_{K_-}g_1 \neq P_{K_-}g_2$, in which case $\|P_{K_-}g - k_2\| < M$, still from strict convexity, contradicting the property of saturation of the constraint.
In either case we conclude that the solution must be unique.

Finally, observe that both existence and uniqueness would directly follow from the observation that S_M is a closed convex subset of the (uniformly convex) Hilbert space H, which ensures that a best approximation projection $H \to S_M$ is available [7].

■

The solution to (BEP) can be given in terms of bounded operators that are closely related to Toeplitz operators.

Theorem 3.2 *The solution to* (BEP) *is given by*

$$h_0 = (I + \lambda T)^{-1}P_{H_+}(k_1 + (1 + \lambda)k_2), \tag{7}$$

where $T : H_+ \to H_+$ *is the self–adjoint operator* $P_{H_+}P_{K_-}$*, and* $\lambda > -1$ *is a constant such that* $\|P_{K_-}h_0 - k_2\| = M$*.*

Note that $0 \leq \langle T h, h\rangle \leq \|h\|^2$ for every $h \in H_+$; thus, the spectrum of T is contained in $[0,1]$, see [24, XI, 8, thm 2]. Hence, $(I+\lambda T)^{-1}$ is continuous on H_+ for every $\lambda > -1$ and

$$1 \leq \|(I+\lambda T)^{-1}\| \leq \frac{1}{1+\lambda}\,,\ \forall \lambda \in (-1,0]\,. \tag{8}$$

In (2), the operator $I - T = P_{H_+} P_{K_+}$ is the Toeplitz operator of symbol χ_K, the characteristic function of $K \subset \mathbb{T}$, see [1, 3], while in (3), it is a band- and time-limiting operator whose spectral properties are discussed in [10, 2.3].

Proof. We use a variational proof, as in [1]. Let $u \in H_+$ be in the (real) tangent space to the set ∂S_M at h_0, i.e.,

$$\mathrm{Re}\langle P_{K_-} h_0 - k_2, u\rangle = 0.$$

Then the optimality implies that

$$\mathrm{Re}\langle P_{K_+} h_0 - k_1, u\rangle = 0.$$

Thus

$$P_{H_+}(P_{K_+} h_0 - k_1) = \mu P_{H_+}(P_{K_-} h_0 - k_2), \tag{9}$$

for some $\mu \in \mathbb{R}$, and hence

$$P_{H_+} k_1 - \mu P_{H_+} k_2 = P_{H_+} P_{K_+} h_0 - \mu P_{H_+} P_{K_-} h_0,$$

so that

$$P_{H_+}(k_1 - \mu k_2) = (h_0 - T h_0) - \mu T h_0.$$

Now

$$h_0 + \lambda T h_0 = P_{H_+}(k_1 + (1+\lambda) k_2), \tag{10}$$

where $\lambda = -\mu - 1$. Using a first order perturbation argument to express the fact that h_0 is a minimum in S_M for the criterion (6) and recalling that $k_1 \notin P_{K_+} S_M$, so that $\beta > 0$ (Proposition 2.2), we get from (9) that $\mu < 0$, ensuring $\lambda > -1$. Hence, (10) implies (7).

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4 Interpolation issues

>From the above implicit approximation formula (7), we can deduce, at least in some cases, an extrapolation result, closely related to the recovery formula of Patil [18] in H^p, that provides a solution to the following question.

Problem 4.1 *Given a decomposition* (1) *satisfying the conditions of Lemma* 2.1*, and a function* $k_1 \in P_{K_+} H_+$*, we seek a function* $f_0 \in H_+$ *such that* $k_1 = P_{K_+} f_0$.

Before solving for Problem 4.1, observe that it raises another issue:

Problem 4.2 *Given a decomposition* (1) *satisfying the conditions of Lemma* 2.1*, characterize those functions* $k_1 \in K_+$ *that belong to* $P_{K_+} H_+$.

We now show that the approximants provided by Theorem 3.2 with $k_2 = 0$ furnish the following solution to Problem 4.2.

Proposition 4.1 *It holds that $k_1 \in P_{K_+} H_+$ if and only if*

$$f_\lambda = (I + \lambda T)^{-1} P_{H_+} k_1 \tag{11}$$

remains bounded as $\lambda \to -1$.

Proof. Assume first that f_λ remains bounded. In this case, up to extracting a subsequence, it weakly converges to some function $f \in H_+$ as $\lambda \to -1$. Thus, $(1 + \lambda T) f_\lambda = P_{H_+} k_1$ weakly converges to $(I - T) f = P_{H_+} P_{K_+} f$ which in fact gives:

$$\langle P_{K_+} f - k_1, \phi \rangle = 0 \ , \ \forall \phi \in H_+. \tag{12}$$

Hence $P_{K_+} f - k_1 \in H_- \cap K_+ = \{0\}$, from Lemma 2.1, and $k_1 \in P_{K_+} H_+$.
Conversely, if $k_1 = P_{K_+} f$ for some $f \in H_+$, then

$$f_\lambda = (I + \lambda T)^{-1} P_{H_+} P_{K_+} f = (I + \lambda T)^{-1} (I - T) f \ ,$$

and

$$f_\lambda - f = -(\lambda + 1)(I + \lambda T)^{-1} T f \ . \tag{13}$$

Thus, it follows from (8) that

$$\| f_\lambda - f \| \le \| f \| \ , \ \forall \lambda \in (-1, 0].$$

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Corollary 4.1 *A solution f_0 to Problem 4.1 is provided by the weak limit of the function f_λ defined by (11):*

$$\langle f_0, \phi \rangle = \lim_{\lambda \to -1} \langle f_\lambda, \phi \rangle \ , \ \forall \phi \in H_+ \ .$$

Proof. It directly follows from Proposition 4.1 and (12).

■

The next result generalizes a theorem of Patil [18] as well as providing a somewhat simpler proof.

Theorem 4.1 *If, in addition to the conditions of Lemma 2.1, the decomposition (1) is such that $K_- \cap H_+ = \{0\}$, then the solution f_0 to Problem 4.1 is unique and*

$$\lim_{\lambda \to -1} \| f_0 - f_\lambda \| = 0 \ ,$$

for f_λ defined by (11).

Proof. Assume first that $f_0, f_0' \in H_+$ verify $k_1 = P_{K_+} f_0 = P_{K_+} f_0'$. Thus, $P_{K_+}(f_0 - f_0') = 0$ which implies that $f_0 - f_0' \in K_- \cap H_+ = \{0\}$, by assumption. This ensures uniqueness. Now, if we let $\lambda = -1 + 1/\alpha$ and $S = I - T$, it is a straightforward consequence of (13) that

$$f_\lambda - f_0 = -(I + (\alpha - 1)S)^{-1} T f_0 \,.$$

It then follows from the spectral theorem [20, 12.24] that:

$$\|f_\lambda - f_0\|^2 = \int_0^1 \frac{d\mu(\lambda)}{(1 + (\alpha - 1)\lambda)^2} \,,$$

if $d\mu(\lambda) = \langle dE_\lambda(Tf_0), Tf_0 \rangle$, E being the spectral decomposition of the (bounded self-adjoint) operator S. As $\alpha \to \infty$, the integrand goes to 0 monotonically, except when $\lambda = 0$. However, the operator $S = P_{H_+} P_{K_+}$ does not have 0 as an eigenvalue because if $Sh_+ = 0$ for some $h_+ \in H_+$, then $P_{K_+} h_+ \in H_- \cap K_+ = \{0\}$, from Lemma 2.1; hence, $h_+ \in K_- \cap H_+ = \{0\}$, by hypothesis. It then follows from [20, 12.29] that $E_0 = \mu(0) = 0$, and that the above integral goes to 0 as $\alpha \to \infty$ by the monotone convergence theorem.

■

The assumption of Theorem 4.1 is easily seen to be satisfied for the decompositions (2), (3); that it is also the case for (4) is a consequence of [11, thm 10.11].

The following corollary, which was already known to be valid in case (2), see [3, prop.2], still holds for more general decompositions (1).

Corollary 4.2 *For every $k_1 \in K_+$, under the assumptions of Theorem 4.1, we have:*

$$\lim_{\lambda \to -1} \|k_1 - P_{K_+} f_\lambda\| = 0 \,. \tag{14}$$

Proof. We get from Theorem 4.1 that (14) holds for every $k_1 \in P_{K_+} H_+$, a dense subset of K_+, from Lemma 2.1. Moreover, $\|P_{K_+} f_\lambda\|$ is uniformly bounded for every $k_1 \in K_+$. Indeed,

$$\begin{aligned} P_{K_+} f_\lambda &= P_{K_+} P_{H_+} (I + \lambda T)^{-1} P_{H_+} k_1 \\ &= (I + \lambda T)^{-1} (I - T) P_{H_+} k_1 = [I - (\lambda + 1)(I + \lambda T)^{-1} T] P_{H_+} k_1 \,, \end{aligned}$$

which is bounded by (13) and the proof of Proposition 4.1.

■

It is also interesting to approach interpolation / extrapolation issues when a norm constraint is added: given $M < \infty$, and a function $k_1 \in K_+$, does k_1 belong to $P_{K_+} S_M$ (take $k_2 = 0$)? This is related to Loewner-type interpolation questions that have been approached in [2, 19, 23] for Hardy spaces and can be stated as follows in the present framework.

Problem 4.3 *Given a decomposition (1) satisfying the conditions of Lemma 2.1 and some constant $M > 0$, characterize those functions $k_1 \in K_+$ that belong to $Q_M = P_{K_+}\{h \in H_+ \,:\, \|h\| \le M\}$.*

Theorem 4.2 *It holds that $k_1 \in Q_M$ if and only if*

$$|\langle k_1, k_+\rangle| \leq M\|P_{H_+}k_+\| \ , \ \forall k_+ \in K_+ \, .$$

Proof. We follow a technique introduced in [23]. The set Q_M is convex and balanced. It is closed (in fact, weakly compact) because $\{h \in H_+ \ : \ \|h\| \leq M\}$ is weakly compact while P_{K_+} is weakly continuous on H_+. Hence the Hahn-Banach Theorem, as stated in [20, 3.7] for example, ensures that $k_1 \in Q_M$ if and only if, $\forall k_+ \in K_+$,

$$|\langle k_1, k_+\rangle| \leq \sup_{q \in Q_M} |\langle q, k_+\rangle| = \sup_{\substack{h \in H_+ \\ \|h\| \leq M}} |\langle h, k_+\rangle| = \sup_{\substack{h \in H_+ \\ \|h\| \leq M}} |\langle h, P_{H_+}k_+\rangle| \leq M\|P_{H_+}k_+\| \, .$$

■

5 Completion problems

Companion to Problem 3.1 will be the *Bounded Completion Problem* (BCP):

Problem 5.1 *Given a decomposition* (1) *satisfying the conditions of Lemma* 2.1*, let $k_2 \in K_-$ and $L \geq 0$, and let R_L be the ball of radius L in K_-, centered at k_2:*

$$R_L = R_L(k_2) = \{k \in K_- : \|k - k_2\| \leq L\} \, . \tag{15}$$

Given $k_1 \in K_+$, we seek $k_0 \in R_L$ such that

$$\text{dist}(k_1 + k_0, H_+) = \inf_{k \in R_L} \text{dist}(k_1 + k, H_+) = \gamma, \text{ say.} \tag{16}$$

The Bounded Completion Problem is close in spirit to the Bounded Extremal Problem: in words, we are given a function in K_+, and we look for a completion in K_- that meets some norm constraint and such that the resulting function is as close to H_+ as possible.

Concerning example (2), where an $L^2(K)$ function is given together with a gauge on the complementary subset, we seek an extended definition to the whole set that meets the gauge and makes the global function as close to an analytic function as possible, see [6].

We recall that

$$\text{dist}(x, H_+) = \|P_{H_-}x\|, \qquad \text{for any } x \in H, \tag{17}$$

a fact which will be used several times in what follows.

Theorem 5.1 *Let $k_1 \in K_+$. Then for every $L > 0$ there is a unique solution $k_0 \in R_L$ to problem* (5.1)*. Moreover, $\|k_0 - k_2\| = L$, unless $k_1 = P_{K_+}h$ for some $h \in H_+$ such that $\|P_{K_-}h - k_2\| < L$. If $k_1 \neq P_{K_+}h$ for some $h \in H_+$ such that $\|P_{K_-}h - k_2\| \leq L$, one can characterize k_0 by*

$$k_0 = (1 + \lambda)k_2 - \lambda P_{K_-}(I + \lambda T)^{-1}P_{H_+}(k_1 + (1 + \lambda)k_2) \, , \tag{18}$$

where $T = P_{H_+}P_{K_-}$ is the operator defined in Theorem 3.2 *and $\lambda > -1$ is a constant such that $\|k_0 - k_2\| = L$.*

Proof. The existence follows by a simple weak compactness argument, as in Theorem 3.1, and we omit it.
The saturation of the constraint holds because $P_{K_-}H_-$ is dense in H_-, so that if $\|k_0 - k_2\| < L$, then we could obtain a contradiction if we could choose $z \in K_-$ sufficiently small that $k_0 + z \in R_L$, and such that

$$\|P_{H_-}(k_1 + k_0 + z)\| < \|P_{H_-}(k_1 + k_0)\|;$$

this we can indeed do, unless $h = k_1 + k_0$ is already in H_+, which means that $k_1 = P_{K_+}h$, where $\|P_{K_-}h - k_2\| = \|k_0 - k_2\| < L$.
The uniqueness follows again from the strict convexity of the norm, as in Theorem 3.1, since the mean $\tilde{k}$ of two optimal vectors k_0 and k_0' can only saturate the constraint if $k_0 - k_2 = k_0' - k_2$. The other case to consider is the case of non-saturation, when

$$k_1 = P_{K_+}(k_1 + k_0) = P_{K_+}(k_1 + k_0');$$

but then $P_{K_-}(k_1 + k_0) = P_{K_-}(k_1 + k_0')$, since otherwise $\mathrm{dist}(k_1 + \tilde{k}, H_+) < \gamma$, which is impossible. Thus $k_0 = k_0'$.
Now the key observation is that:

$$\mathrm{Re}\langle P_{H_-}(k_1 + k_0), u\rangle = 0$$

for all u in the tangent space at k_0 to the sphere

$$S = \{k \in K_- : \|k - k_2\| = L\}.$$

As in [6], this leads us to conclude that there exists $\mu \in \mathbb{R}$ such that

$$P_{K_-}P_{H_-}(k_1 + k_0) = \mu(k_0 - k_2). \tag{19}$$

Equivalently, writing $\rho = 1 - \mu$, we have

$$P_{K_-}P_{H_+}(k_1 + k_0) = \rho k_0 + (1 - \rho)k_2, \tag{20}$$

since adding (19) to (20) gives the tautology $k_0 = k_0$. Observe that the perturbation argument already used in the proof of Theorem 3.2 implies that $\mu \leq 0$; it is furthermore a direct consequence of (19) that $\mu < 0$ unless $\gamma = 0$ which holds if and only if $k_1 = P_{K_+}h$ for some $h \in H_+$ such that $\|P_{K_-}h - k_2\| \leq L$. If we now rule out this case, we get that $\rho > 1$. >From (19), we have:

$$TP_{H_+}(k_1 + k_0) = \rho P_{H_+}k_0 + (1 - \rho)P_{H_+}k_2 = \rho P_{H_+}(k_0 + k_1) + P_{H_+}((1 - \rho)k_2 - \rho k_1),$$

and then

$$P_{H_+}(k_1 + k_0) = (\rho I - T)^{-1}P_{H_+}((\rho - 1)k_2 + \rho k_1),$$

which, together with (20), gives

$$\rho k_0 = (\rho - 1)k_2 + P_{K_-}(\rho I - T)^{-1}P_{H_+}((\rho - 1)k_2 + \rho k_1).$$

Taking now $\lambda = -1/\rho > -1$ gives (18).

■

We note that the solutions h_0 and k_0 to Problems 3.1 and 5.1 are linked together by

$$k_0 = (1 + \lambda)k_2 - \lambda P_{K_-}h_0,$$

whenever $L = |\lambda|M$. In particular, if $k_2 = 0$, then k_0 in fact belongs to the subspace $P_{K_-}H_+$ of K_-.

6 Conclusion

We have considered a number of approximation and interpolation problems in a context which unifies several examples of Hilbert function spaces. They have wide applications, notably to Hardy and Paley-Wiener spaces.
Other commonly used decompositions can be put in the same framework, as for example, with $H = L^2(\mathbb{T})$, $H_+ = \{e^{ik\theta}\}_{k=0}^{n}$ and $K_+ = \{e^{ik\theta}\}_{k=0}^{N}$, $N \le n$, or $H_+ = jH^2$, for some inner function j, with $K_\pm$ as in (2). Yet another example, mainly of theoretical interest, is given by $H = L^2(\mathbb{D})$, H_+ being the Bergman space of the disk [17], and $K_+ = L^2(K)$ where $K \subset \mathbb{D}$ is a compact set of positive planar Lebesgue measure with empty interior and $\mathbb{D} \setminus K$ connected. Further examples can be constructed from other reproducing kernel Hilbert spaces.
There are other extremal problems for which the above approach is likely to be of use, such as various questions relating the time- and band-limiting of signals, see for example [22]. It would be interesting to know whether a similar approach can be taken in some other Banach function spaces. Some promising possibilities are afforded by decomposing Banach algebras [9, 14] which have found uses in spectral factorization problems, uniform algebras [12, 13], and uniformly convex Banach spaces [7].

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