Constrained extremal problems in H^2 and Carleman's formulas

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Abstract: we consider the extremal problem of best approximation to some function f in $L^2(I)$, with I a subset of the circle, by the trace of a Hardy function whose modulus is bounded pointwise by some gauge function on the complementary subset.

1 Introduction

If D is a simply connected plane domain with rectifiable boundary ∂D, a holomorphic function f in the Smirnov class $\mathcal{E}^1(D)$ can be recovered from its boundary values by the Cauchy formula. When the boundary values are only known on a strict subset I of ∂D having positive linear measure, they still define f uniquely but the recovery cannot be achieved in closed form. In fact, recovery becomes then a special case of a classical ill-posed issue, namely the Cauchy problem for the Laplace equation. This issue is quite important in physics and engineering [26, 2, 30, 21].
Following an original idea of Carleman, one approach to the recovery of f from its knowledge on I is to introduce an auxiliary "quenching" function φ, holomorphic and bounded in D, such that $|\varphi| \equiv 1$ a.e. on $\partial D \setminus I$ and $|\varphi| > 1$ in D; such a function is easily constructed by solving a Dirichlet problem for $\log|\varphi|$. In [20], it was proven by Goluzin and Krylov that

$$f(z) = \lim_{n\to\infty} f_n(z), \qquad \text{where} \quad f_n(z) \triangleq \frac{1}{2i\pi}\int_I \left(\frac{\varphi(\xi)}{\varphi(z)}\right)^n \frac{f(\xi)}{\xi - z}\, d\xi, \qquad z \in D, \tag{1}$$

the convergence being locally uniform in D. Cauchy integrals like those defining f_n in (1) are called *Carleman's formulas*. More precisely, for w an outer function (see definitions below), we call an expression of the form

$$g_w = \frac{1}{2i\pi}\int_I \frac{w(\xi)}{w(z)}\frac{f(\xi)}{\xi - z}\, d\xi \tag{2}$$

a Carleman formula for f, which produces an analytic function g_w to approximate f in some way. Expression (2) may also be viewed as a (complex) normalized Cauchy transform.
On the unit disk $\mathbb{D}$ where $\mathcal{E}^p(\mathbb{D})$ coincides with the Hardy class H^p, Patil proved that if $f \in H^p$ with $1 < p < \infty$, then the convergence in (1) actually holds in H^p [31].
Two questions arise naturally, namely what is the meaning of f_n for *fixed* n, and what is its asymptotic behaviour if $f \in L^p(I)$ is *not* the trace of a Hardy function? On $\mathbb{D}$, when $f \in L^2(I)$ and φ is a quenching function with constant modulus a.e. on I, it was proven in [7] that the restriction $(f_n)_{|_I}$ is closest to f in $L^2(I)$-norm among all $g \in H^2$ such that $\|g\|_{L^2(\mathbb{T}\setminus I)} \le \|f_n\|_{L^2(\mathbb{T}\setminus I)}$, where $\mathbb{T}$ denotes the unit circle. Also, the results of the present paper entail that if φ is any holomorphic function which is bounded on $\mathbb{D}$ together with its inverse, and if the boundary of I in $\mathbb{T}$ has linear measure 0, then $(f_n)_{|_I}$ is closest to f in weighted $L^2(|\varphi_{|_I}|^{2n}, I)$-norm among all $g \in H^2$ such that $|g| \le |f_n|$ a.e. on $\mathbb{T}\setminus I$. These extremal properties of f_n are all the more remarkable than Carleman's formulas were originally introduced without reference to optimization. They are, however, implicit in that the constraint on $\mathbb{T}\setminus I$ depends on $|f_n|$ itself. To move on firmer ground, we make a slight twist and we rather investigate the following extremal problem. Let $I \subset \mathbb{T}$ be a subset of positive Lebesgue measure and set $J = \mathbb{T}\setminus I$ for the complementary subset. The question that we raise is :

Given $f \in L^2(I)$ and $M \in L^2(J)$, $M \ge 0$, find $g_0 \in H^2$ such that $|g_0(e^{i\theta})| \le M(e^{i\theta})$ a.e. on J and

$$\|f - g_0\|_{L^2(I)} = \min_{\substack{g\in H^2\\ |g|\le M \text{ a.e. on } J}} \|f - g\|_{L^2(I)}\,. \tag{3}$$

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This should be compared with the so-called *bounded extremal problems* studied in [3, 8, 7] for $1 \leq p \leq \infty$:

$\mathrm{BEP}\big(L^p(I), L^p(J)\big)$:

Given $f \in L^p(I)$, $\psi \in L^p(J)$ and a constant $C > 0$, find $g_0 \in H^p$ such that $\|g_0 - \psi\|_{L^p(J)} \leq C$ and

$$\|f - g_0\|_{L^p(I)} = \min_{\substack{g \in H^p \\ \|g-\psi\|_{L^p(J)} \leq C}} \|f - g\|_{L^p(I)} . \tag{4}$$

Note that in (3), we did not introduce a reference function ψ on J as in (4). While it is straightforward to handle such a generalization when ψ is the trace on J of a H^2-function, the general case holds further difficulties which left here for further research.

When I is of full measure, both problem (3) and $\mathrm{BEP}\big(L^p(I), L^p(J)\big)$ reduce to classical extremal problems, see *e.g.* [17, 19]. Therefore we limit our discussion to the case where J has positive measure.

The first reference dealing with bounded extremal problems seems to be [24], where $\mathrm{BEP}\big(L^2(I), L^2(J)\big)$ is studied when $f = 0$ and I an interval, on the half-plane rather than the disk. The case $\psi = 0$ is solved in [3] using Toeplitz operators, and the general version of $\mathrm{BEP}\big(L^2(I), L^2(J)\big)$ is taken up in [7] where the link with Carleman's formulas is pointed at. Error rates when C goes large and I is an arc can be found in [6], and an extension to the case where additional interpolation conditions are imposed in $\mathbb{D}$ is carried out in [28]. Existence and uniqueness results for $\mathrm{BEP}\big(L^p(I), L^p(J)\big)$ in the range $1 \leq p < \infty$ are also presented in [7]. Reformulations of $\mathrm{BEP}\big(L^p(I), L^p(J)\big)$ in an abstract setting involving Hilbert or smooth Banach spaces were carried out in [27, 13, 36, 14], leading to the construction of backward minimal vectors and hyperinvariant subspaces for certain classes of operators that need not be compact nor quasinilpotent, thereby generalizing [4]. Versions of $\mathrm{BEP}\big(L^2(I), L^2(J)\big)$ where the constraint bears on the imaginary part rather than the modulus, which is useful among other things to approach inverse Dirichlet-Neumann problems, are presented in [22]. Problem $\mathrm{BEP}\big(L^\infty(I), L^\infty(J)\big)$ was studied in [8, 9], together with its meromorphic generalization and related completion issues.

An initial incentive to study such problems lies with engineering issues, more precisely with linear system identification and design. This motivation is explicit in [24], and further discussed in [3, 8, 9, 35, 6], the results of which have been effective to identify hyperfrequency filters [5]. This connection is more transparent on the half plane, where f represents the so-called transfer-function of a linear dynamical system which is measured pointwise in a frequency band I of the imaginary axis, using harmonic identification techniques. Recall that a linear dynamical system is just a convolution operator on $\mathbb{R}$ (identified with the time axis), and that its transfer function is the Fourier-Laplace transform of its kernel [16]. By the Paley-Wiener and the Hausdorff-Young theorems, the causality and the stability of the system from $L^r(\mathbb{R})$ to $L^s(\mathbb{R})$ imply that f belongs to the Hardy class $\mathcal{H}^p$ of the right half plane with $1/p = 1/r - 1/s$, as soon as the latter is less than or equal to $1/2$. Because f can only be estimated up to modelling and measurement errors, one is led to approximate the data on I by a $\mathcal{H}^p$ function while controlling its deviation from some reference behaviour ψ outside I, which is precisely the analog of (4) on the half-plane. This problem can be mapped to $\mathrm{BEP}\big(L^p(I), L^p(J)\big)$ *via* the isometry $g \mapsto (1+w)^{-2/p} g((w-1)/(w+1))$ from H^p onto $\mathcal{H}^p$. More on the relations between Hardy spaces, system identification and control can be found in [18, 30, 29]. Note that it is indeed essential here to bound the behaviour of g_0 on J, for traces of Hardy functions are dense in $L^p(I)$ (in $C(I)$ if $p = \infty$) so that $\mathrm{BEP}\big(L^p(I), L^p(J)\big)$ has no solution if $C = \infty$ unless f is already the trace of a Hardy function. In practice, since modelling and measurement errors will prevent this from ever happening, the error $\|f - g\|_{L^p(I)}$ can be made arbitrarily small at the cost of $\|g\|_{L^p(J)}$ becoming arbitrarily large, which is a version in this context of the classical trade-off between precision and robustess. Motivated by the fact that the transfer function sometimes has to meet uniform bounds for physical reasons (for instance it should be less than 1 in modulus when dealing with passive systems), $\mathrm{BEP}\big(L^p(I), L^p(J)\big)$-like problems with a pointwise constraint on the modulus of the approximant were considered in [34], when the approximated function f and the constraint M are assumed to be continuous on I and $\mathbb{T}$, respectively.

The present paper seems to be first to deal with a mixed situation, where an integral criterion is minimized on I under a pointwise constraint on J. Beyond the noteworthy connection with Carleman's formulas already mentioned, one motivation to study mixed norms stems again from system identification. Indeed, the L^2 norm on I has a probabilistic interpretation as the variance of the output when the input of the system is a noise whose spectrum is uniformly distributed in the bandwidth; more general spectra can also be handled by weighting the L^2-norm in (3) with a boundedly invertible weight, which is but a small modification. If one requires in addition that the system to be identified is passive at higher

frequencies, as is the case for instance with microwave circuits, one is led to consider problem (3) with $M \equiv 1$. A quantitative study of problems like (3) seems also relevant to estimate the growth of orthogonal polynomials, including for weights outside the Steklov class, see the discussion in [15].
Problem (3) is considerably more difficult to investigate than $\mathrm{BEP}\big(L^2(I), L^2(J)\big)$, due to the fact that pointwise evaluation is not smooth –actually not even defined– in $L^2(J)$, and the analysis depends in a crucial manner on the multiplicative structure of Hardy functions. Beyond existence and uniqueness, our results hold under the extra-assumption that the boundary of I has measure zero. We do not know the extend to which this assumption can be relaxed.
The organization of the paper is as follows. In section 2 we set up some notation and recall standard properties of H^p-spaces and conjugate functions. Section 3 deals with existence and uniqueness issues, along with saturation of the constraint. In section 4 we establish an analog, in the present nonsmooth and infinite-dimensional context, of the familiar critical point equation from convex analysis. It gives rise in Section 5 to a dual formulation of the problem which makes connection with Carleman's formulas and turns it into an unconstrained concave maximization issue. We express the derivative under mild assumptions which may be used to design an ascent algorithm.

2 Notations and preliminaries

Let $\mathbb{T}$ be the unit circle endowed with the normalized Lebesgue measure ℓ, and I a subset of $\mathbb{T}$ such that $\ell(I) > 0$ with complementary subset $J = \mathbb{T} \setminus I$. To avoid dealing with trivial instances of problem (3) *we assume throughout that* $\ell(J) > 0$.
If h_1 (resp. h_2) is a function defined on a set containing I (resp. J), we use the notation $h_1 \vee h_2$ for the concatenated function, defined on the whole of $\mathbb{T}$, which is h_1 on I and h_2 on J.
For $E \subset \mathbb{T}$, we let ∂E and $\overset{\circ}{E}$ denote respectively the boundary and the interior of E when viewed as a subset of $\mathbb{T}$; we also write χ_E for the characteristic function of E and $h_{|E}$ to mean the restriction to E of a function h defined on a set containing E.
When $1 \leq p \leq \infty$, we write $L^p(E)$ for the familiar Lebesgue space of (equivalence classes of a.e. coinciding) complex-valued measurable functions on E with finite L^p norm, and we indicate by $L^p_{\mathbb{R}}(E)$ the real subspace of real-valued functions. Likewise $C(E)$ stands for the space of complex-valued continuous functions on E, while $C_{\mathbb{R}}(E)$ indicates real-valued continuous functions. The norm on $L^p(E)$ is denoted by $\| \ \|_{L^p(E)}$, and if h is defined on a set containing E we write for simplicity $\|h\|_{L^p(E)}$ to mean $\|h_{|E}\|_{L^p(E)}$. When E is compact the norm on $C(E)$ is the *sup* norm.
Recall that the Hardy space H^p is the closed subspace of $L^p(\mathbb{T})$ consisting of functions whose Fourier coefficients of strictly negative index do vanish. These are the nontangential limits of functions analytic in the unit disk $\mathbb{D}$ having uniformly bounded L^p means over all circles centered at 0 of radius less than 1. The correspondence is one-to-one and, using this identification, we alternatively regard members of H^p as holomorphic functions in the variable $z \in \mathbb{D}$. This extension is obtained from the values on $\mathbb{T}$ through a Cauchy as well as a Poisson integral [33, ch. 17, thm 11], namely if $g \in H^p$ then :

$$g(z) = \frac{1}{2\,i\,\pi} \int_{\mathbb{T}} \frac{g(\xi)}{\xi - z}\, d\xi\,, \quad \text{and also} \quad g(z) = \frac{1}{2\,\pi} \int_{\mathbb{T}} \mathrm{Re}\left\{\frac{e^{i\theta}+z}{e^{i\theta}-z}\right\} g(e^{i\theta})\, d\theta\,, \quad z \in \mathbb{D}. \tag{5}$$

Because of this Poisson representation, $g(re^{i\theta})$ converges to $g(e^{i\theta})$ in $L^p(\mathbb{T})$ as soon as $1 \leq p < \infty$. Moreover, (5) entails that, for $1 \leq p \leq \infty$, a Hardy function g is uniquely determined, up to a purely imaginary constant, by its real part h on $\mathbb{T}$:

$$g(z) = i\mathrm{Im}g(0) + \frac{1}{2\,\pi} \int_{\mathbb{T}} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, h(e^{i\theta})\, d\theta\,, \quad z \in \mathbb{D}. \tag{6}$$

The integral in the right-hand side of (6) is the *Riesz-Herglotz transform* of h and, whenever $h \in L^1_{\mathbb{R}}(\mathbb{T})$, it defines a holomorphic function in $\mathbb{D}$ which is real at 0 and whose nontangential limit exists a.e. on $\mathbb{T}$ with real part equal to h. However, only if $1 < p < \infty$ is it guaranteed that $g \in H^p$ when $h \in L^p_{\mathbb{R}}(\mathbb{T})$. In fact, the Riesz-Herglotz transform assumes the form $h(e^{i\theta}) + i\widetilde{h}(e^{i\theta})$ a.e. on $\mathbb{T}$, where the real-valued function $\widetilde{h}$ is said to be *conjugate* to h, and the property that $\widetilde{h} \in L^p_{\mathbb{R}}(\mathbb{T})$ whenever $h \in L^p_{\mathbb{R}}(\mathbb{T})$ holds true for $1 < p < \infty$ but not for $p = 1$ nor $p = \infty$. The map $h \to \widetilde{h}$ is called the *conjugation operator*, and for $1 < p < \infty$ it is bounded $L^p_{\mathbb{R}}(\mathbb{T}) \to L^p_{\mathbb{R}}(\mathbb{T})$ by a theorem of M. Riesz [19, chap. III, thm 2.3]; in this range

of exponents, we will denote its norm by K_p. It follows easily from Parseval's relation that $K_2 = 1$, but it is rather subtle that $K_p = \tan(\pi/(2p))$ for $1 < p \le 2$ while $K_p = \cot(\pi/(2p))$ for $2 \le p < \infty$ [32].
A sufficient condition for $\widetilde{h}$ to be in $L^1(\mathbb{T})$ is that h belongs to the the so-called *Zygmund class* $L\log^+ L$, consisting of measurable functions ϕ such that $\phi \log^+ |\phi| \in L^1(\mathbb{T})$ where we put $\log^+ t = \log t$ if $t \ge 1$ and 0 otherwise. More precisely, if we denote by m_h the *distribution function* of h defined on $\mathbb{R}^+$ with values in $[0,1]$ according to the formula

$$m_h(\tau) = \ell\left(\{\xi \in \mathbb{T};\ |h(\xi)| > \tau\}\right),$$

and if we further introduce the non-increasing rearrangement of h given by

$$h^*(t) = \inf\{\tau;\ m_h(\tau) \le t\}, \qquad t \ge 0,$$

it turns out that $h \in L\log^+ L$ if and only if the quantity

$$\|h\|_{L\log^+ L} \triangleq \int_0^1 h^*(t)\log(1/t)\,dt \tag{7}$$

is finite [10, lem. 6.2.], which makes $L\log^+ L$ into a Banach function space. Then, it is a theorem of Zygmund [10, cor. 6.9.] that

$$\|\widetilde{h}\|_{L^1(\mathbb{T})} \le C_0 \|h\|_{L\log^+ L} \tag{8}$$

for some universal constant C_0. A partial converse, due to M. Riesz, asserts that if a real-valued h is bounded from below and if moreover $\widetilde{h} \in L^1(\mathbb{T})$, then $h \in L\log^+ L$ [10, cor. 6.10].
We mentioned already that $\widetilde{h}$ needs not be bounded if $h \in L^\infty_{\mathbb{R}}(\mathbb{T})$. In this case all one can say in general is that $\widetilde{h}$ has *bounded mean oscillation*, meaning that $\widetilde{h} \in L^1(\mathbb{T})$ and

$$\|\widetilde{h}\|_{BMO} \triangleq \sup_E \frac{1}{\ell(E)} \int_E |\widetilde{h} - \widetilde{h}_E|\,d\theta < \infty, \qquad \text{with } \widetilde{h}_E \triangleq \frac{1}{\ell(E)} \int_E \widetilde{h}\,d\theta,$$

where the *supremum* is taken over all subarcs $E \subset \mathbb{T}$. Actually [19, chap. VI, thm 1.5], there is a universal constant C_1 such that

$$\|\widetilde{h}\|_{BMO} \le C_1 \|h\|_{L^\infty(\mathbb{T})}. \tag{9}$$

The subspace of $L^1(\mathbb{T})$ consisting of functions whose BMO-norm is finite is called BMO for short. Notice that $\|\ \|_{BMO}$ is a genuine norm modulo additive constants only. A theorem of F. John and L. Nirenberg [19, ch. VI, thm. 2.1] asserts there are positive constants C, c, such that, for each real-valued $\varphi \in BMO$, every arc $E \subset \mathbb{T}$, and any $x > 0$,

$$\frac{\ell\left(\{t \in E\ :\ |\varphi(t) - \varphi_E| > x\}\right)}{\ell(E)} \le C \exp\left(\frac{-cx}{\|\varphi\|_{BMO}}\right). \tag{10}$$

Conversely, if (10) holds for some finite $A > 0$ in place of $\|\varphi\|_{BMO}$, every arc E and any $x > 0$, then $\varphi \in BMO$ and $A \sim \|\varphi\|_{BMO}$. The John-Nirenberg theorem easily implies that $BMO \subset L^p$ for all $p < \infty$. The space of H^1-functions whose boundary values lie in BMO will be denoted by $BMOA$, and $BMOA/\mathbb{C}$ is a Banach space equipped with the BMO-norm. Clearly $BMOA \subset H^p$ for $1 \le p < \infty$, and $h + i\widetilde{h} \in BMOA$ whenever $h \in L^\infty(\mathbb{T})$. A sufficient condition for the boundedness of $\widetilde{h}$ is that h be Dini-continuous; recall that a function h defined on $\mathbb{T}$ is said to be Dini-continuous if $\omega_h(t)/t \in L^1([0,\pi])$, where

$$\omega_h(t) = \sup_{|\theta_1 - \theta_2| \le t} \left|h\left(e^{i\theta_1}\right) - h\left(e^{i\theta_2}\right)\right|, \qquad t \in [0,\pi],$$

is the modulus of continuity of h. Specifically [19, chap. III, thm 1.3], it holds that

$$\omega_{\widetilde{h}}(\rho) \le C_2 \left(\int_0^\rho \frac{\omega_h(t)}{t}\,dt + \rho \int_\rho^\pi \frac{\omega_h(t)}{t^2}\,dt\right) \tag{11}$$

where C_2 is a constant independent of f. From (11) it follows easily that $\widetilde{h}$ is continuous if h is Dini-continuous, and moreover that

$$\|\widetilde{h}\|_{L^\infty(\mathbb{T})} \le \omega_{\widetilde{h}}(\pi) \le C_2 \int_0^\pi \frac{\omega_h(t)}{t}\,dt, \tag{12}$$

where the first inequality comes from the fact that $\widetilde{h}$ is continuous on $\mathbb{T}$ and therefore vanishes at some point since it has zero-mean.
We turn to multiplicative properties of Hardy functions. It is well-known (see *e.g.* [17, 19, 23]) that a nonzero $f \in H^p$ can be uniquely factored as $f = jw$ where

$$w(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log|f(e^{i\theta})|\,d\theta\right\} \tag{13}$$

belongs to H^p and is called the *outer factor* of f, while $j \in H^\infty$ has modulus 1 a.e. on $\mathbb{T}$ and is called the *inner factor* of f. The latter may be further decomposed as $j = bS_\mu$, where

$$b(z) = e^{i\theta_0} z^k \prod_{z_l \neq 0} \frac{-\bar{z}_l}{|z_l|}\,\frac{z - z_l}{1 - \bar{z}_l z} \tag{14}$$

is the *Blaschke product*, with order $k \geq 0$ at the origin, associated to the sequence $z_l \in \mathbb{D} \setminus \{0\}$ and to the constant $e^{i\theta_0} \in \mathbf{T}$, while

$$S_\mu(z) = \exp\left\{-\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(\theta)\right\} \tag{15}$$

is the *singular inner factor* associated with μ, a positive measure on $\mathbb{T}$ which is singular with respect to Lebesgue measure. The z_l are of course the zeros of f in $\mathbb{D} \setminus \{0\}$, counted with their multiplicities, while k is the order of the zero at 0. If there are infinitely many zeros, the convergence of the product $b(z)$ in $\mathbb{D}$ is ensured by the condition $\sum_l (1 - |z_l|) < \infty$ which holds automatically when $f \in H^p \setminus \{0\}$. If there are only finitely many z_l, we say that (14) is a finite Blaschke product; note that a finite Blaschke product may alternatively be defined as a rational function of the form q/q^R, where q is an algebraic polynomial whose roots lie in $\mathbb{D}$ and q^R indicates the *reciprocal polynomial* given by $q^R(z) = z^n \overline{q(1/\bar{z})}$ if n is the degree of q. The integer n is also called the degree of the Blaschke product.
That $w(z)$ in (13) is well-defined rests on the fact that $\log|f| \in L^1$ if $f \in H^1 \setminus \{0\}$; this also entails that a H^p function cannot vanish on a subset of strictly positive Lebesgue measure on $\mathbb{T}$ unless it is identically zero. For simplicity, we often say that a function is outer (resp. inner) if it is equal to its outer (resp. inner) factor.
Intimately related to Hardy functions is the Nevanlinna class N^+ consisting of holomorphic functions in $\mathbb{D}$ that can be factored as jE, where j is an inner function and E an outer function of the form

$$E(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log\rho(e^{i\theta})\,d\theta\right\}, \tag{16}$$

with ρ a positive function such that $\log\rho \in L^1(\mathbb{T})$ (although ρ itself need not be summable). Such a function again has nontangential limits of modulus ρ a.e. on $\mathbb{T}$ that serve as a definition of its boundary values. The Nevanlinna class will be instrumental to us in that $N^+ \cap L^p(\mathbb{T}) = H^p$, see for example [17, thm 2.11] or [19, 5.8, ch.II]. Thus, formula (16) defines a H^p-function if and only if $\rho \in L^p(\mathbb{T})$. A useful consequence is that, whenever $g_1 \in H^{p_1}$ and $g_2 \in H^{p_2}$, we have $g_1g_2 \in H^{p_3}$ if, and only if $g_1g_2 \in L^{p_3}$. In particular $g_1g_2 \in H^{p_3}$ if $1/p_1 + 1/p_2 = 1/p_3$.
It is a classical fact [19, ch. II, sec. 1] that a function f holomorphic in the unit disk belongs to H^p if, and only $|f|^p$, which is subharmonic in $\mathbb{D}$, has a harmonic majorant there. This makes for a conformally invariant definition of Hardy spaces over general domains in $\overline{\mathbb{C}}$. In this connection, the Hardy space $\bar{H}^p$ of $\overline{\mathbb{C}} \setminus \mathbb{D}$ can be given a treatment parallel to H^p using the conformal map $z \mapsto 1/z$. Specifically, $\bar{H}^p$ consists of L^p functions whose Fourier coefficients of strictly positive index do vanish; these are, a.e. on $\mathbb{T}$, the complex conjugates of H^p-functions, and they can also be viewed as nontangential limits of functions analytic in $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ having uniformly bounded L^p means over all circles centered at 0 of radius bigger than 1. We also set $\overline{BMOA} = \bar{H}^1 \cap BMO$. We further single out the subspace $\bar{H}^p_0$ of $\bar{H}^p$, consisting of functions vanishing at infinity or, equivalently, having vanishing mean on $\mathbb{T}$. Thus, a function belongs to $\bar{H}^p_0$ if, and only if, it is a.e. on $\mathbb{T}$ of the form $e^{-i\theta}\overline{g(e^{i\theta})}$ for some $g \in H^p$. For $G \in \bar{H}^p_0$, the Cauchy formula assumes the form :

$$G(z) = \frac{1}{2\,i\,\pi}\int_{\mathbb{T}} \frac{G(\xi)}{z - \xi}\,d\xi\,, \qquad z \in \overline{\mathbb{C}} \setminus \mathbb{D}. \tag{17}$$

If E is a measurable subset of $\mathbb{T}$, we set

$$< f, g >_E = \frac{1}{2\pi} \int_E f(e^{i\theta}) \overline{g(e^{i\theta})}\, d\theta \tag{18}$$

whenever $f \in L^p(E)$ and $g \in L^q(E)$ with $1/p + 1/q = 1$. If f and g are defined on a set containing E, we often write for simplicity $< f, g >_E$ to mean $< f_{|E}, g_{|E} >$.
The duality product $< \ , \ >_{\mathbb{T}}$ makes H^p and $\bar{H}_0^q$ orthogonal to each other, and reduces to the familiar scalar product on $L^2(\mathbb{T}) \times L^2(\mathbb{T})$. We note in particular the orthogonal decomposition :

$$L^2(\mathbb{T}) = H^2 \oplus \bar{H}_0^2. \tag{19}$$

For $f \in C(\mathbb{T})$ and $\nu \in \mathcal{M}$, the space of complex Borel measures on $\mathbb{T}$, we set

$$\nu.f = \int_{\mathbb{T}} f(e^{i\theta})\, d\nu(\theta) \tag{20}$$

and this pairing induces an isometric isomorphism between $\mathcal{M}$ (endowed with the norm of the total variation) and the dual of $C(\mathbb{T})$ [33, thm 6.19]. If we let $\mathcal{A} \subset H^\infty$ designate the disk algebra of functions analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, and if $\mathcal{A}_0$ indicates those functions in $\mathcal{A}$ vanishing at zero, it is easy to see that $\mathcal{A}_0$ is the orthogonal space under (20) to those measures whose Fourier coefficients of strictly negative index do vanish. Now, it is a fundamental theorem by F. and M. Riesz that such measures assume the form $d\nu(\theta) = g(e^{i\theta})\, d\theta$ with $g \in H^1$, so the Hahn-Banach theorem implies that H^1 is dual *via* (20) to the quotient space $C(\mathbb{T})/\mathcal{A}_0$ [19, chap. IV, sec. 1]. Equivalently, $\bar{H}_0^1$ is dual to $C(\mathbb{T})/\overline{\mathcal{A}}$ under the pairing arising from the line integral :

$$(\dot{f}, F) = \frac{1}{2i\pi} \int_{\mathbb{T}} f(\xi) F(\xi)\, d\xi\,, \tag{21}$$

where F belongs to $\bar{H}_0^1$ and $\dot{f}$ indicates the equivalence class of $f \in C(\mathbb{T})$ modulo $\overline{\mathcal{A}}$. This entails that, contrary to $L^1(\mathbb{T})$, the spaces H^1 and $\bar{H}_0^1$ enjoy a weak-* compactness property of their unit ball.
Finally, we define the analytic and anti-analytic projections $\mathbf{P}_+$ and $\mathbf{P}_-$ on Fourier series by :

$$\mathbf{P}_+\left(\sum_{n=-\infty}^{\infty} a_n e^{in\theta}\right) = \sum_{n=0}^{\infty} a_n e^{in\theta}, \quad \mathbf{P}_-\left(\sum_{n=-\infty}^{\infty} a_n e^{in\theta}\right) = \sum_{n=-\infty}^{-1} a_n e^{in\theta}\,.$$

Equivalent to the M. Riesz theorem is the fact that $\mathbf{P}_+ : L^p \to H^p$ and $\mathbf{P}_- : L^p \to \bar{H}_0^p$ are bounded for $1 < p < \infty$, in which case they coincide with the Cauchy projections:

$$\mathbf{P}_+(h)(z) = \frac{1}{2i\pi} \int_{\mathbb{T}} \frac{h(\xi)}{\xi - z}\, d\xi, \quad z \in \mathbb{D}, \qquad \mathbf{P}_-(h)(s) = \frac{1}{2i\pi} \int_{\mathbb{T}} \frac{h(\xi)}{s - \xi}\, d\xi, \quad s \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}. \tag{22}$$

When restricted to $L^2(\mathbb{T})$, the projections $\mathbf{P}_+$ and $\mathbf{P}_-$ are just the orthogonal projections onto H^2 and $\bar{H}_0^2$ respectively. Likewise $\mathbf{P}_+ : L^\infty \to BMOA$ and $\mathbf{P}_- : L^\infty \to \overline{BMOA}$ are also bounded.
Although $\mathbf{P}_\pm(h)$ needs not be the Fourier series of a function when h is merely in $L^1(\mathbb{T})$, it is nevertheless Abel summable almost everywhere to a function lying in $L^s(\mathbb{T})$ for $0 < s < 1$, and it can still be interpreted as the trace of an analytic function in the Hardy space of exponent s that we did not introduce [17, cor. to thm 3.2]. To us it will be sufficient, when $h \in L^1$, to regard $\mathbf{P}_\pm(f)$ as the Fourier series of a distribution.
We record for further reference the following elementary fact :

Lemma 1 *Let $v \in L^1(J)$ be such that $\mathbf{P}_+(0 \vee v) \in L^2(\mathbb{T})$. Then, whenever $g \in H^2$ is such that $g \in L^2(I) \vee L^\infty(J)$, it holds that*

$$< \mathbf{P}_+(0 \vee v)\,,\, g >_{\mathbb{T}} = < v\,,\, g >_J\,.$$

Proof.
Since $\mathbf{P}_+(0 \vee v) \in L^2(\mathbb{T})$ by hypothesis, we may define $u \in L^1(\mathbb{T})$ by the formula :

$$u = (0 \vee v) - \mathbf{P}_+(0 \vee v)\,,$$

and by the very definition of u all its Fourier coefficients of non-negative index do vanish hence $u \in \bar{H}^1_0$. Clearly $u_{|_I} \in L^2(I)$ and consequently, if $g \in H^2$ is such that $g \in L^2(I) \vee L^\infty(J)$, we have upon checking summability on I and J separately that $u\bar{g} \in \bar{H}^1_0$. Therefore we get :

$$\begin{aligned} < v\,,\, g >_J \; &= < v\bar{g}\,,\, 1 >_J = < (0 \vee v)\,\bar{g}\,,\, 1 >_{\mathbb{T}} = < u\,\bar{g}\,,\, 1 >_{\mathbb{T}} + < \mathbf{P}_+(0 \vee v)\,\bar{g}\,,\, 1 >_{\mathbb{T}} \\ &= < \mathbf{P}_+(0 \vee v)\,\bar{g}\,,\, 1 >_{\mathbb{T}} = < \mathbf{P}_+(0 \vee v)\,,\, g >_{\mathbb{T}} \end{aligned} \tag{23}$$

where the next-to-last equality uses that the mean of the $\bar{H}^1_0$-function $u\bar{g}$ is zero. ■

3 A bounded extremal problem and its well posedness

We first reduce problem (3) to a standard form where $M \equiv 1$. As the log-modulus of a nonzero Hardy function is integrable, we will safely assume that $\log M \in L^1(J)$ for otherwise the zero function is the only candidate approximant. Then, letting w_M be the outer function with modulus 1 on I and M on J, we have that g belongs to H^2 and satisfies $|g| \leq M$ a.e. on J if, and only if g/w_M lies in H^2 and satisfies $g/w_M \leq 1$ a.e. on J; it is so because g/w_M lies by construction in the Nevanlinna class N^+ whose intersection with $L^2(\mathbb{T})$ is H^2. Altogether, upon replacing f by f/w_M and g by g/w_M, we see that Problem (3) is equivalent to the following special case which is the one we shall really work with :

Normalized Problem

Given $f \in L^2(I)$, find $g_0 \in H^2$ such that $|g_0(e^{i\theta})| \leq 1$ a.e. on J and

$$\|f - g_0\|_{L^2(I)} = \min_{\substack{g \in H^2 \\ |g| \leq 1 \text{ a.e. on } J}} \|f - g\|_{L^2(I)}\,. \tag{24}$$

We begin with a basic existence and uniqueness result :

Theorem 1 *Problem* (24) *has a unique solution* g_0*, and necessarily* $\|g_0\|_{L^2(I)} \leq \|f\|_{L^2(I)}$*. Moreover* $\|g_0\|_{L^\infty(J)} = 1$ *unless* $f = g_{|_I}$ *for some* $g \in H^2$ *such that* $\|g\|_{L^\infty(J)} < 1$.

Corollary 1 *Problem* (3) *has a unique solution.*

Proof of Theorem 1.
Define a convex subset of $L^2(I)$ by $\mathcal{C} \triangleq \{g_{|_I};\;\; g \in H^2,\; \|g\|_{L^\infty(J)} \leq 1\}$. *We claim* that $\mathcal{C}$ is closed. Indeed, let $\{g_n\}$ be a sequence in H^2, with $\|g_n\|_{L^\infty(J)} \leq 1$, that converges in $L^2(I)$ to some ϕ. Clearly $\{g_n\}$ is bounded in $L^2(\mathbb{T})$, therefore some subsequence g_{k_n} converges weakly to $g \in H^2$. Since $|g_{k_n}| \leq 1$ on J, we may assume upon refining the subsequence further that it converges weak-* in $L^\infty(J)$ to a limit which can be none but $g_{|_J}$. By weak-* compactness of balls in $L^\infty(J)$, we get $\|g\|_{L^\infty(J)} \leq 1$, hence $g_{|_I} \in \mathcal{C}$. But $g_{k_n|_I}$ *a fortiori* converges weakly to $g_{|_I}$ in $L^2(I)$, thus $\phi = g_{|_I} \in \mathcal{C}$ *as claimed.* By standard properties of the projection on a non-empty closed convex set in a Hilbert space (note that $0 \in \mathcal{C}$), we deduce that the solution g_0 to (24) uniquely exists, and is characterized by the variational inequality [12, thm V.2.]:

$$g_{0|_I} \in \mathcal{C} \quad \text{ and } \quad \mathrm{Re} < f - g_0\,,\, \phi - g_0 >_I \leq 0, \qquad \forall \phi \in \mathcal{C}. \tag{25}$$

Using $\phi = 0$ in (25) and applying the Schwarz inequality yields $\|g_0\|_{L^2(I)} \leq \|f\|_{L^2(I)}$.
Assume finally that $\|g_0\|_{L^\infty(J)} < 1$. Given $h \in H^\infty$, $g_0 + th$ is a candidate approximant for small $t \in \mathbb{R}$ hence the map $t \mapsto \|f - g_0 - th\|^2_{L^2(I)}$ has a minimum at $t = 0$. Differentiating under the integral sign and equating the derivative to zero yields $2\mathrm{Re} < f - g_0, h >_I = 0$ whence $< f - g_0, h >_I = 0$ upon replacing h by ih. Letting $h = e^{ik\theta}$ for $k \in \mathbb{N}$ we see that $(f - g_0) \vee 0$ lies in $\bar{H}^2_0$, hence it is identically zero because it vanishes on J. Thus $f = g_{0|_I}$ as was to be shown. ■

Theorem 1 entails that the constraint $\|g\|_{L^\infty(J)} \leq 1$ in Problem (24) is saturated (meaning it is an equality) unless $f = g_{0|_I}$. If the boundary of I has measure zero, more in fact is true :

Theorem 2 *Assume that* $\ell(\partial I) = 0$ *and let* g_0 *be the solution to Problem* (24)*. Then* $|g_0| = 1$ *a.e. on* J *unless* $f = g_{|_I}$ *for some* $g \in H^2$ *such that* $\|g\|_{L^\infty(J)} \leq 1$.

It would be interesting to know how much the assumption $\ell(\partial I)=0$ can be relaxed in the above statement. Reducing Problem (3) to Problem (24) as before, we obtain as a corollary :

Corollary 2 *Assume that $\ell(\partial I)=0$ and let g_0 be the solution to Problem (3). If $\log M \in L^1(J)$, then $|g_0(e^{i\theta})| = M(e^{i\theta})$ a.e. on J unless $f = g_{|_I}$ for some $g \in H^2$ such that $|g(e^{i\theta})| \leq M(e^{i\theta})$ a.e. on J.*

To prove Theorem 2 we establish three lemmas, the second of which will be of later use in the paper.

Lemma 2 *Let $E \subset \mathbb{T}$ be infinite and $K_1 \subset \mathbb{T}$ be a compact set such that $\overline{E} \cap K_1 = \emptyset$. If we define a collection $\mathcal{R}$ of rational functions in the variable z by*

$$\mathcal{R} = \{\, c_0 + i \sum_{k=1}^{n} c_k \frac{e^{i\,\psi_k}+z}{e^{i\,\psi_k}-z}\,;\quad c_0\,,\ c_k \in \mathbb{R}\,,\quad e^{i\,\psi_k} \in E\,,\quad 1 \leq k \leq n,\quad n \in \mathbb{N}\,\}\,, \tag{26}$$

then $\mathcal{R}_{|_{K_1}}$ is uniformly dense in $C_{\mathbb{R}}(K_1)$.

Proof.
It is elementary to check that members of $\mathcal{R}$ are real-valued a.e. on $\mathbb{T}$. Also, it is enough to assume that E consists of a sequence $\{e^{i\psi_k}\}_{k\in\mathbb{N}}$ that converges in $\mathbb{T}$ to some $e^{i\psi_\infty}$. We work over the real axis where computations are slightly simpler, and for this we consider the Möbius transform :

$$\varphi(z) = i\,\frac{e^{i\psi_\infty}+z}{e^{i\psi_\infty}-z}\,,$$

that maps $\mathbb{T}$ onto $\mathbb{R}\cup\{\infty\}$ with $\varphi(e^{i\psi_\infty})=\infty$. Set $K_2=\varphi(K_1)$, and note that it is compact in $\mathbb{R}$ since $e^{i\psi_\infty} \notin K_1$. Let $\mathcal{R}_{\mathbb{R}}$ denote the collection of all functions $r\circ\varphi^{-1}$ as r ranges over $\mathcal{R}$. We are now left to prove that the restrictions to K_2 of functions in $\mathcal{R}_{\mathbb{R}}$ are uniformly dense in $C_{\mathbb{R}}(K_2)$. For this, we put $t_k=\varphi(e^{i\psi_k})$ and, denoting by $t=\varphi^{-1}(z)$ the independent variable in $\mathbb{R}$, we compute from (26) that

$$\mathcal{R}_{\mathbb{R}} = \{a_0 + \sum_{k=1}^{n} \frac{b_k}{t-t_k}\,,\ a_0\,,\ b_k \in \mathbb{R}\,,\quad 1 \leq k \leq n\,,\quad n \in \mathbb{N}\},$$

that is to say $\mathcal{R}_{\mathbb{R}}$ is the set of real rational functions bounded at infinity, each pole of which is simple and coincides with some t_k. Thus if $P_{\mathbb{R},n}$ stands for the space of real polynomials of degree at most n, we get

$$\mathcal{R}_{\mathbb{R}} = \left\{ \frac{p_n(t)}{\prod_{k=1}^{n}(t-t_k)}\,,\ p_n \in P_{\mathbb{R},n},\quad 1 \leq k \leq n,\quad n \in \mathbb{N} \right\}\,,$$

where the empty product is 1. We claim that to each $\epsilon>0$ and $p \in P_{\mathbb{R},n}$ there exists $r \in \mathcal{R}_{\mathbb{R}}$ such that

$$||r-p||_{L^\infty(K_2)} \leq \epsilon,$$

and this will achieve the proof since $P_{\mathbb{R},n}$ is dense in $C_{\mathbb{R}}(K_2)$ by the Stone-Weierstrass theorem. To establish the claim, let U be a neighborhood of 0 in $\mathbb{R}^n$ such that

$$\forall (x_1\dots x_n)\in U,\qquad \left|1-\frac{1}{\prod_{k=1}^{n}(1-x_k)}\right| \leq \frac{\epsilon}{1+||p||_{L^\infty(K_2)}}.$$

Next, pick n distinct numbers $t_{k_1},\dots,t_{k_n}$ so large in modulus that $t/t_{k_j} \in U$ for $t \in K_2$ and $1 \leq j \leq n$; this is certainly possible since K_2 is compact whereas $|t_k|$ tends to ∞ because $e^{i\psi_k} \to e^{i\psi_\infty}$. Finally, set

$$r(t) = \frac{p(t)}{\prod_{j=1}^{n}(1-\frac{t}{t_{k_j}})}.$$

Clearly r belongs to $\mathcal{R}_{\mathbb{R}}$, and

$$||p-r||_{L^\infty(K_2)} \leq ||p||_{L^\infty(K_2)} \left\| 1-\frac{1}{\prod_{j=1}^{n}(1-\frac{t}{t_{k_j}})} \right\|_{L^\infty(K_2)} \leq \epsilon$$

as claimed. ■

Lemma 3 *Let $f \in L^2(I)$ and g_0 be the solution to problem* (24). *For h a real-valued Dini-continuous function on $\mathbb{T}$* supported on the interior $\overset{\circ}{I}$ of I, *let*

$$b(z) = \frac{1}{2\pi} \int_I \frac{e^{it} + z}{e^{it} - z}\, h(e^{it})\, dt\,, \qquad z \in \mathbb{D}, \tag{27}$$

be the Riesz-Herglotz transform of h. Then b is continuous on $\overline{\mathbb{D}}$, and moreover

$$\mathrm{Re}\, < (f - g_0)\, \overline{g}_0\,,\, b >_I = 0\,. \tag{28}$$

Proof.
It follows from (11) that b continuous on $\overline{\mathbb{D}}$. For $\lambda \in \mathbb{R}$, consider the function

$$\omega_\lambda(z) = \exp \lambda\, b(z)\,, \qquad z \in \mathbb{D},$$

which is the outer function in H^∞ whose modulus is equal to $\exp \lambda\, h$. Since $|\omega_\lambda| = 1$ on J, the function $g_0\, \omega_\lambda$ is a candidate approximant in problem (24) thus $\lambda \to \|f - g_0\, \omega_\lambda\|^2_{L^2(I)}$ reaches a minimum at $\lambda = 0$. By the boundedness of b, we may differentiate this function with respect to λ under the integral sign, and equating the derivative to 0 at $\lambda = 0$ yields (28). ■

Lemma 4 *Let $f \in L^2(I)$ and g_0 be the solution to Problem* (24). *Then $(f - g_0)\, \overline{g}_0$ has real mean on I :*

$$\mathrm{Re}\, < (f - g_0)\, \overline{g}_0\,,\, i >_I = 0. \tag{29}$$

Proof.
For each $\alpha \in [-\pi, \pi]$, the function $g_0\, e^{i\alpha}$ belongs to H^2 and is a candidate approximant in (24) since it has the same modulus as g_0. Hence the function $\alpha \to \|f - g_0\, e^{i\alpha}\|_{L^2(I)}$ reaches a minimum at $\alpha = 0$, and differentiating under the integral sign yields (29). ■

Proof of Theorem 2.
Since $\partial J = \partial I$ has measure zero, it is equivalent to show that $|g_0| = 1$ a.e. on $\overset{\circ}{J}$. Let

$$E = \{e^{i\theta} \in \overset{\circ}{J}\,,\ |g_0(e^{i\theta})| < 1\}\,,$$

and assume for a contradiction that $\ell(E) > 0$. By countable additivity, there is $\varepsilon > 0$ such that

$$E_\varepsilon = \{e^{i\theta} \in \overset{\circ}{J}\,,\ |g_0(e^{i\theta})| \leq 1 - \varepsilon\}$$

has strictly positive measure. Hence by inner regularity of Lebesgue measure, there is a compact set $K \subset E_\varepsilon$ such that $\ell(K) > 0$, and since $K \subset \overset{\circ}{J}$ it is at distance $\eta > 0$ from I. For $\lambda \in \mathbb{R}$ and F a measurable subset of K, let $w_{\lambda,F}$ be the outer function whose modulus is $\exp \lambda$ on F, and 1 on $\mathbb{T} \setminus F$. By definition $w_{\lambda,F}(z) = \exp\{\lambda\, A_F(z)\}$, where

$$A_F(z) = \frac{1}{2\pi} \int_F \frac{e^{it} + z}{e^{it} - z}\, dt\,, \qquad z \in \mathbb{D} \tag{30}$$

is the Riesz-Herglotz transform of χ_F. For $\lambda < \log(1/(1 - \varepsilon))$ the function $g_0\, w_{\lambda,F}$ belongs to H^2 and satisfies $|g_0\, w_{\lambda,F}| \leq 1$ a.e. on J so that, by definition of g_0, the function $\lambda \to \|f - g_0\, w_{\lambda,F}\|_{L^2(I)}$ reaches a minimum at $\lambda = 0$. From (30), we see that A_F is uniformly bounded on I because $|e^{it} - e^{i\theta}| \geq \eta > 0$ whenever $e^{it} \in F$ and $e^{i\theta} \in I$. Therefore we may differentiate under the integral sign to compute the derivative of $\|f - g_0\, w_{\lambda,F}\|^2_{L^2(I)}$ with respect to λ, which gives us

$$-2\mathrm{Re}\, < f - g_0\, \exp\{\lambda\, A_F\}\,,\, g_0\, A_F\, \exp\{\lambda\, A_F\} >_I\ .$$

Since the latter must vanish at $\lambda = 0$ we obtain

$$\mathrm{Re}\, < f - g_0\,,\, g_0\, A_F >_I = \mathrm{Re}\, < (f - g_0)\, \overline{g}_0\,,\, A_F >_I = 0\,. \tag{31}$$

Let e^{it_0} be a density point of K and I_l denote the arc centered at e^{it_0} of length l, so that $\ell(I_l \cap K)/l \to 1$ as $l \to 0$. In particular $\ell(I_l \cap K) \neq 0$ for sufficiently small l. Noting that

$$\left| \frac{e^{it}+e^{i\theta}}{e^{it}-e^{i\theta}} - \frac{e^{it_0}+e^{i\theta}}{e^{it_0}-e^{i\theta}} \right| \leq 2l/\eta^2 \quad \text{for } e^{it} \in I_l \cap K, \quad e^{i\theta} \in I, \tag{32}$$

and observing that $(f-g_0)\overline{g}_0 \in L^1(I)$, we get from (31)-(32) that

$$\operatorname{Re} < (f-g_0)\overline{g}_0 \,, \frac{e^{it_0}+e^{i\theta}}{e^{it_0}-e^{i\theta}} >_I = \lim_{l\to 0} \operatorname{Re} < (f-g_0)\,\overline{g}_0 \,, \frac{2\pi}{\ell(I_l \cap K)} A_{I_l \cap K} >_I = 0\,. \tag{33}$$

Thus, if we let $\mathcal{D}_K$ denote the set of density points of K, we may recap (33) and (29) by saying that $(f-g_0)\,\overline{g}_0$ is orthogonal to the *real* vector space

$$\mathcal{S}_K = \{ i\,c_0 + \sum_{k=1}^{n} c_k \frac{e^{i\,\phi_k}+z}{e^{i\,\phi_k}-z} \,, \; c_0\,, \; c_k \in \mathbb{R}\,, \; e^{i\,\phi_k} \in \mathcal{D}_K\,, \quad 1 \leq k \leq n, \quad n \in \mathbb{N} \}$$

for the *real* scalar product $\operatorname{Re} < \;,\; >_I$. Since $\ell(\partial I) = 0$ we can replace I by $\bar{I}$ in this product :

$$\operatorname{Re} < (f-g_0)\,\overline{g}_0, \, r >_{\bar{I}} = 0\,, \qquad \forall\, r \in \mathcal{S}_K\,. \tag{34}$$

As $\ell(K) > 0$ and almost every point of K is a density point by Lebesgue's theorem [33, sec. 7.12], the set $\mathcal{D}_K$ is certainly infinite. Moreover, since $K \subset \mathring{J}$, we have that $\overline{I} \cap \overline{\mathcal{D}}_K = \emptyset$. Now, Lemma 2 with $E = \mathcal{D}_K$ and $K_1 = \overline{I}$ implies in view of (34) that

$$\operatorname{Re} < (f-g_0)\,\overline{g}_0 \,, \, i\phi >_{\bar{I}} = 0\,, \qquad \forall \phi \in C_{\mathbb{R}}(\bar{I}), \tag{35}$$

which entails that $(f-g_0)\,\overline{g}_0$ is real-valued a.e. on $\bar{I}$. In particular, if h is a Dini-continuous real function supported on $\mathring{I}$, (35) holds with $\phi = \widetilde{h}_{|_{\bar{I}}}$. Hence by Lemma 3 (where I may be replaced by $\bar{I}$), we get that

$$< (f-g_0)\,\overline{g}_0 \,, \, h >_{\bar{I}} = 0. \tag{36}$$

However, by regularization, Dini-continuous functions are uniformly dense in the space of continuous functions with compact support on $\mathring{I}$ [25, chap. 1, prop. 8]. Therefore (36) in fact holds for every continuous h supported on $\mathring{I}$. Consequently $(f-g_0)\,\overline{g}_0$ must vanish a.e. on $\mathring{I}$ thus also on I. Hence, either $g_0 = f$ a.e. on I or $g_0 = 0$ on a set of positive measure, in which case $g_0 = 0$. In any case, by Theorem 1, f is the trace on I of a H^2-function with modulus at most 1 on J. ■

We now consider the continuity of the solution to problem (24) with respect to the data.

Theorem 3 *Let $f \in L^2(I)$ and g_0 be the solution to problem (24). Assume that $f^{\{n\}}$ converges to f in $L^2(I)$ as $n \to \infty$, and let $g_0^{\{n\}}$ indicate the corresponding solution to problem (24). Then ${g_0^{\{n\}}}_{|_I}$ converges to ${g_0}_{|_I}$ in $L^2(I)$ and ${g_0^{\{n\}}}_{|_J}$ converges weak-* to ${g_0}_{|_J}$ in $L^\infty(J)$. If moreover $\ell(\partial I) = 0$ and f is not the trace on I of a H^2-function less than 1 in modulus a.e. on J, then $g_0^{\{n\}}$ converges to g_0 in $L^2(\mathbb{T})$.*

Proof. By definition $\|g_0^{\{n\}}\|_{L^\infty(J)} \leq 1$, and by Theorem 1 $\|g_0^{\{n\}}\|_{L^2(I)} \leq \|f^{\{n\}}\|_{L^2(I)}$, hence $g_0^{\{n\}}$ is a bounded sequence in H^2. Let g_∞ be a weak accumulation point and $g_0^{\{k_n\}}$ a subsequence converging weakly to g_∞ in H^2; *a fortiori* ${g_0^{\{k_n\}}}_{|_I}$ converges weakly to ${g_\infty}_{|_I}$ in $L^2(I)$. By weak (resp. weak-*) compactness of balls in $L^2(I)$ (resp. $L^\infty(J)$), we get $|g_\infty| \leq 1$ a.e. on J and

$$\|f - g_\infty\|_{L^2(I)} \leq \liminf_{n\to\infty} \|f^{\{k_n\}} - g_0^{\{k_n\}}\|_{L^2(I)}.$$

In particular g_∞ is a candidate approximant, so one has inequalities :

$$\|f-g_0\|_{L^2(I)} \leq \|f-g_\infty\|_{L^2(I)} \leq \liminf_{n\to\infty} \|f^{\{k_n\}} - g_0^{\{k_n\}}\|_{L^2(I)} \leq \limsup_{n\to\infty} \|f^{\{k_n\}} - g_0^{\{k_n\}}\|_{L^2(I)}. \tag{37}$$

If one of these were strict, there would exist $\varepsilon > 0$ such that

$$\|f - g_0\|_{L^2(I)} + \varepsilon \;\leq\; \|f^{\{k_n\}} - g_0^{\{k_n\}}\|_{L^2(I)} \tag{38}$$

for infinitely many n. But $\|f - f^{\{k_n\}}\|_{L^2(I)} < \varepsilon/2$ for large n, thus (38) yields

$$\|f^{\{k_n\}} - g_0\|_{L^2(I)} + \varepsilon/2 \;\leq\; \|f^{\{k_n\}} - g_0^{\{k_n\}}\|_{L^2(I)}$$

contradicting the definition of $g_0^{\{k_n\}}$. Therefore equality holds throughout in (37), whence $g_\infty = g_0$ by the uniqueness part of Theorem 1. Equality in (37) is also to the effect that

$$\lim_{n\to\infty} f^{\{k_n\}} - g_0^{\{k_n\}} = f - g_0 \quad \text{in} \;\; L^2(I)$$

because the norm of the weak limit is not less than the limit of the norms. Refining k_n if necessary, we can assume in addition that ${g_0^{\{k_n\}}}_{|J}$ converges weak-* to some h in $L^\infty(J)$, and since we already know that it converges weakly to ${g_0}_{|J}$ in $L^2(J)$ we get $h = {g_0}_{|J}$. Finally if $\ell(\partial I) = 0$, we deduce from Theorem 2 that $|g_0| = 1$ a.e. on J hence ${g_0^{\{k_n\}}}_{|J}$ converges to ${g_0}_{|J}$ in $L^2(J)$, again because the norm of the weak limit is not less than the limit of the norms. Altogether we have shown that any sequence meeting the assumptions contains a subsequence satisfying the conclusions, which is enough to prove the theorem. ■

To conclude this section, we show that if f has more summability than required, then so does g_0.

Proposition 1 *Assume that $f \in L^p(I)$ for some finite $p > 2$. If g_0 denotes the solution to problem (24) and if $\ell(\partial I) = 0$, then $g_0 \in H^p$ and $\|g_0\|_{L^p(I)} \leq (1 + K_{p/2})\|f\|_{L^p(I)}$.*

Proof. Let h be a Dini-continuous real-valued function supported in $\overset{\circ}{I}$, and b his Riesz-Herglotz transform. Since b has real part h on $\mathbb{T}$, Lemma 3 gives us

$$< |g_0|^2 \,,\, h >_I = \text{Re} \, < f\overline{g}_0 \,,\, b >_I \,. \tag{39}$$

Using Hölder's inequality in (39) and observing that $\|g_0\|_{L^2(I)} \leq \|f\|_{L^2(I)} \leq \|f\|_{L^p(I)}$ in view of Theorem 1 and the fact that $p > 2$ while $\ell(I) < 1$, we obtain

$$\left|< |g_0|^2 \,,\, h >_I\right| \;\leq\; \|f\|_{L^p(I)} \, \|g_0\|_{L^2(I)} \, \|b\|_{L^s(I)} \leq \|f\|^2_{L^p(I)} \, \|b\|_{L^{s_0}(I)}, \qquad 1/p + 1/2 + 1/s_0 = 1.$$

Thus, because the conjugation operator has norm K_{s_0} on $L^{s_0}(\mathbb{T})$ while h is supported on I, we get *a fortiori*

$$\left|< |g_0|^2 \,,\, h >_I\right| \;\leq\; (1 + K_{s_0})\|f\|^2_{L^p(I)} \, \|h\|_{L^{s_0}(I)}. \tag{40}$$

Now, Dini-continuous functions supported on $\overset{\circ}{I}$ are dense in $L^{s_0}(\overset{\circ}{I})$, hence also in $L^{s_0}(I)$ as $\ell(\partial I) = 0$. Therefore (40) implies by duality

$$\|g_0\|_{L^{p_1}(I)} \leq (1 + K_{s_0})^{1/2}\|f\|_{L^p(I)}, \qquad 1/p_1 = (1/p + 1/2)/2. \tag{41}$$

Hölder's inequality in (39), using this time (41) instead of $\|g_0\|_{L^2(I)} \leq \|f\|_{L^p(I)}$, strengthens (40) to

$$\left|< |g_0|^2 \,,\, h >_I\right| \;\leq\; (1 + K_{s_0})^{1/2}(1 + K_{s_1})\|f\|^2_{L^p(I)} \, \|h\|_{L^{s_1}(I)}, \qquad 1/p + 1/p_1 + 1/s_1 = 1,$$

which gives us by duality

$$\|g_0\|_{L^{p_2}(I)} \leq (1 + K_{s_0})^{1/4} \, (1 + K_{s_1})^{1/2} \, \|f\|_{L^p(I)}, \qquad 1/p_2 = (1/p + 1/p_1)/2.$$

Set $1/p_k = (1/p + 1/p_{k-1})/2$ and $1/p + 1/p_k + 1/s_k = 1$. Iterating this reasoning yields by induction

$$\|g_0\|_{L^{p_k}(I)} \leq \|f\|_{L^p(I)} \; \Pi_{j=0}^{k-1}(1 + K_{s_j})^{1/2^{k-j}} \,. \tag{42}$$

As k goes large p_k increases to p and $K_{s_k} = K_{p_{k+1}/2}$ decreases to $K_{p/2}$. Hence the product on the right of (42) becomes arbitrarily close to $1 + K_{p/2}$, and the result now follows on letting $k \to +\infty$. ■

In problem (24), it would be interesting to know whether $g_0 \in BMOA$ when $f \in L^\infty(I)$ and $\ell(\partial I) = 0$.

4 The critical point equation

In any convex minimization problem, the solution is characterized by a variational inequality saying that the *criterium* increases with admissible increments of the variable. If the problem is smooth, infinitesimal increments span a half-space whose boundary hyperplane is tangent to the admissible set, and the variational inequality becomes an equality asserting that the derivative of the objective function is zero on that hyperplane. This equality, sometimes called a *critical point equation*, expresses that the gradient of the objective function in the ambient space is a vector lying orthogonal to the constraint; this vector is an implicit parameter of the critical point equation, known as a *Lagrange parameter.*
In problem (24) the variational inequality is (25), and the non-smoothness of the L^∞-norm makes it *a priori* unclear whether a critical point equation exists. It turns out that it does, at least when $\ell(\partial I) = 0$.

Theorem 4 *Assume that $f \in L^2(I)$ is not the trace on I of a H^2-function of modulus less than or equal to 1 a.e on J, and suppose further that $\ell(\partial I) = 0$. Then, $g_0 \in H^2$ is the solution to problem (24) if, and only if, the following two conditions hold :*

(i) $|g_0(e^{i\theta})| = 1$ for a.e. $e^{i\theta} \in J$,

(ii) there exists a non-negative function $\lambda \in L^1_{\mathbb{R}}(J)$ such that,

$$(g_{0|_I} - f) \vee \lambda\, g_{0|_J} \in \bar{H}^1_0. \tag{43}$$

Moreover, if $f \in L^p(I)$ for some p such that $2 < p < \infty$, then $\lambda \in L^p(J)$.

Remark: note that (43) is equivalent to saying that $(g_{0|_I} - f) \vee \lambda\, g_{0|_J} \in L^1(\mathbb{T})$ and

$$\mathbf{P}_+ \Big((g_{0|_I} - f) \vee \lambda\, g_{0|_J} \Big) = 0 \tag{44}$$

which is the critical point equation proper, with Lagrange parameter λ. Observe that $\log \lambda \in L^1(J)$, otherwise the $\bar{H}^1_0$-function $(g_{0|_I} - f) \vee (\lambda\, g_{0|_J})$ would be zero hence $f = g_{0|_I}$, contrary to the hypothesis. To prove Theorem 4, we need two lemmas the first of which stands somewhat dual to Lemma 3 :

Lemma 5 *Let $f \in L^2(I)$ and g_0 be the solution to problem (24). If h is a non-negative function in $L^\infty(\mathbb{T})$ which is supported on $\mathring{J}$, and if*

$$a(z) = \frac{1}{2\pi} \int_J \frac{e^{i\theta} + z}{e^{i\theta} - z}\, h(e^{i\theta})\, d\theta\,, \qquad z \in \mathbb{D}, \tag{45}$$

denotes its Riesz-Herglotz transform, then a is continuous on $\overline{I}$ and we have that

$$\mathrm{Re} < (f - g_0)\, \overline{g}_0\,,\, a >_I \;\geq\; 0\,. \tag{46}$$

Proof. Since h is supported in $\mathring{J}$, it is clear from the definition that a is continuous on $\overline{I}$. For $t \in \mathbb{R}$, let us put

$$w_t(z) = \exp t\, a(z)\,, \qquad z \in \mathbb{D},$$

which is the outer function in H^∞ with modulus $\exp\{t\, h\}$. As $h \geq 0$, the function $g_0\, w_t$ is a candidate approximant in problem (24) when $t \leq 0$. Since $t \to \|f - g_0\, w_t\|^2_{L^2(I)}$ can be differentiated with respect to t under the integral sign by the boundedness of a on I, its derivative at $t = 0$ must be non-positive by the minimizing property of g_0. But this derivative is just $-2\mathrm{Re} < (f - g_0)\, \overline{g}_0\,,\, a >_I$. ∎

Our second preparatory result is of technical nature :

Lemma 6 *Assume that $f \in L^2(I)$ and let g_0 be the solution to problem* (24)*. If $f \neq {g_0}_{|_I}$ and $\ell(\partial I) = 0$, then there exists a unique $\lambda \in L^1_{\mathbb{R}}(J)$ such that*

$$({g_0}_{|_I} - f)\, \overline{g}_{0|_I} \vee \lambda \ \in \bar{H}^1_0 \,. \tag{47}$$

Necessarily $\lambda \geq 0$ a.e. on J, and if $f \in L^\infty(I)$ then $\lambda \in L^p(J)$ for $1 < p < \infty$. If $f^{\{n\}} \in L^\infty(I)$ converges to f in $L^2(I)$ while $g_0^{\{n\}}$ is the corresponding solution to problem (24)*, and if we write by* (47)

$$\left({g_0}^{\{n\}}_{|_I} - f^{\{n\}}\right) \overline{g}_{0}{}^{\{n\}}_{|_I} \vee \lambda^{\{n\}} \ \in \bar{H}^1_0, \qquad \text{with} \ \ \lambda^{\{n\}} \in L^1_{\mathbb{R}}(J), \tag{48}$$

then the sequence of concatenated functions in (48) *converges weak-* in $\bar{H}^1_0$ to the function* (47).

Proof. The uniqueness of λ is clear because if $\lambda_1 \in L^1_{\mathbb{R}}(J)$ satisfies (47), then $0 \vee (\lambda - \lambda_1) \in \bar{H}^1_0$ so that $\lambda = \lambda_1$. To prove the existence of λ, assume first that $f \in L^\infty(I)$ and fix $p \in (2, \infty)$. By proposition 1 and Hölder's inequality, we know that $(g_0 - f)\,\overline{g}_0 \in L^p(I)$. For h a real-valued function in $L^q(J)$ where $1/q = 1 - 1/p$, let a be the Riesz-Herglotz transform of $0 \vee h$ given by (45) and put

$$\mathcal{L}(h) = \mathrm{Re} < (f - g_0)\,\overline{g}_0 \,,\, a >_I \,. \tag{49}$$

As $0 \vee h$ vanishes on I by construction, it is clear that

$$\mathcal{L}(h) = \mathrm{Re} < (f - g_0)\,\overline{g}_0 \,,\, \widetilde{0 \vee h} >_I,$$

and since the conjugation operator is bounded by K_q on $L^q_{\mathbb{R}}(\mathbb{T})$, we obtain from Hlder's inequality

$$|\mathcal{L}(h)| \ \leq \ K_q \,\|(f - g_0)\,\overline{g}_0\|_{L^p(I)} \,\|h\|_{L^q(J)} \,.$$

Thus $\mathcal{L}$ is a continuous linear form on $L^q_{\mathbb{R}}(J)$ and there exists $\lambda \in L^p_{\mathbb{R}}(J)$ such that

$$\mathcal{L}(h) =< \lambda \,,\, h >_J, \qquad h \in L^q(J). \tag{50}$$

By Lemma 5, $\mathcal{L}$ is a positive functional on bounded functions supported on $\mathring{J}$. Hence $\lambda \geq 0$ a.e. on $\mathring{J}$ thus also on J since $\ell(\partial J) = \ell(\partial I) = 0$. As $\mathrm{Re}\, a = h$ and λ is real-valued, equation (50) gives us

$$\mathcal{L}(h) = \mathrm{Re} < \lambda \,,\, a >_J, \qquad h \in L^q(J), \tag{51}$$

and therefore, substracting (49) from (51), we get

$$\mathrm{Re} < ({g_0}_{|_I} - f)\,\overline{g}_{0|_I} \vee \lambda \ ,\ a >_{\mathbb{T}} = \ 0 \tag{52}$$

whenever a is the Riesz-Herglotz transform of some $h \in L^q_{\mathbb{R}}(J)$.
By regularization Dini-continuous functions are dense in continuous functions with compact support in $\mathring{I}$, so they are dense in $L^q(I)$ since $\ell(\partial I) = 0$. Hence it follows from Lemma 3 and the boundedness of the conjugation operator in $L^q_{\mathbb{R}}(\mathbb{T})$ that

$$\mathrm{Re}\ < (g_0 - f)\,\overline{g}_0 \,,\, b >_I = 0\,. \tag{53}$$

whenever b is the Riesz-Herglotz transform of some $\phi \in L^q_{\mathbb{R}}(I)$. As λ is real-valued and $\mathrm{Re}\, b = 0$ a.e. on J, we may rewrite (53) in the form

$$\mathrm{Re} < ({g_0}_{|_I} - f)\,\overline{g}_{0|_I} \vee \lambda \ ,\ b >_{\mathbb{T}} = \ 0. \tag{54}$$

Now, by (6), every H^q-function is the sum of three terms : a pure imaginary constant, the Riesz-Herglotz transform of $\phi \vee 0$ for some $\phi \in L^q_{\mathbb{R}}(I)$, and the Riesz-Herglotz transform of $0 \vee h$ for some $h \in L^q_{\mathbb{R}}(J)$. Therefore by (54), (52), (29) and the realness of λ, we obtain

$$\mathrm{Re} < ({g_0}_{|_I} - f)\,\overline{g}_{0|_I} \vee \lambda \ ,\ g >_{\mathbb{T}} = \ 0\,, \qquad \forall g \in H^q.$$

Changing g into ig we see that the real part is superfluous and letting $g(e^{i\theta}) = e^{ik\theta}$ for $k \in \mathbb{N}$ we get

$$({g_0}_{|_I} - f)\,\overline{g}_{0|_I} \vee \lambda \ \in \bar{H}^p_0 \,. \tag{55}$$

If f is now an arbitrary function in $L^2(I)$ and $f^{\{n\}}$, $g_0^{\{n\}}$ are as indicated in the statement of the lemma, we know from (55), since $f^{\{n\}} \in L^\infty(I)$, that there is a unique $\lambda^{\{n\}}$ meeting (48). By Theorem 3 we have that $g_0^{\{n\}} \to g_0$ in H^2, hence by the Schwarz inequality

$$\lim_{n\to\infty} \left\| \left(g_0^{\{n\}} - f^{\{n\}}\right) \overline{g}_0^{\{n\}} - (g_0 - f)\,\overline{g}_0 \right\|_{L^1(I)} = 0. \tag{56}$$

Besides, since $\lambda^{\{n\}} \geq 0$ and the mean on $\mathbb{T}$ of a $\bar{H}_0^1$-function is zero, (48) implies

$$\left\|\lambda^{\{n\}}\right\|_{L^1(J)} = \int_J \lambda^{\{n\}}(t)\,dt = \int_I \left(f^{\{n\}} - g_0^{\{n\}}\right) \overline{g}_0^{\{n\}}(t)\,dt \leq \left\| \left(g_0^{\{n\}} - f^{\{n\}}\right) \overline{g}_0^{\{n\}} \right\|_{L^1(I)},$$

and in view of (56) we deduce that $\left\|\lambda^{\{n\}}\right\|_{L^1(J)}$ is bounded independently of n. Consequently the sequence

$$\left(g_0^{\{n\}}{}_{|_I} - f^{\{n\}}\right) \overline{g}_0^{\{n\}}{}_{|_I} \vee \lambda^{\{n\}} \tag{57}$$

has a weak-* convergent subsequence to some F in $\bar{H}_0^1$, regarding the latter as dual to $C(\mathbb{T})/\mathcal{A}$ under the pairing $<\ ,\ >_{\mathbb{T}}$. Checking this convergence on continuous functions supported on the interior of I, we conclude from (56) that $F_{|_{\mathring{I}}} = (g_{0|_I} - f)\,\overline{g}_{0|_I}$ a.e. on $\mathring{I}$ thus also on I. Therefore if we let $\lambda = F_{|_J}$, we meet (47). Checking the same convergence on positive functions supported on $\mathring{J}$, we deduce since $\lambda^{\{n\}} \geq 0$ that $F_{|_J}$ is non-negative. Finally, since F is determined by its trace $(g_{0|_I} - f)\,\overline{g}_{0|_I}$ on I, there is a unique weak-* accumulation point of the bounded sequence (57) which is thus convergent. ■

Proof of Theorem 4.
To prove sufficiency, assume that $g_0 \in H^2$ satisfies $(i) - (ii)$, and let $u \in H^2$ be such that $\|u\|_{L^\infty(J)} \leq 1$. From (44) we get

$$\mathbf{P}_+\left(0 \vee \lambda\, g_{0|_J}\right) = \mathbf{P}_+\left((f - g_{0|_I}) \vee 0\right) \in H^2,$$

thus applying Lemma 1 with $v = \lambda\, g_{0|_J}$ and $g = u - g_0$, we obtain

$$< \lambda g_0\,,\, u - g_0 >_J = - < \mathbf{P}_+\left((f - g_{0|_I}) \vee 0\right)\,,\, u - g_0 >_{\mathbb{T}} = - < f - g_0\,,\, u - g_0 >_I\,. \tag{58}$$

Since Re $< \lambda g_0\,,\, u - g_0 >_J =$ Re $< \lambda\,,\, u\bar{g}_0 - 1 >_J$ is non-negative because $\lambda \geq 0$ and $\mathrm{Re}(u\bar{g}_0) \leq |u| \leq 1$, we see from (58) that (25) is met.
Proving necessity is a little harder. For this, let g_0 solve problem (24) and observe from Theorem 2 that (i) holds. Thus we are left to prove (ii); in fact, we will show that the function λ from Lemma 6 meets (43).
Assume first that $f \in L^\infty(I)$. From Proposition 1 we get in particular $g_0 \in H^4$, and by Lemma 6 there is $\lambda \geq 0$ in $L^2_{\mathbb{R}}(J)$ such that (47) holds with $\bar{H}_0^1$ replaced by $\bar{H}_0^2$. Using (i), we may rewrite this as

$$\left((g_{0|_I} - f) \vee \lambda\, g_{0|_j}\right) \overline{g}_0 \,=\, F, \qquad F \in \bar{H}_0^2\,. \tag{59}$$

Let $g_0 = jw$ be the inner-outer factorization of g_0. We will show that $F \in \bar{j}\bar{H}_0^2$, *and this will achieve the proof when* $f \in L^\infty(I)$. Indeed, dividing (59) by $\overline{g}_0$ then yields

$$(g_{0|_I} - f) \vee \lambda\, g_{0|_j} \,\in\, \bar{w}^{-1}\bar{H}_0^2 \tag{60}$$

which means that the concatenated function in (60) is of the form $e^{-i\theta}\overline{g(e^{i\theta})/w(e^{i\theta})}$ for some $g \in H^2$. However, g/w belongs to the Nevanlinna class N^+ by definition, and it also lies in $L^2(\mathbb{T})$ because so does the function on the left-hand side of (60) (recall $|g_0| = 1$ a.e. on J). Hence $g/w \in H^2$, implying that $e^{-i\theta}\overline{g(e^{i\theta})/w(e^{i\theta})} \in \bar{H}_0^2 \subset \bar{H}_0^1$, as desired.
Let $j = bS_\mu$ where b is the Blaschke product defined by (14) and S_μ the singular inner factor defined by (15). To prove that $F \in \bar{j}\bar{H}_0^2$, it is enough by uniqueness of the inner-outer factorization to establish separately that $F \in \bar{b}\bar{H}_0^2$ and $F \in \bar{S}_\mu\bar{H}_0^2$. To establish the former, it is sufficient to show that $F \in \bar{b}_1\bar{H}_0^2$ whenever b_1 is a finite Blaschke product dividing b, *i.e.* such that $b = b_1 b_2$ with b_2 a Blaschke product.

Pick such a b_1 and put for simplicity $\gamma_0 = b_2 S_\mu w$, so that $g_0 = b_1\gamma_0$. We can write $b_1 = q/q^R$, where q is an algebraic polynomial and $q^R = z^n\overline{q(1/\bar{z})}$ its reciprocal. We may assume that q is monic and $\deg q > 0$:

$$q(z) = z^n + \alpha_{n-1}z^{n-1} + \alpha_{n-2}z^{n-2} + \ldots + \alpha_0\,, \qquad \text{for some } n \in \mathbb{N}\setminus\{0\}.$$

When the set of monic polynomials of degree n gets identified with $\mathbb{C}^n$, taking as coordinates all the coefficients except the leading one, the subset Ω of those polynomials whose roots lie in $\mathbb{D}$ is open. Now, if $Q \in \Omega$ and $b_Q = Q/Q^R$ denotes the associated Blaschke product, the function $g = b_Q\gamma_0$ is a candidate approximant in Problem (24) since $|g| = |g_0|$ on $\mathbb{T}$, thus the map

$$Q \to \|f - \gamma_0\, b_Q\|^2_{L^2(I)} \tag{61}$$

reaches a minimum on Ω at $Q = q$. Let us write a generic $Q \in \Omega$ as

$$Q(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_0.$$

Because $b_Q(e^{i\theta})$ is a rational function in the variables a_j whose denominator is locally uniformly bounded away from 0 on $\mathbb{T}$, we may differentiate (61) under the integral sign with respect to $\mathrm{Re}\, a_j$, $\mathrm{Im}\, a_j$. Since q is a minimum point, equating these partial derivatives to zero at $(a_l) = (\alpha_l)$ yields

$$-2\mathrm{Re}\, < (f - g_0)\,\overline{\gamma}_0\,, \left(x_j \frac{\partial b_Q(e^{i\theta})}{\partial \mathrm{Re}\, a_j} + y_j \frac{\partial b_Q(e^{i\theta})}{\partial \mathrm{Im}\, a_j}\right)\Bigg|_{\substack{a_l = \alpha_l \\ 0\le l\le n-1}} >_I = 0\,, \qquad \forall\, x_j\,,\ y_j \in \mathbb{R}$$

for every $j \in \{0, \ldots, n-1\}$. After a short computation, this gives us

$$\mathrm{Re}\, < (f - g_0)\,\overline{\gamma}_0\ ,\ \frac{(x_j + iy_j)e^{ij\theta}}{q^R(e^{i\theta})} - \frac{(x_j - iy_j)e^{i(n-j)\theta}q(e^{i\theta})}{\left(q^R(e^{i\theta})\right)^2} >_I =\ 0\,, \qquad \forall\, x_j\,,\ y_j \in \mathbb{R}\,,$$

where the second argument in the above scalar product is a function of $e^{i\theta} \in I$. Multiplying both arguments of this product by the unimodular function $\overline{b_1(e^{i\theta})} = q^R/q(e^{i\theta})$ does not affect its value, thus

$$\mathrm{Re}\, < (f - g_0)\,\overline{g}_0\ ,\ \frac{(x_j + iy_j)e^{ij\theta}}{q(e^{i\theta})} - \frac{(x_j - iy_j)e^{i(n-j)\theta}}{q^R(e^{i\theta})} >_I =\ 0\,, \qquad \forall\, x_j\,,\ y_j \in \mathbb{R}\,. \tag{62}$$

In another connection, by the very definition of q^R, we have that

$$\frac{e^{i(n-j)\theta}}{q^R(e^{i\theta})} = \frac{e^{i(n-j)\theta}}{e^{in\theta}\overline{q(e^{i\theta})}} = \overline{\left(\frac{e^{ij\theta}}{q(e^{i\theta})}\right)}$$

hence the second argument of $<\ ,\ >_I$ in (62) is pure imaginary on $\mathbb{T}$, and since λ is real a.e. on J

$$\mathrm{Re}\, < \lambda\ ,\ \frac{(x_j + iy_j)e^{ij\theta}}{q(e^{i\theta})} - \frac{(x_j - iy_j)e^{i(n-j)\theta}}{q^R(e^{i\theta})} >_J =\ 0\,, \qquad \forall\, x_j\,,\ y_j \in \mathbb{R}\,. \tag{63}$$

Therefore, substracting (62) from (63), we obtain from (i) and (59) that

$$\mathrm{Re}\, < F\ ,\ \frac{(x_j + iy_j)e^{ij\theta}}{q(e^{i\theta})} - \frac{(x_j - iy_j)e^{i(n-j)\theta}}{q^R(e^{i\theta})} >_{\mathbb{T}} =\ 0\,, \qquad \forall\, x_j\,,\ y_j \in \mathbb{R}\,. \tag{64}$$

The roots of q^R are reflected from those of q across $\mathbb{T}$, thus lie outside $\overline{\mathbb{D}}$. Hence $e^{i(n-j)\theta}/q^R(e^{i\theta}) \in H^2$, and since $F \in \bar{H}^2_0$ we see from (19) that (64) simplifies to

$$\mathrm{Re}\, < F\ ,\ \frac{(x_j + iy_j)e^{ij\theta}}{q(e^{i\theta})} >_{\mathbb{T}} =\ 0\,, \qquad \forall\, x_j\,,\ y_j \in \mathbb{R}\,.$$

As $x_j + iy_j$ is an arbitrary complex number, the symbol "Re" is redundant in this equation, therefore $< F\,,\, e^{ij\theta}/q(e^{i\theta}) >_{\mathbb{T}} = 0$ for all $j \in \{0, \ldots, n-1\}$ and combining linearly these n equations gives us

$$< F\,,\ \frac{p(e^{i\theta})}{q(e^{i\theta})} >_{\mathbb{T}} =\ 0\,, \qquad \forall p \in P_{n-1}\,, \tag{65}$$

where P_{n-1} is the space of algebraic polynomials of degree at most $n-1$. Now, it is elementary that

$$\bar{b}_1 \bar{H}_0^2 = \frac{q^R}{q}\,\bar{H}_0^2 = \left(\frac{P_{n-1}}{q}\right)^{\perp} \qquad \text{in} \quad \bar{H}_0^2, \tag{66}$$

and consequently from (65) and (66), we see that $F \in \bar{b}_1 \bar{H}_0^2$ as desired.
We turn to the proof that $F \in \bar{S}_\mu \bar{H}_0^2$, assuming that μ is not the zero measure otherwise it is trivial. We need introduce the inner divisors of S_μ which, by uniqueness of the inner-outer factorization, are just the singular factors S_{μ_0} where μ_0 is a positive measure on $\mathbb{T}$ such that $\mu - \mu_0$ is still positive. Pick such a μ_0, and set $\beta_0 = bS_{\mu-\mu_0}w$ so that $g_0 = S_{\mu_0}\beta_0$. For $a \in \mathbb{D}$, consider the function

$$j_a(z) \;=\; \frac{S_{\mu_0}(z) + a}{1 + \bar{a}S_{\mu_0}(z)}\,, \qquad z \in \mathbb{D}.$$

It is elementary to check that j_a is inner, so that $\beta_0 j_a$ is a candidate approximant in problem (24) because $|\beta_0 j_a| = |g_0|$ a.e. on $\mathbb{T}$. Therefore the map

$$a \to \|f - \beta_0\, j_a\|^2_{L^2(I)} \tag{67}$$

reaches a minimum on $\mathbb{D}$ at $a = 0$. Since

$$\frac{\partial j_a(z)}{\partial \mathrm{Re}\, a} \;=\; \frac{1}{1 + \bar{a}S_{\mu_0}(z)} - \frac{S_{\mu_0}(z)(S_{\mu_0}(z) + a)}{(1 + \bar{a}S_{\mu_0}(z))^2}\,, \qquad \frac{\partial j_a(z)}{\partial \mathrm{Im}\, a} \;=\; \frac{i}{1 + \bar{a}S_{\mu_0}(z)} + \frac{iS_{\mu_0}(z)(S_{\mu_0}(z) + a)}{(1 + \bar{a}S_{\mu_0}(z))^2}\,,$$

are bounded for $z \in \mathbb{T}$, locally uniformly with respect to $a \in \mathbb{D}$, we may differentiate (67) under the integral sign with respect to $\mathrm{Re}\, a$ and $\mathrm{Im}\, a$, and equating both partial derivatives to zero at $a = 0$ yields

$$\mathrm{Re} < (f - g_0)\,\overline{\beta}_0 \;,\; (x + iy) - (x - iy)S_{\mu_0}^2 >_I = \,0\,, \qquad \forall x\,,\; y \in \mathbb{R}\,.$$

Multiplying both arguments of $<\;,\;>_I$ by the unimodular function $\overline{S}_{\mu_0}$ we get

$$\mathrm{Re} < (f - g_0)\,\overline{g}_0 \;,\; (x + iy)\,\overline{S}_{\mu_0} - (x - iy)S_{\mu_0} >_I = \,0\,, \qquad \forall x\,,\; y \in \mathbb{R}\,. \tag{68}$$

In another connection, as $(x + iy)\,\overline{S}_{\mu_0} - (x - iy)S_{\mu_0}$ is pure imaginary on $\mathbb{T}$ while λ is real-valued,

$$\mathrm{Re} < \lambda \;,\; (x + iy)\,\overline{S}_{\mu_0} - (x - iy)S_{\mu_0} >_J = \,0\,, \qquad \forall x\,,\; y \in \mathbb{R}\,. \tag{69}$$

Substracting (68) from (69), we deduce from (i) and (59) that

$$\mathrm{Re} < F \;,\; (x + iy)\,\overline{S}_{\mu_0} - (x - iy)S_{\mu_0} >_{\mathbb{T}} = \,0\,, \qquad \forall x\,,\; y \in \mathbb{R}.$$

Since $F \in \bar{H}_0^2$ while $S_{\mu_0} \in H^2$, this simplifies to

$$\mathrm{Re} < F \;,\; (x + iy)\,\overline{S}_{\mu_0}) >_{\mathbb{T}} = \,0\,, \qquad \forall x\,,\; y \in \mathbb{R}\,.$$

But $x + iy$ is arbitrary in $\mathbb{C}$, so the symbol "Re" is redundant in the above equation and we obtain

$$< F \;,\; \overline{S}_{\mu_0}) >_{\mathbb{T}} = \,0. \tag{70}$$

Put $F(e^{i\theta}) = e^{-i\theta}\overline{g(e^{i\theta})}$ with $g \in H^2$, and take conjugates in (70) after multiplying both arguments by $e^{i\theta}$:

$$< g\,,\, e^{-i\theta}S_{\mu_0} >_{\mathbb{T}} = \,0. \tag{71}$$

As S_μ is a nontrivial singular inner factor, it follows from [1, cor. 6.1.] that the closed linear span of the functions $\mathbf{P}_+(e^{-i\theta}S_{\mu_0})$ when S_{μ_0} ranges over all inner divisors of S_μ is equal to $(S_\mu H^2)^\perp$ in H^2. Hence (71) implies that $g \in S_\mu H^2$, and therefore $F \in \overline{S}_\mu\, \bar{H}_0^2$ as announced.
Having completed the proof of necessity when $f \in L^\infty(I)$, we now remove this restriction. Let $f \in L^2(I)$ and $f^{\{n\}} \in L^\infty(I)$ converge to f in $L^2(I)$. Adding to $f^{\{n\}}$ a small $L^2(I)$-function that goes to zero with n if necessary, we may assume that $f^{\{n\}} \notin H^2_{|_I}$. With the notations of Lemma 6, let us put for simplicity

$$F^{\{n\}} \triangleq \left(g_{0|_I}^{\{n\}} - f^{\{n\}}\right)\,\overline{g}_{0|_I}^{\{n\}} \vee \lambda^{\{n\}}, \qquad F \triangleq \left(g_{0|_I} - f\right)\,\overline{g}_{0|_I} \vee \lambda. \tag{72}$$

By the first part of the proof, we can write

$$F^{\{n\}} = \bar{g}_0^{\{n\}} G^{\{n\}}\,, \qquad \text{where } \; G^{\{n\}} \triangleq \left((g_0{}_{|_I}^{\{n\}} - f^{\{n\}}) \vee \lambda^{\{n\}}\, g_0{}_{|_J}^{\{n\}} \right) \in \bar{H}_0^1. \tag{73}$$

Note that $\|G^{\{n\}}\|_{L^1(\mathbb{T})}$ is bounded since $\|f^{\{n\}} - g_0^{\{n\}}\|_{L^2(I)} \le \|f^{\{n\}}\|_{L^2(I)}$ (for the zero function is a candidate approximant) and $\|\lambda^{\{n\}} g_0^{\{n\}}\|_{L^1(J)} = \|\lambda^{\{n\}}\|_{L^1(J)}$ is bounded by Lemma 6. Thus, extracting a subsequence if necessary, we may assume that $G^{\{n\}}$ converges weak-* to some $G \in \bar{H}_0^1$, and then $G^{\{n\}}(z) \to G(z)$ for fixed $z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ by (17). Moreover, still from Lemma 6, we know that $F^{\{n\}}$ converges to F weak-* in $\bar{H}_0^1$, so we get by (17) again that $F^{\{n\}}(z) \to F(z)$ for fixed $z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Finally Theorem 3 entails that $\bar{g}_0^{\{n\}} \to \bar{g}_0$ in $\bar{H}^2$, hence using (17) once more we get that $\bar{g}_0^{\{n\}}(z) \to \bar{g}_0(z)$ for fixed $z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Altogether, in view of (73), we obtain:

$$F(z) = \lim_{n \to \infty} F^{\{n\}}(z) = \bar{g}_0(z)\, G(z)\,, \qquad z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}\,,$$

showing that $F/\bar{g}_0 = G \in \bar{H}_0^1$. By (i) and the definition (72) of F, this yields (43) and achieves the proof. ∎

Using Theorem 4 it is easy to characterize the solution to problem (3). For this, we write $L^1(M^2 d\theta, J)$ to mean those functions h on J such that $hM^2 \in L^1(J)$.

Corollary 3 *Assume that $M \in L^2(J)$ is non-negative with $\log M \in L^1(J)$, and that $f \in L^2(I)$ is not the trace on I of an H^2-function of modulus less than or equal to M a.e on J; suppose further that $\ell(\partial I) = 0$. Then, for $g_0 \in H^2$ to be the solution to problem (3), it is necessary and sufficient that the following two properties hold :*

(i) $|g_0(e^{i\theta})| = M(e^{i\theta})$ for a.e. $e^{i\theta} \in J$,

(ii) there exists a non-negative measurable function $\lambda \in L^1(M^2 d\theta, J)$, such that :

$$(g_0{}_{|_I} - f) \vee \lambda\, g_0{}_{|_J} \in \bar{w}_M^{-1} \bar{H}_0^1, \tag{74}$$

where w_M designates the outer function with modulus 1 a.e. on I and modulus M a.e. on J. In particular if $1/M \in L^\infty(J)$ (more generally if $\lambda M \in L^1(J)$), then (74) amounts to :

$$(g_0{}_{|_I} - f) \vee \lambda\, g_0{}_{|_J} \in \bar{H}_0^1. \tag{75}$$

Proof. Clearly (i) is equivalent to $|g_0/w_M| = 1$ a.e. on J, and since $|w_M|^2 = 1 \vee M^2$ we see on multiplying (74) by $\bar{w}_M$ that it is equivalent to

$$\left(\frac{g_0{}_{|_I}}{w_M} - \frac{f}{w_M} \right) \vee \left(\lambda\, M^2 \right) \frac{g_0{}_{|_J}}{w_M} \in \bar{H}_0^1.$$

The conclusion now follows from Theorem 4 and the reduction of problem (3) to problem (24) given in section 3. If $\lambda M \in L^1(J)$ so does $\lambda g_0{}_{|_J}$ by (i), and the function (74) lies in $e^{-i\theta}\overline{N^+} \cap L^1(\mathbb{T}) = \bar{H}_0^1$. ∎
Relation (75) can be recast as a spectral equation for a Toeplitz operator, which should be compared with those in [3, 8] that form the basis of a constructive approach to BEP$\big(L^2(I), L^2(J)\big)$. There, λ is a constant and the operators involved are continuous. In our case we consider the Toeplitz operator $\phi_{0\vee(\lambda-1)}$ having symbol $0 \vee (\lambda - 1)$, with values in H^2 and domain $\mathcal{D} = \{g \in H^2;\ \lambda g_{|_J} \in L^1(J),\ \mathbf{P}_+(0 \vee \lambda g_{|_J}) \in H^2\}$:

$$\phi_{0\vee(\lambda-1)}(g) = \mathbf{P}_+ \left(0 \vee (\lambda - 1) g_{|_J} \right).$$

By Beurling's theorem [19, chap. II, cor. 7.3] $\phi_{0\vee(\lambda-1)}$ is densely defined, for $\mathcal{D}$ contains $w_\rho H^2$ where w_ρ is the outer function with modulus $1 \vee \min(1, 1/\lambda)$. Note also that $I + \phi_{0\vee(\lambda-1)}$ is injective, because if $g_{|_I} \vee \lambda g_{|_J} \in \bar{H}_0^2$ for some $g \in \mathcal{D}$ we may multiply it by $\bar{g}$ to obtain a $\bar{H}_0^1$-function h which is real-valued on $\mathbb{T}$ and thus identically zero by Poisson representation of $\overline{h(1/\bar{z})} \in e^{i\theta} H^1$.

Corollary 4 *Let $M \in L^2(J)$ be non-negative and $1/M \in L^\infty(J)$. Assume $f \in L^2(I)$ is not the trace on I of a H^2-function of modulus less than or equal to M a.e on J; suppose further that $\ell(\partial I) = 0$. If g_0 is the solution to problem* (3) *and λ is as in (74), then*

$$g_0 = \left(I + \phi_{0\vee(\lambda-1)}\right)^{-1} \mathbf{P}_+(f \vee 0) \,. \tag{76}$$

Proof. From (75) we see that $\lambda {g_0}_{|_J} \in L^1(J)$ and that $\mathbf{P}_+(0 \vee \lambda {g_0}_{|_J}) = \mathbf{P}_+((f - {g_0}_{|_I}) \vee 0) \in H^2$, hence $g_0 \in \mathcal{D}$. Using that $g_0 = \mathbf{P}_+(g_0)$, we now obtain (76) on rewriting (75) as

$$\mathbf{P}_+\Big(g_0 \,+\, 0 \vee (\lambda - 1){g_0}_{|_J} \,-\, f \vee 0\Big) = 0,$$

■

Further smoothness properties of $\lambda M^2 \in L^1(J)$ follow from the next representation formula.

Proposition 2 *Let $M \in L^2(J)$ be non-negative with $\log M \in L^1(J)$, and assume that $f \in L^2(I)$ is not the trace on I of an H^2-function of modulus less than or equal to M a.e. on J. Suppose also that $\ell(\partial I) = 0$. If g_0 denotes the solution to problem* (3) *and $\lambda \in L^1(M^2 d\theta, J)$ is the non-negative function such that (74) holds, then λM^2 extends across $\overset{\circ}{J}$ to a holomorphic function F on $\overline{\mathbb{C}} \setminus \overline{I}$ satisfying*

$$F\,(1/\bar{z}) \;=\; \overline{F(z)}, \qquad z \in \overline{\mathbb{C}} \setminus \overline{I}. \tag{77}$$

Moreover, we have the Herglotz-type representation :

$$F(z) \;=\; -\frac{1}{2i\pi} \int_I \frac{e^{i\theta} + z}{e^{i\theta} - z} \,\mathrm{Im}\left\{f(e^{i\theta})\,\overline{g_0(e^{i\theta})}\right\}\, d\theta\,, \qquad z \in \overline{\mathbb{C}} \setminus \overline{I}. \tag{78}$$

Proof. By (i) of Corollary 3 we know that $|g_0| = M$ a.e. on J, hence multiplying (74) by $\bar{g}_0$ we get

$$\left(|{g_0}_{|_I}|^2 - f\,{\bar{g}_0}_{|_I}\right) \,\vee\, \lambda\, M^2 \;\in\; e^{-i\theta}\, \overline{N}^+ \cap L^1(\mathbb{T}) = \bar{H}^1_0. \tag{79}$$

Call F the concatenated function on the left of (79), so that $H(z) = i\,\overline{F(1/\bar{z})}$ lies in H^1 and vanishes at zero since it has zero mean on $\mathbb{T}$. Clearly H has real part $-\mathrm{Im} f\,{\bar{g}_0}_{|_I} \vee 0$ on $\mathbb{T}$, so the Riesz-Herglotz representation (6) yields :

$$i\,\overline{F(1/\bar{z})} \;=\; H(z) = -\frac{1}{2\pi} \int_I \frac{e^{i\theta} + z}{e^{i\theta} - z} \,\mathrm{Im}\left\{f(e^{i\theta})\,\overline{g_0(e^{i\theta})}\right\}\, d\theta\,, \qquad z \in \mathbb{D}\,,$$

and upon conjugating and changing z into $1/\bar{z}$ we obtain (78) for $z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. As the right-hand side extends analytically to $\mathbb{D}$ across $\overset{\circ}{J}$ by reflection, (77) follows. ■

The interpretation of λ as a Lagrange parameter is justified by the duality relation below. For convenience, we write $L^1_+(M^2 d\theta, J)$ for the set of non-negative functions in $L^1(M^2 d\theta, J)$ whose logarithm lies in $L^1(J)$.

Proposition 3 *Assume that $M \in L^2(J)$ is non-negative with $\log M \in L^1(J)$, and that $f \in L^2(I)$ is not the trace on I of an H^2-function of modulus less than or equal to M a.e on J. Suppose further that $\ell(\partial I) = 0$, and let $g_0 \in H^2$ be the solution to Problem* 3 *with λ as in* (74)*. Then, it holds that*

$$\begin{aligned} \|f - g_0\|^2_{L^2(I)} &= \sup\nolimits_{\mu \in L^1_+(M^2 d\theta, J)} \,\inf\nolimits_{g \in H^2} \Big(\|f - g\|^2_{L^2(I)} + \textstyle\int_J \mu\,(|g|^2 - M^2)\,d\theta\Big) \\ &= \inf\nolimits_{g \in H^2} \,\sup\nolimits_{\mu \in L^1_+(M^2 d\theta, J)} \Big(\|f - g\|^2_{L^2(I)} + \textstyle\int_J \mu\,(|g|^2 - M^2)\,d\theta\Big). \end{aligned} \tag{80}$$

Moreover, the $\sup\inf$ *and the* $\inf\sup$ *in* (80) *are attained for $g = g_0$ and $\mu = \lambda$.*

Proof. Let A, B respectively denote the $\sup\inf$ and the $\inf\sup$ in (80). Setting $g = g_0$ for each μ, we get $\|f - g_0\|^2_{L^2(I)} \geq A$ from Corollary 3-(i). For the reverse inequality, we fix $\mu = \lambda$ and we show that

$$\min_{g \in H^2} \;\|f - g\|^2_{L^2(I)} + \int_J \lambda\,(|g|^2 - M^2)\,d\theta$$

is attained at g_0. Clearly, it is enough to minimize over those $g \in H^2$ such that $\lambda|g|^2 \in L^1(J)$. Pick such a g, and for $t \in \mathbb{R}$ let $g_t = g_0 + t(g - g_0)$. The function

$$\Psi(t) = \|f - g_t\|^2_{L^2(I)} + \int_J \lambda\left(|g_t|^2 - M^2\right) d\theta,$$

is convex and continuously differentiable on $\mathbb{R}$. Differentiating under the integral sign, we get

$$\Psi'(t) = 2\mathrm{Re}\left(< g_t - f,\, g - g_0 >_I + < \lambda g_t,\, g - g_0 >_J\right),$$

and in particular

$$\Psi'(0) = 2\mathrm{Re}\left(< (g_{0|_I} - f) \vee \lambda g_{0|_J}\,,\, g - g_0 >_{\mathbb{T}}\right) = 2\mathrm{Re}\left(< \left((g_{0|_I} - f) \vee \lambda g_{0|_J}\right)(\overline{g} - \overline{g_0})\,,\, 1 >_{\mathbb{T}}\right). \quad (81)$$

Now $(g_0 - f) \vee \lambda g_0 \in e^{-i\theta}\overline{N^+}$ by (74), and since $g - g_0 \in H^2$ it also holds that $\overline{g} - \overline{g_0} \in \overline{N^+}$. Therefore

$$\left((g_{0|_I} - f) \vee \lambda g_{0|_J}\right)(\overline{g} - \overline{g_0}) \in e^{-i\theta}\overline{N^+},$$

and since it belongs to $L^1(\mathbb{T})$ because $\lambda^{1/2} g_{0|_J}$ and $\lambda^{1/2} g_{|_J}$ both lie in $L^2(J)$, we deduce that it is also in $\bar{H}^1_0$. Consequently it has zero mean on $\mathbb{T}$, and we see from (81) that $\Psi'(0) = 0$, hence Ψ meets a minimum at 0 by convexity. Expressing that $\|f - g_0\|^2_{L^2(I)} = \Psi(0) \le \Psi(1)$ for each $g \in H^2$ such that $\lambda|g|^2 \in L^1(J)$ leads us to $\|f - g_0\|^2_{L^2(I)} \le A$, as desired. Thus we have proven the first equality in (80) and we have also shown it is attained for $g = g_0$ and $\mu = \lambda$.
To establish that $\|f - g_0\|^2_{L^2(I)} = B$, observe first that

$$\sup_{\mu \in L^1_+(M^2 d\theta, J)} \left(\|f - g\|^2_{L^2(I)} + \int_J \mu\left(|g|^2 - M^2\right) d\theta\right) = +\infty$$

unless $|g| \le M$ a.e. on J; indeed if $|g| > M$ on a set $E \subset J$ of strictly positive measure, we can set $\mu = \rho\chi_E + \varepsilon$ for fixed $\varepsilon > 0$ and arbitrarily large ρ. Therefore we may restrict the minimization in the second line of (80) to those g such that $|g| \le M$ a.e. on J. For such g the supremum is $\|f - g\|^2_{L^2(I)}$, for the integral term is nonpositive and we can pick μ to be a positive but arbitrary small function. As g_0 minimizes $g \mapsto \|f - g\|^2_{L^2(I)}$ by definition, and since the integral term is always 0 if we put $g = g_0$ by Corollary 3-(i), we may set $g = g_0$ and $\mu = \lambda$ to attain the inf sup. This achieves the proof. ∎
Note that Proposition 3 would still hold if we dropped the log-integrability requirement in the definition of $L^1_+(M^2 d\theta, J)$, for the latter was never needed in the proof. However, this requirement conveniently restricts the maximization space in (80) to a class of μ for which one can form the outer function w_μ, and this will be of use in what follows.

5 The dual functional and Carleman's formulas

For $M \in L^2(J)$ a non-negative function such that $\log M \in L^1(J)$ and $f \in L^2(I)$ which is not the trace on I of a H^2-function of modulus less than or equal to M a.e. on J, we denote by Φ_M the *dual functional* of problem (3) which acts on $L^1_+(M^2 d\theta, J)$ as follows (compare [11, sec. 4.3]):

$$\Phi_M(\mu) = \inf_{g \in H^2}\left(\|f - g\|^2_{L^2(I)} + \int_J \mu\left(|g|^2 - M^2\right) d\theta\right), \qquad \mu \in L^1_+(M^2 d\theta, J). \quad (82)$$

As an *infimum* of affine functions, Φ_M is concave and upper semi-continuous with respect to $\mu \in L^1_+(M^2 d\theta, J)$, when endowed with the natural norm (*i.e.* the L^1 -norm on J with weight M^2). Note that the extra-condition $\log\mu \in L^1(J)$ makes $L^1_+(M^2 d\theta, J)$ non-complete. In view of (80), solving problem (3) amounts to maximize Φ_M over the convex set $L^1_+(M^2 d\theta, J)$. As we shall see momentarily, the true nature of Carleman-type formulas in this context is that they solve for g_0 in (74) as a function of f and λ, and more generally for the optimal g in (82), whenever the inf is attained (*cf.* Proposition 4).

Theorem 5 *Let $M \in L^2(J)$ be non-negative with $\log M \in L^1(J)$, and assume that $f \in L^2(I)$ is not the trace on I of a H^2-function of modulus less than or equal to M a.e. on J. Suppose that $\ell(\partial I) = 0$, and* *let g_0 be the solution to problem (3) with $\lambda \in L^1(M^2 d\theta, J)$ the non-negative function such that (74) holds. Write $w_{\lambda^{1/2}}$ for the outer function with modulus $\lambda^{1/2}$ a.e. on J and modulus 1 a.e. on I. Then*

$$g_0(z) = \frac{1}{2i\pi\, w_{\lambda^{1/2}}(z)} \int_I \frac{w_{\lambda^{1/2}}(\xi)\, f(\xi)}{\xi - z}\, d\xi, \qquad z \in \mathbb{D}. \tag{83}$$

Conversely, if λ is a positive function on J such that $\log \lambda \in L^1(J)$ and if g_0 defined by (83) lies in H_2, then g_0 is the solution to problem (3) where $M = |g_0|_{|_J}$. In this case λ is the function appearing in (74).

Proof. Assume g_0 is the solution to problem (3) so that (i) and (ii) of Corollary 3 hold. Dividing (74) by $\bar{w}_{\lambda^{1/2}}$ and using that $|w_{\lambda^{1/2}}|^2 = 1 \vee \lambda$, we deduce

$$w_{\lambda^{1/2}}(g_0 - (f \vee 0)) \;\in\; \bar{w}_{\lambda^{1/2}}^{-1}\, \bar{w}_M^{-1}\, \bar{H}_0^1.$$

Since $\lambda \in L^1(M^2 d\theta, J)$, the left-hand side lies in $L^2(\mathbb{T})$ and therefore it belongs to $\bar{H}_0^2$ because the right-hand side is in $e^{-i\theta}\overline{N^+}$ by construction. In particular

$$\mathbf{P}_+\left(w_{\lambda^{1/2}}(g_0 - (f \vee 0))\right) = 0. \tag{84}$$

But $w_{\lambda^{1/2}} g_0 \in H^2$ because it clearly belongs to $N^+ \cap L^2(\mathbb{T})$, so that (84) implies:

$$w_{\lambda^{1/2}} g_0 = \mathbf{P}_+(w_{\lambda^{1/2}} g_0) = \mathbf{P}_+\left(w_{\lambda^{1/2}}(f \vee 0)\right).$$

Now, (83) follows from this and (22). Conversely, assume that g_0 defined by (83) lies in H_2 and set $M = |g_0|_{|_J}$. Since $f w_{\lambda^{1/2}} \in L^2(I)$, we see from (83) and (22) that $g_0 w_{\lambda^{1/2}} \in H^2$ and that

$$g_0 w_{\lambda^{1/2}} = \mathbf{P}_+\left(f w_{\lambda^{1/2}} \vee 0\right)$$

which implies (84). Thus $w_{\lambda^{1/2}}(g_0 - (f \vee 0)) \in \bar{H}_0^2$ and multiplying by $\bar{w}_M \bar{w}_{\lambda^{1/2}} \in \bar{H}^2$ yields

$$\bar{w}_M\, |w_{\lambda^{1/2}}|^2\, (g_0 - (f \vee 0)) = \bar{w}_M \left((g_{0|_I} - f) \vee \lambda g_{0|_J}\right) \in \bar{H}_0^1$$

from which (74) follows. As (i) of Corollary 3 is met by definition, g_0 indeed solves for problem (3). ■

Theorem 5 justifies one assertion made in the introduction. Namely if w is outer, we may write it as $w_I w_J$ where w_I (resp. w_J) has modulus 1 on I (resp. J). Then, expression (2) coincides with formula (83) if f gets replaced by $f w_I$, g_0 by $g_w w_I$, and $w_{\lambda^{1/2}}$ by w_J. So, if w is invertible in H^∞, it follows from the theorem that g_w is a best approximant to f in $L^2(|w|_I^2 d\theta, I)$ among those H^2 functions not exceeding $|g_w|$ in modulus, pointwise on J.
Next, we compute the dual functional $\Phi_M(\mu)$ introduced in (82).

Proposition 4 *Let $M \in L^2(J)$ be non-negative with $\log M \in L^1(J)$, and assume that $f \in L^2(I)$ is not the trace on I of an H^2-function of modulus less than or equal to M a.e. on J. Suppose further that $\ell(\partial I) = 0$ and let $\mu \in L^1_+(M^2 d\theta, J)$. Write $w_{\mu^{1/2}}$ for the outer function with modulus $\mu^{1/2}$ a.e. on J and modulus 1 a.e. on I. Then, the function $\Phi_M(\mu)$ defined by (82) can be expressed as*

$$\Phi_M(\mu) = \left\|\mathbf{P}_-\left(f w_{\mu^{1/2}} \vee 0\right)\right\|^2_{L^2(\mathbb{T})} - \left\|\mu^{1/2} M\right\|^2_{L^2(J)}. \tag{85}$$

Moreover, if we set

$$g_\mu(z) = \frac{1}{2i\pi\, w_{\mu^{1/2}}(z)} \int_I \frac{w_{\mu^{1/2}}(\xi)\, f(\xi)}{\xi - z}\, d\xi, \qquad z \in \mathbb{D}, \tag{86}$$

then the infimum in the right-hand side of (82) is attained at $g = g_\mu$, whenever the latter belongs to H^2.

Proof. Assume first that μ is such that $g_\mu \in H^2$; this holds in particular when $1/\mu \in L^\infty(J)$, because then $1/w_{\mu^{1/2}} \in H^\infty$ while (22) shows that the integral in (86) lies in H^2. From Theorem 5 it follows that g_μ is the solution to problem (3) where M gets replaced by $|g_\mu|$, and μ plays the role of λ in (74). Hence Proposition 3 implies that g_μ is an infimizer of

$$\inf_{g \in H^2} \left(\|f - g\|^2_{L^2(I)} + \int_J \mu\,(|g|^2 - |g_\mu|^2)\, d\theta\right),$$

and since μ is kept fixed g_μ is clearly also an infimizer of

$$\inf_{g\in H^2}\Big(\|f-g\|^2_{L^2(I)}+\int_J \mu\,(|g|^2-M^2)\,d\theta\Big)$$

which is but the right-hand side of (82). This proves the second assertion of the proposition.
By (86)) and (22), taking into account that $|w_{\mu^{1/2}}|=1\vee\mu^{1/2}$, what precedes can be reformulated as

$$\begin{aligned}\Phi_M(\mu) &= \|f-g_\mu\|^2_{L^2(I)}+\textstyle\int_J \mu\,(|g_\mu|^2-M^2)\,d\theta\\ &= \left\|(w_{\mu^{1/2}}f\vee 0)-w_{\mu^{1/2}}g_\mu\right\|^2_{L^2(\mathbb{T})}-\textstyle\int_J \mu\,M^2\,d\theta\\ &= \left\|P_{\bar H^2_0}\left(fw_{\mu^{1/2}}\vee 0\right)\right\|^2_{L^2(\mathbb{T})}-\left\|\mu^{1/2}M\right\|^2_{L^2(J)}.\end{aligned}$$

This proves (85) when $g_\mu\in H^2$. To get it in general, we apply what we just did to the sequence $\mu_n=\mu+1/n$, observing that $g_{\mu_n}\in H^2$ because $1/\mu_n\in L^\infty(J)$. By monotone convergence we obtain

$$\lim_{n\to\infty}\left\|\mu_n^{1/2}M-\mu^{1/2}M\right\|_{L^2(J)}=0. \tag{87}$$

Moreover, as $\log\mu_n$ decreases to $\log\mu$, we certainly have on putting $\log^-(x)=\max\{-\log x,0\}$ and $\log^+(x)=\max\{\log x,0\}$ that

$$\begin{aligned}\log^-\mu_n &\le \log^-\mu\le|\log\mu|\in L^1(J),\\ \log^+\mu_n &\le \log^+\left(\mu_nM^2\right)+\left|\log M^2\right|\le\left|\mu_nM^2-1\right|+2|\log M|\\ &\le (\mu+1)M^2+1+2|\log M|\in L^1(J),\end{aligned}$$

and therefore, by dominated convergence as applied to $\log\mu_n=\log^+\mu_n-\log^-\mu_n$, we obtain

$$\lim_{n\to\infty}\exp\left\{\frac{1}{4\pi}\int_J\frac{e^{it}+z}{e^{it}-z}\log\mu_n dt\right\}=\exp\left\{\frac{1}{4\pi}\int_J\frac{e^{it}+z}{e^{it}-z}\log\mu dt\right\},\qquad z\in\overset{\circ}{I}.$$

In other words, $w_{\mu_n^{1/2}}$ converges pointwise to $w_{\mu^{1/2}}$ on $\overset{\circ}{I}$ and therefore almost everywhere on I since $\ell(\partial I)=0$. Thus, appealing to dominated convergence once more, we get

$$\lim_{n\to\infty}\left\|fw_{\mu_n^{1/2}}-fw_{\mu^{1/2}}\right\|_{L^2(I)}=0, \tag{88}$$

and from (87), (88), and (85) which is known to hold with μ replaced by μ_n, we see that

$$\lim_{n\to\infty}\Phi_M(\mu_n)=\left\|\mathbf{P}_-\left(fw_{\mu^{1/2}}\vee 0\right)\right\|^2_{L^2(\mathbb{T})}-\left\|\mu^{1/2}M\right\|^2_{L^2(J)}. \tag{89}$$

In another connection, it is plain that

$$\limsup_{n\to\infty}\Phi_M(\mu_n)\le\Phi_M(\mu)\le\liminf_{n\to\infty}\Phi_M(\mu_n), \tag{90}$$

where the first inequality comes from (87) and the upper semi-continuity of Φ_M in $L^1_+(M^2d\theta,J)$ while the second inequality is obvious from (82), (87), and the fact that $\mu\le\mu_n$. Now (85) follows from (89) and (90). ■

We mentioned early in Section 3 that problem (3) reduces to the case where $M\equiv 1$. If moreover $f\in L^\infty(I)$ and we let $A_{f\vee 0}:H^2\to\bar H^2_0$ denote the Hankel operator with symbol $f\vee 0$ defined by $A_{f\vee 0}(u)=\mathbf{P}_-((f\vee 0)u)$, Proposition 4 yields a formula for the value of the criterion which may be compared with the Nehari theorem (see *e.g.* [29, Thm. 1.3.2]):

Corollary 5 *When $f\in L^\infty(I)\setminus H^2_{|_I}$ and $\ell(\partial I)=0$, the squared value of problem* (24) *is:*

$$\sup_{u\in H^2,\ |u_{|_I}|\equiv 1}\Big(\|A_{f\vee 0}(u)\|^2_{L^2(\mathbb{T})}-\|u\|^2_{L^2(\mathbb{T})}+\ell(I)\Big). \tag{91}$$

Proof. This is straightforward from (85) and the first half of (80), except that the maximization bears on outer u only. However, since for every inner function Θ it holds that

$$\|\mathbf{P}_-((f\vee 0)\Theta u)\|_2 = \|\mathbf{P}_-(\Theta(\mathbf{P}_-((f\vee 0)u))\|_2 \leq \|\Theta(\mathbf{P}_-((f\vee 0)u)\|_2 = \|\mathbf{P}_-((f\vee 0)u)\|_2,$$

maximizing over all $u\in H^2$ having modulus 1 a.e. on I does not increase the value of the problem. ■

Being concave on the convex set $L^1_+(M^2d\theta, J)$, the functional Φ_M has a directional derivative at every point in each admissible direction. Here, a direction h is said to be admissible at $\mu\in L^1_+(M^2d\theta,J)$ if $\mu+th\in L^1_+(M^2d\theta,J)$ as soon as $t\geq 0$ is small enough. From a constructive viewpoint, computing this derivative is important when designing ascent algorithms to maximize Φ_M and thus numerically solve for problem (3). It also sheds light on the role of λ as a "pointwise" Lagrange multiplier. The next proposition does such a computation, under mild assumptions on f, in those directions h such that $h/\mu\in L^\infty(J)$. Note, since $\mu\neq 0$ a.e. (for $\log\mu\in L^1(J)$), that such directions are dense in the set of all admissible directions, hence this result allows one indeed to find a direction of ascent for Φ_M.

Proposition 5 *Assumptions and notations being as in Proposition* 4*, suppose in addition that* $|f|\in L^p(I)$ *for some* $p>2$*. Let further* h *be a real function on* J *such that* $\|h/\mu\|_{L^\infty(J)}<1$*. Then* $\mu+h\in L^1_+(M^2d\theta,J)$ *and* $h\in L^1(M^2d\theta,J)$*. Moreover, defining* g_μ *as in* (86)*, it holds that* $h|g_\mu|^2\in L^1(J)$ *and that*

$$\left|\Phi_M(\mu+h)-\Phi_M(\mu)-\int_J h(|g_\mu|^2-M^2)\,d\theta\right| = \mathbf{o}\left(\|h/\mu\|_{L^\infty(J)}\right), \tag{92}$$

where the function $\mathbf{o}K$ *is a little "o" of its argument near* 0*, uniformly with respect to* μ*.*

Proof. Clearly $\mu+h=\mu(1+h/\mu)\in L^1_+(M^2d\theta,J)$ whenever $\|h/\mu\|_{L^\infty(J)}<1$, which in turn entails $h\in L^1(M^2d\theta,J)$. Using (22), we may rewrite (86) in the following form:

$$w_{\mu^{1/2}}g_\mu = \mathbf{P}_+(fw_{\mu^{1/2}}\vee 0), \tag{93}$$

and since $|w_{\mu^{1/2}}|=1$ on I we get that $w_{\mu^{1/2}}g_\mu\in H^2$ with norm at most $\|f\|_{L^2(I)}$ because $\mathbf{P}_+$ is a contraction in $L^2(\mathbb{T})$. As $|w_{\mu^{1/2}}|^2=\mu$ on J, we thus have that $h|g_{\mu|_J}|^2=(h/\mu)|w_{\mu^{1/2}}g_\mu|^2_{|_J}\in L^1(J)$ with norm bounded by $\|f\|^2_{L^2(I)}$ when $\|h/\mu\|_{L^\infty(J)}<1$. In particular, the integral in the left-hand side of (92) is well-defined. Next, multiplying the $\bar{H}^2_0$-function $w_{\mu^{1/2}}g_\mu-(w_{\mu^{1/2}}f\vee 0)=-\mathbf{P}_-(w_{\mu^{1/2}}f\vee 0)$ by the $\bar{H}^2$-function $\overline{w_{\mu^{1/2}}g_\mu}$ (compare (84)) yields that

$$(|g_{\mu|_I}|^2 - f\bar{g}_{\mu|_I})\vee\mu|g_{\mu|_J}|^2\in\bar{H}^1_0,$$

with norm at most $\|f\|^2_{L^2(I)}$. Therefore the conjugate function of $(|g_{\mu_{|_I}}|^2-\mathrm{Re}(\bar{f}g_{\mu|_I}))\vee\mu|g_{\mu_{|_J}}|^2$ lies in $L^1(\mathbb{T})$, and by Zygmund's theorem so does the conjugate function of $|f|^2\vee 0$ since the latter is nonnegative and lies in $L^{p/2}(\mathbb{T})$ by assumption, thus *a fortiori* in H^1 because $p>2$. Adding up yields

$$\overbrace{\left(\frac{|g_\mu|^2_{|_I}+|f|^2}{2}+\frac{|g_{\mu|_I}-f|^2}{2}\right)}^{\sim}\vee\mu|g_\mu|^2_{|_J}\in L^1(\mathbb{T}),$$

with norm bounded by some constant $C(f)$ depending only on f, and since the function under brace is positive it lies in $L\log^+ L$ with norm bounded by some constant $C'(f)$, thanks to the M. Riesz theorem (*cf.* (8) and the remark thereafter). *A fortiori* then,

$$\left|\mathbf{P}_-(fw_{\mu^{1/2}|_I}\vee 0)\right|^2 = \left|(fw_{\mu^{1/2}|_I}\vee 0)-w_{\mu^{1/2}}g_\mu\right|^2 = |g_{\mu|_I}-f|^2\vee\mu|g_\mu|^2_{|_J}\in L\log^+ L \tag{94}$$

with norm bounded by $C''(f)$, where we used (93) again. Let us write

$$w_{(\mu+h)^{1/2}}(z) = w_{\mu^{1/2}}(z)\exp\left\{\frac{1}{4\pi}\int_J\frac{e^{i\theta}+z}{e^{i\theta}-z}\log(1+h/\mu)(e^{i\theta})\,d\theta\right\} = w_{\mu^{1/2}}(z)\,e^{\Delta_h(z)},$$

where we have put for simplicity

$$\Delta_h(z) = \frac{1}{4\pi}\int_J\frac{e^{i\theta}+z}{e^{i\theta}-z}\log(1+h/\mu)(e^{i\theta})\,d\theta, \qquad z\in\mathbb{D}. \tag{95}$$

Note that $\Delta_h \in BMOA$ and $e^{\Delta_h} \in H^\infty$ since $\log(1+h/\mu) \in L^\infty(J)$ for $\|h/\mu\|_{L^\infty(J)} < 1$. Now, it is straightforward that

$$\begin{aligned} \left\|\mathbf{P}_-\left(fw_{(\mu+h)^{1/2}} \vee 0\right)\right\|^2_{L^2(\mathbb{T})} &- \left\|\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right\|^2_{L^2(\mathbb{T})} = \left\|\mathbf{P}_-\left(fw_{\mu^{1/2}}(e^{\Delta_h}-1) \vee 0\right)\right\|^2_{L^2(\mathbb{T})} \\ &+ 2\mathrm{Re} < \mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right), \mathbf{P}_-\left(fw_{\mu^{1/2}}(e^{\Delta_h}-1) \vee 0\right) >_{\mathbb{T}}, \end{aligned} \tag{96}$$

and our next goal is to prove that

$$\left| 2\mathrm{Re} < \mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right) \ , \ \mathbf{P}_-\left(fw_{\mu^{1/2}}(e^{\Delta_h}-1) \vee 0\right) >_{\mathbb{T}} - \int_J h|g_\mu|^2 \, d\theta \right| = \mathbf{o}\left(\|h/\mu\|_{L^\infty(J)}\right). \tag{97}$$

For this, since $\mathbf{P}_+ + \mathbf{P}_- = \mathrm{id}$ (the identity operator), we observe from (19) that

$$\begin{aligned} < \mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right), \mathbf{P}_-\left(fw_{\mu^{1/2}}(e^{\Delta_h}-1) \vee 0\right) >_{\mathbb{T}} &= < \mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right), (e^{\Delta_h}-1)(fw_{\mu^{1/2}} \vee 0) >_{\mathbb{T}} \\ =< \mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right), (e^{\Delta_h}-1)\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right) >_{\mathbb{T}} &= < \left|\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2, e^{\Delta_h}-1 >_{\mathbb{T}} \end{aligned}$$

where we used in the second equality that $(e^{\Delta_h}-1)\mathbf{P}_+\left(fw_{\mu^{1/2}} \vee 0\right) \in H^2$ for $e^{\Delta_h}-1 \in H^\infty$. Besides,

$$\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right) + \mathbf{P}_+\left(fw_{\mu^{1/2}} \vee 0\right) = 0 \quad a.e. \text{ on } J,$$

which implies in view of (93) that

$$\int_J h|g_\mu|^2 = \int_J \frac{h}{\mu}\left|\mathbf{P}_+\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2 =< \left|\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2, 0 \vee h/\mu >_{\mathbb{T}}.$$

Altogether, the expression inside absolute values on the left-hand side of (97) is therefore equal to

$$< \left|\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2, \mathrm{Re}\left(2(e^{\Delta_h}-1) - (0 \vee h/\mu)\right) >_{\mathbb{T}}.$$

By (95), it holds on $\mathbb{T}$ that $2\Delta_h = 0 \vee \log(1+h/\mu) + i\varphi$ where φ denotes the conjugate function of $0 \vee \log(1+h/\mu)$. Thus, the quantity above can be rewritten as $Q_1 + Q_2$ with

$$\begin{aligned} Q_1 &\triangleq 2 < \left|\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2 \ , \ (\cos(\varphi/2)-1)\left(1 \vee (1+h/\mu)^{1/2}\right) >_{\mathbb{T}}, \\ Q_2 &\triangleq 2 < \left|\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2 \ , \ (1+h/\mu)^{1/2} - 1 - h/(2\mu) >_J. \end{aligned}$$

We prove separately that both Q_1 and Q_2 are $\mathbf{o}\left(\|h/\mu\|_{L^\infty(J)}\right)$; hereafter, we use the same symbol $\mathbf{o}$ for different functions as this causes no confusion. On the one hand, there is an absolute constant C such that $\left|(1+h/\mu)^{1/2} - 1 - h/(2\mu)\right| < C\|h/\mu\|^2_{L^\infty(J)}$ for $\|h/\mu\|_{L^\infty(J)} < 1$, therefore

$$|Q_2| \le 2C\|f\|^2_{L^2(\mathbb{T})}\|h/\mu\|^2_{L^\infty(J)} \tag{98}$$

which is indeed $\mathbf{o}\left(\|h/\mu\|_{L^\infty(J)}\right)$, where "$\mathbf{o}$" is independent of μ. On the other hand, as $\cos(\varphi/2)-1 \le 0$, it holds for $\|h/\mu\|_{L^\infty(J)} < 1$ that

$$|Q_1| \le 2\sqrt{2} < \left|\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2 \ , \ 1 - \cos(\varphi/2) >_{\mathbb{T}}. \tag{99}$$

Put for simplicity $F \triangleq \left|\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2$ and $B(\varphi) \triangleq (1-\cos(\varphi/2))/\varphi$, noting that $|B(\varphi)|$, hence also $|B(\varphi)|\log^+|B(\varphi)|$, is bounded above independently of φ. Then, it holds that

$$< \left|\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2 \ , \ 1 - \cos(\varphi/2) >_{\mathbb{T}} \ = \ < B(\varphi)F \ , \ \varphi >_{\mathbb{T}}. \tag{100}$$

Now, by (9), we have that $\|\varphi\|_{BMO} \le C_1\|\log(1+h/\mu)\|_{L^\infty(J)}$, and by (8) the function $B(\varphi)F + i\widetilde{B(\varphi)F}$ lies in H^1 with norm at most $C_0\|B(\varphi)F\|_{L\log^+ L}$. Therefore, by Fefferman's duality [19, Ch. VI, Thm. 4.4], we get that

$$|< B(\varphi)F \ , \ \varphi >_{\mathbb{T}}| \le C_4\|B(\varphi)F\|_{L\log^+ L}\|\log(1+h/\mu)\|_{L^\infty(J)} \tag{101}$$

for some absolute constant C_4. Hence, by the inequality $|\log(1+h/\mu)| \le 2|h/\mu|$ which is valid for $|h/\mu| \le 1/2$, we obtain from (100)-(101) that

$$< \left|\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2 \ , \ 1 - \cos(\varphi/2) >_{\mathbb{T}} \ \le \ 2\left\|B(\varphi)\left(\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right)^2\right\|_{L\log^+ L}\|h/\mu\|_{L^\infty(J)}$$

as soon as $\|h/\mu\|_{L^\infty(J)} < 1/2$. Therefore, to prove that (99) is $\mathbf{o}(\|h/\mu\|_{L^\infty(J)})$, it is enough to show that

$$\lim_{\|h/\mu\|_{L^\infty(J)}\to 0} \left\| B_h \left(\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right)^2 \right\|_{L\log^+ L} = 0. \tag{102}$$

But $F = \left|\mathbf{P}_-\left(fw_{\mu^{1/2}} \vee 0\right)\right|^2$ is bounded in $L^{p/2}(\mathbb{T})$-norm, independently of μ, by $K_p\|f\|_p^2$ in view of the M. Riesz theorem and because $w_{\mu^{1/2}}$ has modulus 1 on I, hence $F\log^+ F$ is bounded in $L^\alpha(\mathbb{T})$ for every $1 < \alpha < p/2$. Moreover, as $B(\varphi)$ is a C^1-smooth function of φ, it holds that $\|B(\varphi)\|_{BMO} \le C_5\|\varphi\|_{BMO}$ for some absolute constant C_5. Consequently, since $\|\varphi\|_{BMO} \le 2C_1\|h/\mu\|_{L^\infty}$ for $|h/\mu| \le 1/2$, the limiting relation (102) follows from Hölder's inequality and the fact that the BMO norm dominates the $L^q(\mathbb{T})$ norm for all $1 < q < \infty$.
In the same vein we show that

$$\left\|\mathbf{P}_-\left(fw_{\mu^{1/2}}(e^{\Delta_h}-1) \vee 0\right)\right\|_{L^2(\mathbb{T})}^2 = \mathbf{o}\left(\|h/\mu\|_{L^\infty(J)}\right). \tag{103}$$

Indeed, since $\mathbf{P}_-$ is a contraction in $L^2(\mathbb{T})$ and $|w_{\mu^{1/2}}| \equiv 1$ on I, we have that

$$\left\|\mathbf{P}_-\left(fw_{\mu^{1/2}}(e^{\Delta_h}-1) \vee 0\right)\right\|_{L^2(\mathbb{T})}^2 \le \; < |f|^2\,,\ \left|e^{\Delta_h}-1\right|^2 >_{L^2(I)} = 2 < |f|^2\,,\ (1-\cos(\varphi/2)) >_{L^2(I)}$$

which can be treated like the right-hand side of (99) to obtain (103). In view of (85), (96), (97) and (103), the proof is complete once we have observed that

$$\left\|(\mu+h)^{1/2}M\right\|_{L^2(J)}^2 - \left\|(\mu)^{1/2}M\right\|_{L^2(J)}^2 = \int_J hM^2. \tag{104}$$

■

Remark: it is unclear whether Proposition 5 holds true as soon as $|f|^2 \in L\log^+ L$. In this case, the difficulty is of course to prove (102). When f merely lies in $L^2(I)$, it is easy to check using (12), (98), (99), and (104) that

$$\left|\Phi_M(\mu+h) - \Phi_M(\mu) - \int_J h(|g_\mu|^2 - M^2)\,d\theta\right| = \mathbf{O}\left(\left(\|h/\mu\|_{L^\infty(J)}^2 + \int_0^\pi \frac{\omega_{0\vee h/\mu}(t)}{t}\,dt\right)^2\right), \tag{105}$$

which is a weak substitute to (92) under the (much stronger) assumption that $0 \vee h/\mu$ is Dini-continuous. When $f \in L^p(I)$ with $p > 2$, Proposition 5 may be used to constructively approach problem (3) from the point of view of convex optimization using an ascent algorithm, thanks to the uniform character of the function $\mathbf{o}$ with respect to μ in (92).

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