Solutions to conjugate Beltrami equations and approximation in generalized Hardy spaces

Y. Fischer and J. Leblond

INRIA Sophia Antipolis Méditérranée, 2004 routes des Lucioles, BP 93, 06902 Sophia Antipolis

Yannick.Fischer@sophia.inria.fr Juliette.Leblond@sophia.inria.fr

Abstract. We present a constructive method for the robust approximation to solutions of some elliptic equations in a plane domain from incomplete and corrupted boundary data. We state this inverse problem in generalized Hardy spaces of functions satisfying the conjugate Beltrami equation, of which we give some properties, in the Hilbertian framework. The issue is then reworded as a constrained approximation (bounded extremal) problem which is shown to be well-posed. A practical motivation comes from modelling plasma confinement in a tokamak reactor. There, the particular form of the conductivity coefficient leads to Bessel-exponential type families of solutions of which we establish density properties.

1 Introduction

We consider the elliptic equation in the open unit disc $\mathbb{D} \subset \mathbb{R}^2$

$$\operatorname{div}(\sigma \nabla u) = 0 \text{ in } \mathbb{D}\,, \tag{1}$$

whenever σ is a real-valued Lipschitz-continuous function on $\mathbb{D}$

$$\sigma \in W^{1,\infty}(\mathbb{D}) \qquad \text{such that} \qquad 0 < c \le \sigma \le C \quad \text{a.e. in} \quad \mathbb{D}\,, \tag{2}$$

for two constants $0 < c < C < +\infty$. Let $\mathbb{T} = \partial\mathbb{D}$ be the unit circle and $\partial_n u$ denote the normal derivative of u on $\mathbb{T}$, w.r.t. the (outer) unit normal vector. Let further $I \subset \mathbb{T}$ be an open (non-empty) subset of $\mathbb{T}$ and $J = \mathbb{T} \setminus \overline{I}$ that we also assume to be of non-empty interior.
Assuming σ satisfying (2) to be known, we look at the following inverse problem *(CP)* of Cauchy type:

(CP) Given u and $\partial_n u$ on the subset $I \subset \mathbb{T}$, recover u solution to (1) and $\partial_n u$ on the complementary subset J.

Even for smooth data on I, these Cauchy-type interpolation issues are ill-posed in the sense of Hadamard. In order to ensure well-posedness for $L^2(I)$ boundary data, we will

turn to a bounded extremal (approximation) problem involving norm constraints on the complementary subset J.
When σ is constant, (1) reduces to the Laplace equation for which such an approach was studied in classical Hardy spaces of analytic functions [1, 6]. When σ satisfies (2), the general diffusion equation (1) was considered in [7]. Related generalized Hardy (Banach) classes were introduced, while associated bounded extremal problems are discussed in [14]. The present work is a sequel to [7], as well as a companion paper to [14], of which it aims at providing a synthetic version in the particular Hilbertian framework.
Further, an important practical physical motivation for such issues comes from plasma confinment in tokamaks when identifying the boundary of a plasma from magnetic measurements on the boundary of the device (from measurements of the poloidal flux and field on the vacuum vessel). The axisymmetric configuration leads to a two-dimensional problem in the meridian plane sections of the device (a torus, see Figures 1, 2). Thanks to the so-called Grad-Shafranov equation [8], the (poloidal) magnetic flux is a solution to the conductivity equation (1) in an annular domain and with a conductivity σ_* that depends only of a single real coordinate. This situation is more precisely studied here, and related particular solutions to (1) are expressed as combinations of functions of Bessel-exponential type.

Let $z = x + iy$ denotes the complex coordinate in the plane $\mathbb{C} \simeq \mathbb{R}^2$. So ∂_x and ∂_y stand for the partial derivatives with respect to x and y, respectively. In the same way $\partial = \frac{1}{2}(\partial_x - i\partial_y)$ and $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$ refer to the derivation operators w.r.t. the complex variables z and $\overline{z}$. We follow the approach of [3, 4, 7] that consists in rephrasing (1) as a complex ($\mathbb{R}$-linear) elliptic equation of order 1, namely the conjugate Beltrami equation

$$\overline{\partial} f = \nu\, \overline{\partial f} \qquad \text{a.e. in} \quad \mathbb{D}\,, \tag{CB}$$

where

$$\nu \in W^{1,\infty}(\mathbb{D}) \qquad \text{is real valued,} \qquad \text{and} \quad \|\nu\|_{L^\infty(\mathbb{D})} \leq \kappa < 1\,, \tag{κ}$$

for some $\kappa \in (0, 1)$, with

$$\nu(z, \bar{z}) = \frac{1 - \sigma(x, y)}{1 + \sigma(x, y)}\,. \tag{3}$$

Generalizing the harmonic / analytic framework ($\sigma = 1$, or constant, $\nu = 0$), we will show that solutions to (1) coincide with real parts of solutions to (*CB*). This link between (1) and (*CB*) is the basis of [7, 14], from which we will recall several results.

In the Hilbertian case, we introduce generalized Hardy spaces H^2_ν of solutions to Beltrami equations (*CB*) and explain how the main features of classical Hardy spaces H^2 do extend to these classes (Section 2). We then establish well-posedness of some bounded extremal problems in H^2_ν (Section 3). Moreover, we caracterize a generalized conjugation operator that extends the classical Hilbert transform (Riesz projection). This allows us to compute solutions to these approximation issues. We therefore consider the particular situation arising from physical issues of plasma confinment, where $\sigma = \sigma_*$ and $\nu = \nu_*$ are the associated conductivity and dilation coefficients (Section 4). This is preliminary done in the situation of a disc, while simply-connected smooth domains can also be considered (see Remark 3.1). The extension to the annular case is briefly described in conclusion (Section 5).

2 Generalized Hardy classes

We assume throughout that $\nu \in W^{1,\infty}_{\mathbb{R}}(\mathbb{D})$ satisfies (κ), or, equivalently, that $\sigma = (1-\nu)/(1+\nu)$ satisfies (2). The subscript $_{\mathbb{R}}$ denotes real-valued functions.

Proposition 2.1 *A function $f = u + iv \in L^2(\mathbb{D})$ is a solution to (CB) if, and only if, $u \in L^2_{\mathbb{R}}(\mathbb{D})$ satisfies (1) while $v \in L^2_{\mathbb{R}}(\mathbb{D})$ satisfies*

$$\operatorname{div}\left(\sigma^{-1}\,\nabla v\right) = 0 \text{ in } \mathbb{D}\,, \tag{4}$$

where $\sigma = (1-\nu)/(1+\nu)$.

Proof. For functions $u, v \in W^{1,2}_{\mathbb{R}}(\mathbb{D})$, the classical Sobolev-Hilbert space, setting $f = u+iv$ and substituting it in (CB) lead to a system of two real elliptic equations of the second order in divergence form. Indeed, u satisfies (1) while v satisfies (4).
Conversely, if $\sigma \in W^{1,\infty}_{\mathbb{R}}(\mathbb{D})$ and $u \in W^{1,2}_{\mathbb{R}}(\mathbb{D})$ are real-valued functions that respectively satisfy (2) and (1) in $\mathbb{D}$, there exists $v \in W^{1,2}_{\mathbb{R}}(\mathbb{D})$ real-valued, unique up to an additive constant, such that $f = u + iv \in W^{1,p}(\mathbb{D})$ satisfies (CB) with ν defined by (3).
This shows the equivalence between (CB) and the system

$$\begin{cases} \partial_x v = -\sigma \partial_y u, \\ \partial_y v = \sigma \partial_x u, \end{cases} \tag{5}$$

which admits a solution because $\partial_y(-\sigma\partial_y u) = \ \partial_x(\sigma\partial_x u)$ from (1).
More generally, this formally holds for solutions u, v, f in the distributional sense, thereby achieving the proof (see [7]). ∎

Remark 2.1 System (5) generalizes Cauchy-Riemann equations insofar as taking $\sigma = 1$ amounts to deal with the classical analytic situation, in which case v is the harmonic conjugate to u, [15].

We define the "generalized" Hardy space $H^2_\nu(\mathbb{D}) = H^2_\nu$ to consist of those Lebesgue measurable functions f on $\mathbb{D}$ such that

$$\operatorname*{ess\,sup}_{0<r<1} \|f\|_{L^2(\mathbb{T}_r)} = \operatorname*{ess\,sup}_{r>1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^2 d\theta \right)^{1/2} < +\infty$$

(which therefore belong to $L^2(\mathbb{D})$) and satisfying (CB) in the sense of distributions on $\mathbb{D}$, see [7]. Equipped with the norm

$$\|f\|_{H^2_\nu(\mathbb{D})} = \operatorname*{ess\,sup}_{0<r<1} \|f\|_{L^2(\mathbb{T}_r)}\,,$$

and with the associated inner product, H^2_ν is a Hilbert space. When $\nu = 0$, then $H^2_0 = H^2 = H^2(\mathbb{D})$ is the classical Hardy space of holomorphic functions on the unit disc (see [15]).
We let $L\,:\ L^2(\mathbb{T}) \to \mathbb{C}$ be the (bounded) mean value operator on $\mathbb{T}$ given by

$$L\,\phi = \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{i\theta})\,d\theta\,, \qquad \forall \phi \in L^2(\mathbb{T})\,.$$

First of all, we state some basic properties of H^2_ν spaces from [7, Prop. 4.3.1, 4.3.2] that extend those of classical Hardy spaces, [15].

Proposition 2.2
- Any function $f \in H^2_\nu$ has a non-tangential limit almost everywhere on $\mathbb{T}$, called the trace of f and denoted by trf; moreover, $trf \in L^2(\mathbb{T})$ and its $L^2(\mathbb{T})$ norm is equivalent to the H^2_ν norm of f.
- The space trH^2_ν is closed in $L^2(\mathbb{T})$.
- If $f \in H^2_\nu$, then trf cannot vanish on a subset of $\mathbb{T}$ having positive Lebesgue measure unless $f \equiv 0$.
- Every $f \in H^2_\nu$ is uniquely determined by the real part of its trace on $\mathbb{T}$, up to an additional imaginary constant.

These generalized Hardy spaces H^2_ν provide a suitable framework for Dirichlet type problems. Indeed, assuming that the boundary data u belongs to $L^2_{\mathbb{R}}(\mathbb{T})$, there exists $f \in H^2_\nu$ of which the real part of the trace coincides with u on $\mathbb{T}$ [7, Thm 4.4.2.1]. Up to the normalization condition $L(Im(trf)) = 0$, such an f is unique. This result was established in [12] for $u \in W^{1/2,2}_{\mathbb{R}}(\mathbb{T})$.

Using Propositions 2.1 and 2.2, we get from (5) that, for $f = u+iv \in H^2_\nu$, then $\partial_\theta v = \sigma \partial_n u$ a.e. on $\mathbb{T}$, if $\partial_\theta v$ stands for the tangential derivative of v on $\mathbb{T}$. Hence, the Cauchy inverse problem *(CP)* we look at can be stated as an extrapolation issue, with data $F_d = u + iv$ given on I:

(CP) Given F_d on the subset $I \subset \mathbb{T}$, recover $f \in H^2_\nu$ such that $F_d = (trf)_{|_I}$.

The solution is then given by $(trf)_{|_J}$. Observe however that the above issue only makes sense for compatible data u, $\partial_n u$ on I, whence $F_d \in (trH^2_\nu)_{|_I}$. If these data are subject to measurement or computation errors, then $F_d \in L^2(I)$ but $F_d \notin (trH^2_\nu)_{|_I}$, which is the classical ill-posedness situation for Cauchy type inverse problems. This is the reason why the issue *(CP)* might better be stated as an approximation one (see Section 3):

(CP) Given $F_d \in L^2(I)$, $I \subset \mathbb{T}$, recover $f \in H^2_\nu$ such that $F_d \simeq (trf)_{|_I}$.

We now define the ν-Hilbert transform as the map

$$\begin{array}{rccc} \mathcal{H}_\nu : & L^2_{\mathbb{R}}(\mathbb{T}) & \to & L^2_{\mathbb{R}}(\mathbb{T}) \\ & Re(tr\ f) = u & \mapsto & Im(tr\ f) = v\,, \quad L(v) = 0 \end{array}$$

It generalizes the harmonic conjugation and thereafter the M. Riesz theorem [15, III, Thm 2.3]. We thus have the following properties (see [7, Cor. 4.4.2.1]).

Proposition 2.3
- The operator $\mathcal{H}_\nu$ is bounded on $L^2_{\mathbb{R}}(\mathbb{T})$.
- $trH^2_\nu = \{u + iv \in L^2(\mathbb{T}); v = \mathcal{H}_\nu u + L(v)\}$.
- $L^2_{\mathbb{R}}(\mathbb{T}) = Re\, trH^2_\nu$.

The operator $\mathcal{H}_\nu$ naturally allows us to define an orthogonal projection from $L^2(\mathbb{T})$ onto trH^2_ν similar to the classical Riesz projection

$$\forall\, f \in L^2(\mathbb{T}), \quad \mathcal{P}_\nu f = \frac{1}{2}\,(I + i\mathcal{H}_\nu)\, f + \frac{1}{2} L(f) \tag{6}$$

Remark 2.2 The definition of $\mathcal{P}_\nu$ requires that of $\mathcal{H}_\nu$ on $L^2(\mathbb{T})$, not only on $L^2_{\mathbb{R}}(\mathbb{T})$. Following [4], this is done by considering $v = \mathcal{H}_\nu u$ as the real part of the function $g = -if$, solution to (CB) with the dilation coefficient $-\nu$, therefore belonging to $H^2_{-\nu}$. This leads to set $\mathcal{H}_\nu(iv) = i\mathcal{H}_{-\nu}v$, the operator $\mathcal{H}_\nu$ being thereby extended to complex-valued functions, but remaining only $\mathbb{R}$-linear.

Finally, we state a density result that is of high importance in extremal problems below (see [7, Thm 4.5.2.1]).

Theorem 2.1 *Let $I \subset \mathbb{T}$ be a measurable subset such that $J = \mathbb{T}\backslash I$ has positive Lebesgue measure. The restrictions to I of traces of H^2_ν-functions form a dense subset of $L^2(I)$.*

As a consequence, for all $\varphi \in L^2(I)$, there exists a sequence (f_k) of functions of H^2_ν such that $tr f_k \to \varphi$ in $L^2(I)$ as $k \to \infty$. There are two possibilities concerning its behaviour:
- either φ is already the trace on I of a function belonging to H^2_ν, whence $\|tr f_k\|_{L^2(J)}$ remains bounded as $k \to \infty$,
- or $\|tr f_k\|_{L^2(J)} \to \infty$ as $k \to \infty$.

Remark 2.3 Indeed, it can be deduced from the second point of Proposition 2.2 that, as a closed convex subspace, trH^2_ν is weakly closed in $L^2(\mathbb{T})$.

Thus, as soon as available data F_d on I do not coincide with the trace of a function in H^2_ν (and this will be the case when it comes to numerically computed quantities or to physical measurements, because the Cauchy data u, $\partial_n u$ will no longer be compatible), an additional constraint is needed for *(CP)* issue to have a bounded solution in H^2_ν. It is therefore natural to put a bound on the $L^2(J)$ norm of the approximant, so as to end up with a well-posed problem.
From (possibly noisy) Cauchy data available on I of solutions to (CB), the issue amounts no more to extrapolation but rather to best $L^2(I)$ approximation, subject to a norm constraint on J. This leads to the bounded extremal problems that are the topic of the next section and allow to properly state and solve the above approximation issue as:

(CP) Given $F_d \in L^2(I)$, $I \subset \mathbb{T}$, recover $f \in H^2_\nu$ that satisfies some norm constraint on J and such that $(trf)_{|I}$ best approximates F_d in $L^2(I)$ among such functions.

3 Bounded extremal problems (BEP) in the disc

Let $I \subset \mathbb{T}$ be a finite union of open connected subsets of $\mathbb{T}$ such that both I and $J = \mathbb{T}\backslash \overline{I}$ have positive Lebesgue measure. For $M \geq 0$, $\phi \in L^2_{\mathbb{R}}(J)$, define

$$\mathcal{C}_M = \left\{ g \in trH^2_\nu;\ \|Re\ g - \phi\|_{L^2(J)} \leq M \right\} |_I \subset L^2(I)\,.$$

We then have the following result, of existence and uniqueness of a best approximant in $\mathcal{C}_M$ to $L^2(I)$ functions, see also [14, Thm 3].

Theorem 3.1 *Fix $M > 0$, $\phi \in L^2_{\mathbb{R}}(J)$. For every function $F_d \in L^2(I)$, there exists a unique function $g_* \in \mathcal{C}_M$ such that*

$$\|F_d - g_*\|_{L^2(I)} = \min_{g \in \mathcal{C}_M} \|F_d - g\|_{L^2(I)}\,. \qquad \text{(BEP)}$$

Moreover, if $F_d \notin \mathcal{C}_M$, then $\|Re\ g_* - \phi\|_{L^2(J)} = M$.

Proof. Existence and uniqueness rely on classical arguments of uniform convexity (provided here by the Hilbertian setting). First, $\mathcal{C}_M$ is a convex subset of $L^2(I)$. We then show that it is closed in $L^2(I)$.
Let $\varphi_{k|_I} \in \mathcal{C}_M$, $\varphi_{k|_I} \to \varphi_I$ in $L^2(I)$ as $k \to \infty$. Put $\varphi_k = u_k + iv_k \in trH^2_\nu$. Then u_k is bounded in $L^2(\mathbb{T})$ by definition, whence also (φ_k) from Propostion 2.3. It follows that (φ_k) weakly converges to $\psi \in L^2(\mathbb{T})$. Since trH^2_ν is weakly closed $L^2(\mathbb{T})$ (see Remark 2.3), we get that $\psi \in trH^2_\nu$ and $\psi_{|_I} = \varphi_I$. The constraint on J is satisfied by $Re\ \varphi_k = u_k$, so does $Re\ \psi$, whence $\varphi_I \in \mathcal{C}_M$.
To summarize, $\mathcal{C}_M$ is a closed convex set of $L^2(I)$. It ensures the existence of a best approximation projection π from $L^2(I)$ on $\mathcal{C}_M$ and $g_* = \pi F_d$ is the unique solution to (BEP), [5, 10].
To prove that the constraint is saturated unless $F_d \in \mathcal{C}_M$, assume for a contradiction that $\|Re\ g_* - \phi\|_{L^2(J)} < M$. By Theorem 2.1, there is a function $h \in trH^2_\nu$ such that

$$\|F_d - g_* - h\|_{L^2(I)} < \|F_d - g_*\|_{L^2(I)} \,,$$

and by the triangle inequality we have

$$\|F_d - g_* - \lambda h\|_{L^2(I)} < \|F_d - g_*\|_{L^2(I)} \,,\ \forall\ 0 < \lambda < 1 \,.$$

Now, λ can be chosen sufficiently small to ensures that

$$\|Re(g_* + \lambda h) - \phi\|_{L^2(J)} < M \,,$$

contradicting the optimality of g_*, since uniqueness holds in $\mathcal{C}_M$. ∎

Towards constructive aspects, if $F_d \notin \mathcal{C}_M$, the solution g^* is given by ([14, Prop. 2])

$$\mathcal{P}_\nu \chi_I g + \gamma\, \mathcal{P}_\nu \chi_J Re g = \mathcal{P}_\nu \chi_I F_d + \gamma\, \mathcal{P}_\nu \chi_J \phi \,, \tag{7}$$

for the unique $\gamma > 0$ such that $\|Reg_* - \phi\|_{L^p(J)} = M$, and the projection operator $\mathcal{P}_\nu$ is defined by (6). The parameter γ is a Lagrange multiplier expressing the dependence between the criterion (error) $e = \|F_d - g_*\|_{L^2(I)}$ and the constraint M. Of course, the implicit character of γ is the main difficulty in computations, but it can be handled by dichotomy procedures, since e is a strictly decreasing smooth function of M and λ, and $M \to \infty$, while $e \to 0$ and $\lambda \to 0$.

Remark 3.1 While the study of Sections 2, 3 was performed in the unit disc $\mathbb{D}$, it may directly be extended to any simply-connected bounded open plane domain with C^1 boundary, using composition by conformal mappings [17].

4 Complete families of solutions

This section is devoted to the study of particular solutions to the conductivity equation (1) linked with the physical background mentionned in the introduction, for particular conductivity coefficients σ_* and ν_*. More precisely, the magnetic poloidal flux generated by a plasma in a tokamak follows equation (1) in poloidal plane sections (annular vacuum region located between the plasma itself and the device's wall). Here, we preliminary look

at the associated Cauchy-type issue *(CP)* in a disc (and briefly mention in Section 5 how it may be extended to the annular situation).
Denoting by (x, y, φ) the classical cylindrical coordinates, the axisymmetric assumption and the toroidal configuration imply that the magnetic quantities do not depend on φ (see Figure 1). In a poloidal plane section where φ is constant, it is known [8] that the

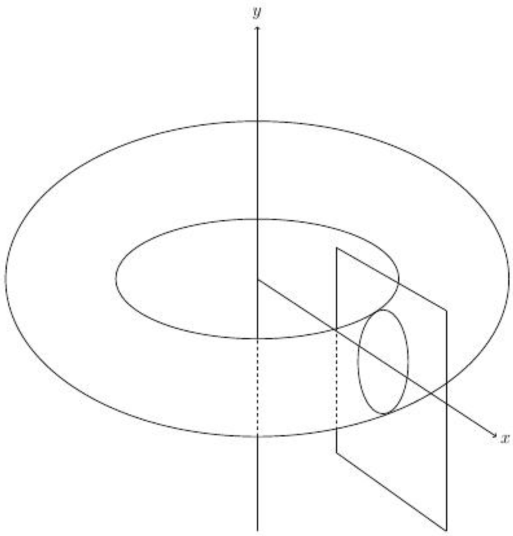

Figure 1: Schematic diagram of a tokamak

poloidal component u of the magnetic flux satisfies equation (1), where the conductivity coefficient $\sigma = \sigma_*$ only depends of the radial coordinate x

$$\sigma_*(x, y) = \frac{1}{x}\,. \tag{8}$$

Observe that the dilation coefficient ν_* associated to σ_* is given by (3)

$$\nu_*(z, \bar{z}) = (z + \bar{z} - 2)/(z + \bar{z} + 2).$$

Though it holds in principle in an annular domain (comprised between the plasma and the chamber), we consider here equation (1) with (8) in the disc $\mathbb{D}_0$ (see Remark 3.1, and Section 5)

$$\mathbb{D}_0 = \{(x, y) \in \mathbb{R}^2; (x - x_0)^2 + y^2 < R^2\} \text{ where } 0 < R < |x_0|.$$

Again $\mathbb{T}_0$ is its boundary. We introduce the rectangle $\mathcal{R}$:

$$\mathcal{R} = [a, b] \times [-c, c], b > a > 0 \text{ and } c > 0$$

with boundary $\partial\mathcal{R}$, so that $\mathbb{D}_0$ has compact closure in $\mathcal{R}$ (see Figure 2).

It is suggested by the particular form (8) of σ_* to seek particular solutions of (1) and (4) in $\mathbb{D}_0$ with separated variables x and y. Setting $u(x, y) = A(x)B(y)$ and $v(x, y) = C(x)D(y)$

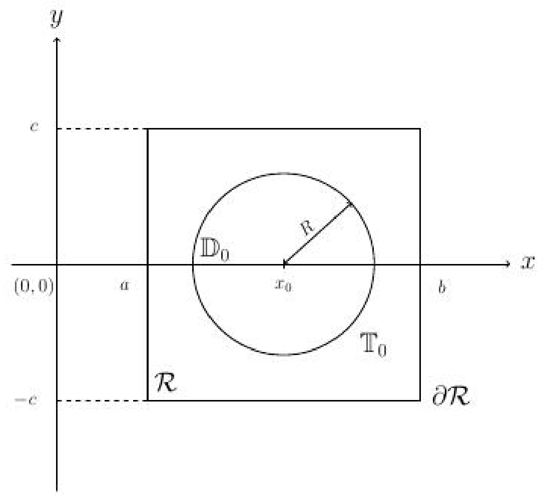

Figure 2: Poloidal section of the torus

in (5), it follows that exact solutions of that kind are expandable in terms of a series of Bessel-exponential type (see [14]) for $(x, y) \in \mathcal{R}$. That is, for every $N \in \mathbb{N}^*$, for every sequences of real-valued coefficients (λ_n), (μ_n), (α_n), (β_n), (γ_n), (δ_n), $n = 1, \cdots, N$, and every $a_0, b_0, c_0 \in \mathbb{R}$:

$$
\begin{aligned}
u(x, y) &= \sum_{n=1}^{N} x J_1(\lambda_n x)[\alpha_n e^{\lambda_n y} + \beta_n e^{-\lambda_n y}] \\
&\quad + \sum_{n=1}^{N} x[\gamma_n I_1(\mu_n x) + \delta_n K_1(\mu_n x)] \sin(\mu_n(y + c)) + a_0 x^2 + b_0 y + c_0.
\end{aligned}
$$

and $(\mathcal{H}_{\nu_*} u)(x, y) = v(x, y) - L\, v$ (see Proposition 2.3) with

$$
\begin{aligned}
v(x, y) &= \sum_{n=1}^{N} J_0(\lambda_n x)[\alpha_n e^{\lambda_n y} - \beta_n e^{-\lambda_n y}] \\
&\quad + \sum_{n=1}^{N} [-\gamma_n I_0(\mu_n x) + \delta_n K_0(\mu_n x)] \cos(\mu_n(y + c)) - b_0 \ln x + 2a_0 y.
\end{aligned}
$$

Here, J_i, I_i, K_i, $i = 0, 1$ respectively denote the Bessel functions of the first kind, and the modified Bessel functions of the first and second kind [19].

An analogous decomposition, with Bessel functions, has already been used for studies of tokamak equilibrium, see [13, 20]. But it seems that its density properties in $L^2_{\mathbb{R}}(\mathbb{T}_0)$ (Proposition 4.2) are new. It is the purpose of the following discussion.
First, note that in the expression of u, (λ_n) and (μ_n) are two sequences of free parameters. Choosing first λ_n to be the positive roots of $J_0(\lambda_n b)$ enables to consider the functions $(\sqrt{x} J_0(\lambda_n x))_{n \geq 1}$ as a complete orthogonal system in $L^2((0, b); x\, dx)$ [9]. Moreover, an appropriate choice of μ_n is $n\pi/2c$, so that the coefficients of the sinus terms of u vanish

at horizontal parts of the boundary $\partial\mathcal{R}$. We then set $\mathcal{B}$ for the space of solutions u as above with free parameters (α_n), (β_n), (γ_n), (δ_n), $n = 1, \cdots, N$, $a_0, b_0, c_0 \in \mathbb{R}$.
Thus, using by turns classical integral properties and appropriated Fourier expansions leads to the result that $\mathcal{B}_{|\partial\mathcal{R}}$ is $W^{1,2}$ dense in $W^{1,2}_{\mathbb{R}}(\partial\mathcal{R})$. Finally, taking into account that $\mathcal{R}$ is a simply-connected plane domain with Lipschitz boundary [16] and since there exists a continuous lifting of the trace from $W^{1/2,2}_{\mathbb{R}}(\partial\mathcal{R})$ to $W^{1,2}_{\mathbb{R}}(\mathcal{R})$, we establish, see [14, Prop. 3]:

Proposition 4.1

$$\textit{The family } \mathcal{B} \textit{ is } W^{1,2}(\mathcal{R}) \textit{ dense in } \mathcal{S}_{\mathcal{R}} = \left\{ u \in W^{1,2}_{\mathbb{R}}(\mathcal{R})\, ; \, \mathrm{div}\left(\frac{1}{x}\nabla u\right) = 0 \right\}.$$

We can prove now the following density result, [14, Prop. 4]:

Proposition 4.2 *The restriction $\mathcal{B}_{|\mathbb{T}_0}$ of $\mathcal{B}$ to $\mathbb{T}_0$ is L^2 dense in $L^2_{\mathbb{R}}(\mathbb{T}_0) = Re\, tr H^2_{\nu_*}(\mathbb{D}_0)$.*

Proof. Let $\psi \in L^2_{\mathbb{R}}(\mathbb{T}_0)$ and $\varepsilon > 0$. There exists $\phi \in W^{1/2,2}_{\mathbb{R}}(\mathbb{T}_0)$ such that

$$\|\psi - \phi\|_{L^2(\mathbb{T}_0)} \leq \varepsilon.$$

For a such a class of smooth boundary data, Dirichlet-type results from [12] ensures the existence of a unique solution $u \in W^{1,2}_{\mathbb{R}}(\mathbb{D}_0)$ to (1) in $\mathbb{D}_0$ such that $u_{|\mathbb{T}_0} = \phi$. Moreover (1) is an elliptic equation with regular coefficients in $\mathcal{R}$ and $\mathbb{D}_0$ is a simply connected domain with compact closure in $\mathcal{R}$. Then approximation results of Browder [11, Thm 5] ensure the existence of $u_{\mathcal{R}} \in \mathcal{S}_{\mathcal{R}}$ such that $u_{\mathcal{R}|\mathbb{D}_0}$ is arbitrarily close to u in the $W^{1,2}_{\mathbb{R}}(\mathbb{D}_0)$-norm. Now, with Proposition 4.1, there exists $b \in \mathcal{B}$ arbitrarily close to $u_{\mathcal{R}}$ in the $W^{1,2}(\mathcal{R})$-norm. Collecting the various approximations leads to

$$\|u - b\|_{W^{1,2}(\mathbb{D}_0)} \leq \varepsilon$$

whence also $\|u - b\|_{L^2(\mathbb{T}_0)} \leq c\varepsilon$, for some $c > 0$, with Sobolev's inequalities ([10, Ch. 9]). Finally, putting all end-to-end we get that $\|\psi - b\|_{L^2(\mathbb{T}_0)} \leq C\varepsilon$ with $C > 0$. This implies that $\mathcal{B}_{|\mathbb{T}_0}$ is dense in $L^2_{\mathbb{R}}(\mathbb{T}_0)$, and the conclusion follows from Proposition 2.3. ■

This result can be illustrated through numerical simulations. The following ones (Figures 3, 4) are obtained with our software, using Matlab (R2008b). The specific choice of parameters is $(x_0, R) = (5, 2)$, with $b = x_0 + R + 1$, and $N = 1$ or $N = 6$ (recall that N indexes the number of functions of each kind involved in $\mathcal{B}$, so that functions in $\mathcal{B}$ are determined by $4N + 3$ real valued coefficients). The circle $\mathbb{T}_0$ is uniformly discretized by $p = \lfloor 2\pi \rfloor / 0.1 = 62$ points. In Figure 3, u is defined on $\mathbb{T}_0$ by $u(x, y) = x^9 y^2 + \ln(x)$ and u_1 and u_6 are the approximants that correspond respectively to $N = 1$ and $N = 6$. Recall that on $\mathbb{T}_0$, $x = x_0 + R\cos\theta$ and $x = R\sin\theta$. At the same time, we show in Figure 4 the behavior of the error $\tilde{e} = \tilde{e}(N)$ between u and its approximants u_N: $\tilde{e}^2 = \sum_p (u - u_N)^2 / \sum_p u^2$.

Figure 4 shows that the error $\tilde{e}$ is a decreasing function of n. This illustrates the density result of Proposition 4.2. Indeed, the larger the number of functions in $\mathcal{B}$, the better the accuracy of the approximant is. On Figure 3 this can be noticed by the fact that u and u_6 are undifferentiated while it is not the case for u_1.

Now, a similar result to Proposition 4.2 can be showed for the family $\{\mathcal{H}_{\nu_*}\mathcal{B} + c; c \in \mathbb{R}\}$.

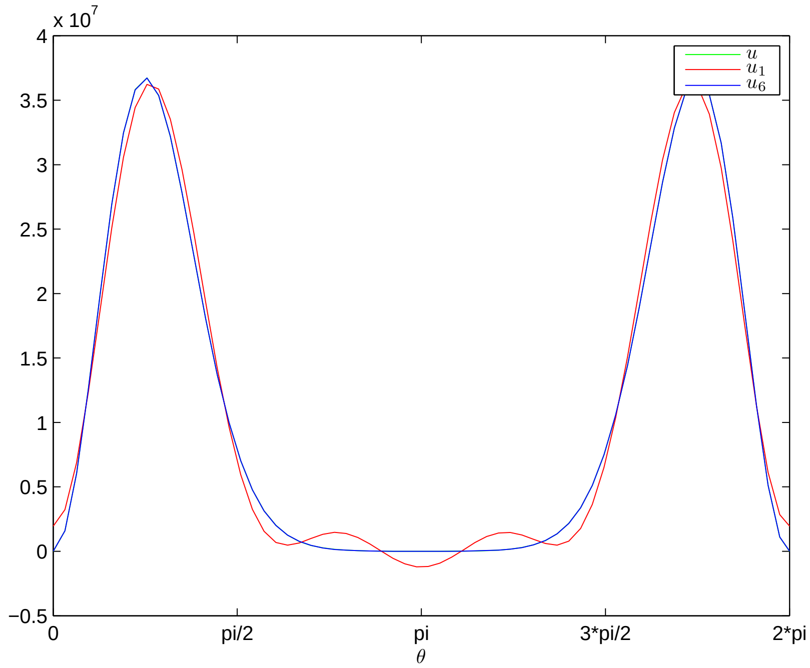

Figure 3: Approximations u_1 and u_6 of $u = x^9y^2 + \ln(x)$

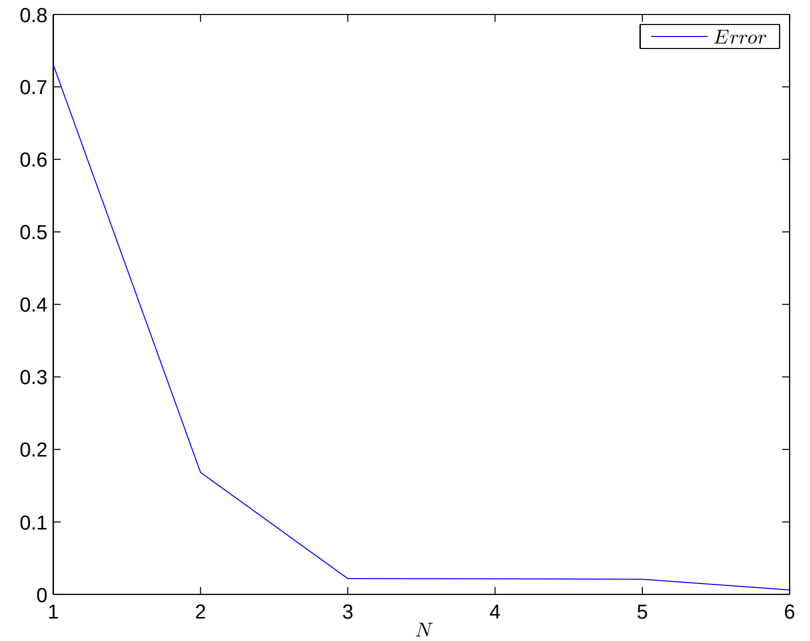

Figure 4: Error $\tilde{e}(N)$ when $u = x^9y^2 + \ln(x)$

This is once again illustrated through numerical simulations (Figures 5, 6). There are obtained with the same parameters as the previous ones. In Figure 5, v is defined on $\mathbb{T}_0$ by $v(x, y) = ye^x - x^4$ and v_1 and v_6 are the approximants that correspond respectively to $N = 1$ and $N = 6$. We show in Figure 6 the behavior of the error $\tilde{e} = \tilde{e}(N)$ between v and its approximants v_N.

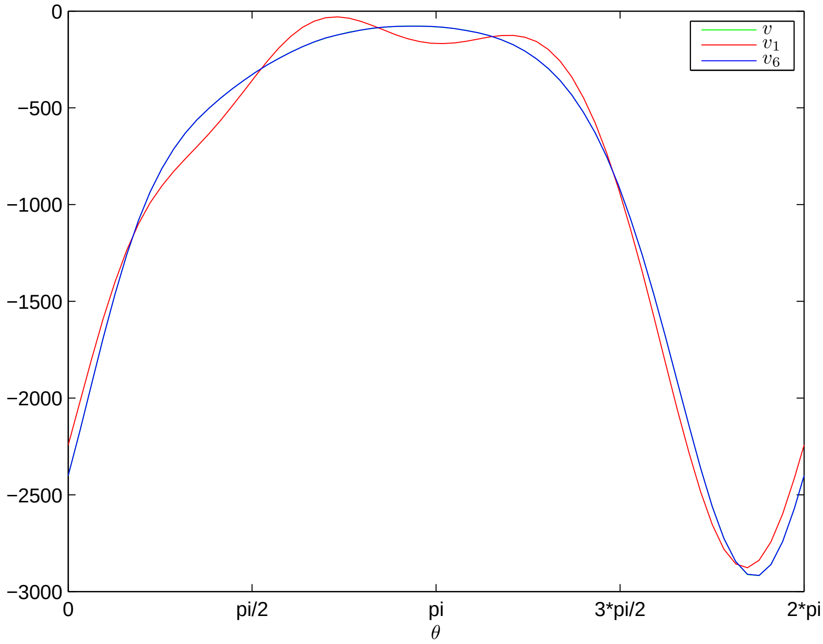

Figure 5: Approximations v_1 and v_6 of $v = ye^x - x^4$

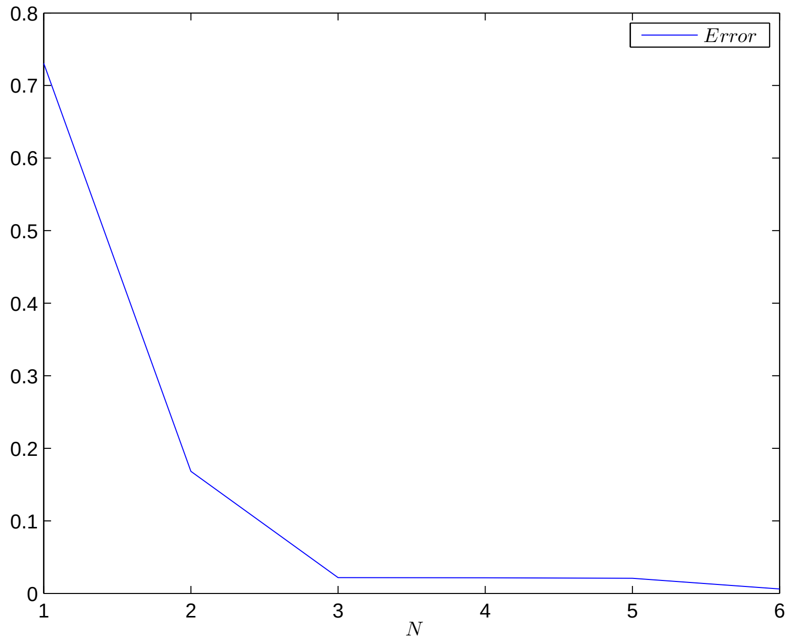

Figure 6: Error $\tilde{e}(N)$ when $v = ye^x - x^4$

Thanks to Figures 5 and 6, we naturally come to the same conclusions as before.

Finally, with Proposition 2.3, we then obtain:

Corollary 4.1
- $\{\mathcal{B} + i\mathcal{H}_{\nu_*}\mathcal{B} + i\mathbb{R}\}$ *is dense in* $L^2(\mathbb{T}_0)$.
- $\{b + i\mathcal{H}_{\nu_*}b + i\mathbb{R}; b \in \mathcal{B}\}$ *is* L^2 *dense in* $trH^2_{\nu_*}(\mathbb{D}_0)$.

The computational and practical uses of these families rely on the equality between the coefficients of expansions of b and $\mathcal{H}_{\nu_*}b$ in $\mathcal{B}$ and $\mathcal{H}_{\nu_*}\mathcal{B}$. Besides, all the problematic remains the same by considering equation (CB) associated this time to the coefficient $-\nu$ in which case system (5) must be reworded with the conductivity σ^{-1}.
For every $f \in L^2(\mathbb{T}_0)$, the operator $\mathcal{P}_{\nu_*}$ is determined by (6):

$$\mathcal{P}_{\nu_*}(f) = \mathcal{P}_{\nu_*}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}\begin{pmatrix} u + Lu - \mathcal{H}_{-\nu_*}v \\ v + Lv + \mathcal{H}_{\nu_*}u \end{pmatrix}.$$

Therefore, by computing the expansions of u in $\mathcal{B}$ and of v in $\mathcal{H}_{\nu_*}\mathcal{B}$, we obtain those of each component of $\mathcal{P}_{\nu_*}$ on these families.
More details concerning the computation of the operator $\mathcal{P}_{\nu_*}$ and the solution to the bounded extremal problem, with numerical examples, may be found in [14].

5 Conclusion

First, recall that the present study directly extends to simply-connected smooth domains, see Remark 3.1. Also, the results of Sections 2 and 3 are valid in L^p uniformly convex Banach spaces L^p, $1 < p < \infty$, where H^p_ν Hardy spaces may be analogously defined, [7, 14].
Next, in order to handle the physical issue concerning plasma confinment in a tokamak, it will be necessary to deal with *(CP)* and the related approximation issues in annular configurations. That is the purpose of a forthcoming work which will be essentially based on a topological decomposition of generalized Hardy spaces of an annulus. More precisely, if the annulus is denoted by $\mathbb{A} = \mathbb{D}\backslash\rho\bar{\mathbb{D}}$, there exists $\nu_i \in W^{1,\infty}_{\mathbb{R}}(\mathbb{D})$ and $\nu_e \in W^{1,\infty}_{\mathbb{R}}(\mathbb{C}\backslash\rho\bar{\mathbb{D}})$ such that $\nu_{i|_{\mathbb{A}}} = \nu_{e|_{\mathbb{A}}} = \nu$ and:

$$H^2_\nu(\mathbb{A}) = H^2_{\nu_i}(\mathbb{D}) \oplus H^2_{\nu_e}(\mathbb{C} \setminus \rho\bar{\mathbb{D}}).$$

To ensure well-posedness of the associated bounded extremal problem, the first step is to extend the density results of Section 2 to the doubly-connected situation.
We finally mention the fact that other choices of appropriate bases may be of interest too. An interesting one, suitable to the tokamaks geometry, could be the toroidal harmonics. The separation of the Laplace's equation has already been studied by several authors (see [2, 18]). Adapting this approach to a more general diffusion equation as (1) should be of high interest.

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