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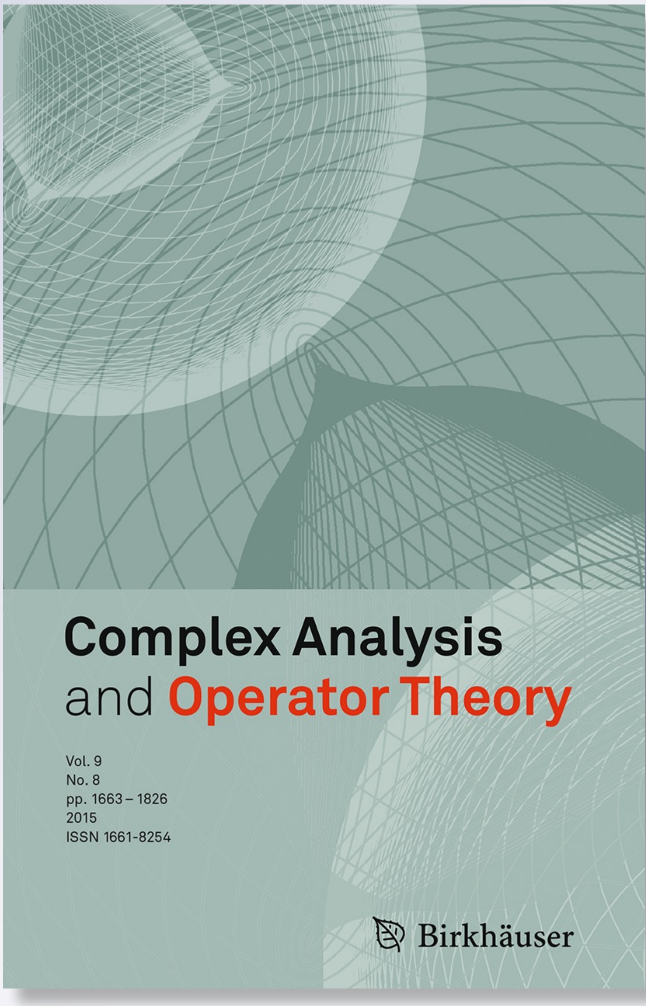

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Abstract We study the composition operators $f \mapsto f \circ \phi$ on generalized analytic function spaces named generalized Hardy spaces, on bounded domains of $\mathbb{C}$, for holomorphic functions ϕ with uniformly bounded derivative. In particular, we provide necessary and/or sufficient conditions on ϕ, depending on the geometry of the domains, ensuring that these operators are bounded, invertible, or isometric.

Keywords Generalized Hardy spaces · Composition operators

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Contents

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1 Introduction

The present work aims at generalizing properties of composition operators on Hardy spaces of domains of the complex plane to the framework of generalized Hardy spaces. Generalized analytic functions, among which pseudo-holomorphic functions, were considered a long time ago, see [9,34]. More recently, they were studied in [22], in particular because of their links with classical partial differential equations (PDEs) in mathematical physics, like the conductivity or Schrödinger equations, see [2], [3, Lem. 2.1]. By generalized analytic functions, we mean solutions (as distributions) to the following $\overline{\partial}$-type equations (real linear conjugate Beltrami and Schrödinger type elliptic PDEs):

$$\overline{\partial} f = \nu \overline{\partial f} \quad \text{or} \quad \overline{\partial} w = \alpha \overline{w},$$

without loss of generality [9]. For specific classes of dilation coefficients ν, α, these two PDEs are equivalent to each other, as follows from a trick going back to Bers and Nirenberg, see (13) below.

They are also related to the complex linear Beltrami equation and to quasi-conformal applications [1]. Properties of associated (normed) Hardy classes H^p_ν and G^p_α have been established in [6,7,14] for $1 < p < \infty$. These classes seem to have been introduced in [26] for simply connected domains. They share many properties of the classical Hardy spaces of analytic (holomorphic) functions (for $\alpha \equiv 0 \equiv \nu$). The proofs of these properties rely on the factorization result from [9], which was extended in [5–7] to G^p_α functions, through classical Hardy spaces H^p. Note that important applications of these classes come from Dirichlet–Neumann boundary value problems and Cauchy type transmission issues for the elliptic conductivity PDE $\nabla \cdot (\sigma \nabla u) = 0$ with conductivity $\sigma = (1-\nu)\ (1+\nu)^{-1}$ in domains of $\mathbb{R}^2 \simeq \mathbb{C}$, see [2,7]. Indeed, on simply-connected domains, solutions u coincide with real–parts of solutions f to $\overline{\partial} f = \nu \overline{\partial f}$. In particular, this links Calderón's inverse conductivity problem to similar issues for the real linear conjugate Beltrami equation, as in [3]. Further, these new Hardy classes furnish a suitable framework in order to state and solve families of best constrained approximation issues (bounded extremal problems) from partial boundary values, see [14,17], that are given by Dirichlet–Neumann boundary conditions, through generalized harmonic conjugation or Hilbert transform.

In the Hilbertian setting $p = 2$, constructive aspects are available for particular conductivity coefficients ν, for which bases of H^2_ν may be explicitly constructed, in the disk or the annulus, see [16,17]. In the annular setting, and in toroidal coordinates, this allows to tackle a free boundary problem related to plasma confinement in tokamaks. Concerning realistic geometries, properties of composition operators on generalized Hardy classes may provide a selection of conformal maps from the disk or circular domains.

The present work is a study of some properties of composition operators on these Hardy classes. Let $\Omega \subset \mathbb{C}$ be a domain. Hardy spaces $H^p_\nu(\Omega)$ of solutions to the conjugate Beltrami equation $\overline{\partial} f = \nu\overline{\partial f}$ a.e. on Ω are first considered when Ω is the unit disc $\mathbb{D}$ or the annulus $\mathbb{A} = \{z \in \mathbb{C} : r_0 < |z| < 1\}$. A way to define those spaces in general bounded Dini-smooth domains is to use their conformal invariance property, see [6]. More precisely, if Ω_1 and Ω_2 are two bounded Dini-smooth domains and ϕ a conformal map from Ω_1 onto Ω_2, then f is in $H^p_\nu(\Omega_2)$, with $\nu \in W^{1,\infty}_{\mathbb{R}}(\Omega_2)$ if and only if $f \circ \phi$ is in $H^p_{\nu\circ\phi}(\Omega_1)$ and $\nu \circ \phi \in W^{1,\infty}_{\mathbb{R}}(\Omega_1)$. In terms of operator, if $\phi : \Omega_1 \to \Omega_2$ is an analytic conformal map, the composition operator $C_\phi : f \longmapsto f \circ \phi$ maps $H^p_\nu(\Omega_2)$ onto $H^p_{\nu\circ\phi}(\Omega_1)$. Similar results hold in G^p_α Hardy spaces of solutions to $\overline{\partial} w = \alpha\overline{w}$.

Suppose now that the composition map $\phi : \Omega_1 \to \Omega_2$ is a function in $W^{1,\infty}(\Omega_1, \Omega_2)$ and analytic in Ω_1, what can we say about $C_\phi(f) = f \circ \phi$ when $f \in H^p_\nu(\Omega_2)$ in terms of operator properties? This operator has been widely studied in the case of analytic Hardy spaces (i.e. $\nu \equiv 0$) when $\Omega_1 = \Omega_2 = \mathbb{D}$ giving characterizations of composition operators that are invertible in [27], isometric in [18], similar to isometries in [8], and compact in [31,33], for example. Fewer results are known concerning composition operators on H^p spaces of an annulus. However, one can find in [11] a sufficient condition on ϕ for the boundedness of C_ϕ and a characterization of Hilbert–Schmidt composition operators.

In more general (non smooth) simply connected domains, boundedness and compactness properties of composition operators on Hardy spaces are established in [19,33]. Note that, in this context, several definitions of Hardy spaces can be considered [13, Ch.10].

The study of composition operators has been generalized to many other spaces of analytic functions, such as Dirichlet or Bergman spaces, see [24,32] and the references therein.

In this paper, we study some properties, namely boundedness, invertibility, isometry, for the composition operator defined on the generalized Hardy space $H^p_\nu(\Omega)$ and $G^p_\alpha(\Omega)$ where Ω is a bounded Dini-smooth domain (most of the time, Ω will be the unit disc $\mathbb{D}$ or the annulus $\mathbb{A}$). Compactness issues will be dealt with in a subsequent paper.

In Sect. 2, we provide some notations. Definitions of generalized Hardy classes for bounded Dini-smooth domains, together with some of their properties are given in Sect. 3. Section 4 is devoted to boundedness results for composition operators on generalized Hardy classes H^p_ν and G^p_α for $1 < p < \infty$. Section 5 is related to their invertibility, while isometric composition operators are studied in Sect. 6 on generalized Hardy classes of the disk and the annulus. There, Theorems 1 and 3 appear to be new in $H^p(\mathbb{A})$ as well. A conclusion is written in Sect. 7 in which we discuss the extension of some results to generalized Hardy spaces over arbitrary domains.

2 Definitions and Notations

In this paper, we will denote by Ω a connected open subset of the complex plane $\mathbb{C}$ (also called a domain of $\mathbb{C}$), by $\partial\Omega$ its boundary, by $\mathbb{D}$ the unit disc and by $\mathbb{T} = \partial\mathbb{D}$ the unit circle. For $0 < r_0 < 1$, let $\mathbb{A}$ be the annulus $\{z \in \mathbb{C} : r_0 < |z| < 1\} = \mathbb{D} \cap (\mathbb{C}\backslash r_0\overline{\mathbb{D}})$, the boundary of which is $\partial\mathbb{A} = \mathbb{T} \cup \mathbb{T}_{r_0}$, where $\mathbb{T}_{r_0}$ is the circle of radius r_0. More generally, we will consider a circular domain $\mathbb{G}$ defined as follows

$$\mathbb{G} = \mathbb{D}\backslash \bigcup_{j=0}^{N-1} (a_j + r_j\overline{\mathbb{D}}), \tag{1}$$

where $N \geq 2$, $a_j \in \mathbb{D}$, $0 < r_j < 1$, $0 \leq j \leq N-1$. Its boundary is

$$\partial\mathbb{G} = \mathbb{T} \cup \bigcup_{j=0}^{N-1} (a_j + \mathbb{T}_{r_j})$$

where the circles $a_j + \mathbb{T}_{r_j}$ for $0 \leq j \leq N-1$ have a negative orientation whereas $\mathbb{T}$ has the positive orientation. Note that for $N = 2$ and $a_0 = 0$, $\mathbb{G}$ is the annulus $\mathbb{A}$.

A domain Ω of $\overline{\mathbb{C}}$ is Dini-smooth if and only if its boundary $\partial\Omega$ is a finite union of Jordan curves with non-singular Dini-smooth parametrization. We recall that a function f is said to be Dini-smooth if its derivative is Dini-continuous, i.e. its modulus of continuity ω_f is such that

$$\int_0^{\varepsilon} \frac{\omega_f(t)}{t} dt < \infty, \quad \text{for some } \varepsilon > 0.$$

Recall that, if Ω is a bounded Dini-smooth domain, there exists a circular domain $\mathbb{G}$ and a conformal map ϕ between $\mathbb{G}$ and Ω which extends continuously to a homeomorphism between $\overline{\mathbb{G}}$ and $\overline{\Omega}$, while the derivatives of ϕ also extend continuously to $\overline{\mathbb{G}}$ [6, Lem. A.1]. If E, F are two Banach spaces, $\mathcal{L}(E, F)$ denotes the space of bounded linear maps from E to F, and $T \in \mathcal{L}(E, F)$ is an isometry if and only if, for all $x \in E$, $\|Tx\|_F = \|x\|_E$.

If $A(f)$ and $B(f)$ are quantities depending on a function f ranging in a set E, we will write $A(f) \lesssim B(f)$ when there is a positive constant C such that $A(f) \leq CB(f)$ for all $f \in E$. We will say that $A(f) \sim B(f)$ if there is $C > 0$ such that $C^{-1}B(f) \leq A(f) \leq CB(f)$ for all $f \in E$.

The Lebesgue measure (on the complex plane or in the 1-dimensional case) will be denoted by m. For $1 \leq p \leq \infty$, $L^p(\Omega)$ designates the classical Lebesgue space of functions defined on Ω with respect to m, equipped with the classical norm.

We denote by $\mathcal{D}(\Omega)$ the space of smooth functions with compact support in Ω. Let $\mathcal{D}'(\Omega)$ be its dual space which is the space of distributions on Ω.

For $1 \leq p \leq \infty$, we recall that the Sobolev space $W^{1,p}(\Omega)$ is the space of all complex valued functions $f \in L^p(\Omega)$ with distributional derivatives in $L^p(\Omega)$. The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\partial f\|_{L^p(\Omega)} + \|\overline{\partial} f\|_{L^p(\Omega)},$$

where the operators ∂ and $\overline{\partial}$ are defined, in the sense of distributions: for all $\phi \in \mathcal{D}(\Omega)$,

$$\langle \partial f, \phi \rangle = -\int f \partial \phi, \quad \langle \overline{\partial} f, \phi \rangle = -\int f \overline{\partial} \phi,$$

with $\partial = \frac{1}{2}(\partial_x - i\partial_y)$ and $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$.

Note that, when Ω is C^1 (in particular, when Ω is Dini-smooth), $W^{1,\infty}(\Omega)$ coincides with the space of Lipschitz functions on Ω [15, Thm 4, Sec 5.8]. We will write $L^p_{\Omega_2}(\Omega)$ and $W^{1,p}_{\Omega_2}(\Omega)$ to specify that the functions have values in $\Omega_2 \subset \mathbb{C}$.

3 Generalized Hardy Spaces

3.1 Hardy Spaces

For a detailed study of classical Hardy spaces $H^p(\Omega)$ of analytic functions in $\Omega \subset \mathbb{C}$, we refer to [13,20], and to [29] for annular domains. Let us briefly recall here some basic facts needed in the sequel, see [23] for more details.

For $1 \leq p \leq \infty$, any function $f \in H^p(\mathbb{D})$ has a non-tangential limit a.e. on $\mathbb{T}$ which we call the trace of f and is denoted by $\operatorname{tr} f$. For all $f \in H^p(\mathbb{D})$, we have that $\operatorname{tr} f \in H^p(\mathbb{T})$ where $H^p(\mathbb{T})$ is the strict subspace of $L^p(\mathbb{T})$:

$$H^p(\mathbb{T}) = \left\{ h \in L^p(\mathbb{T}), \hat{h}(n) = \frac{1}{2\pi} \int_0^{2\pi} h(e^{it}) e^{-int} dt = 0,\ n < 0 \right\}.$$

More precisely, $H^p(\mathbb{D})$ is isomorphic to $H^p(\mathbb{T})$ and $\|f\|_{H^p(\mathbb{D})} = \|\operatorname{tr} f\|_{L^p(\mathbb{T})}$, which allows us to identify the two spaces $H^p(\mathbb{D})$ and $H^p(\mathbb{T})$.

The space $H^p(\mathbb{A})$ can be identified to $H^p(\partial\mathbb{A})$ *via* the isomorphic isomorphism $f \in H^p(\mathbb{A}) \longmapsto \operatorname{tr} f \in H^p(\partial\mathbb{A})$ and thus $\|f\|_{H^p(\mathbb{A})} = \|\operatorname{tr} f\|_{L^p(\partial\mathbb{A})}$. Likewise, following [29], the space $H^p(\mathbb{A})$ of the annulus $\mathbb{A} = \mathbb{D} \backslash r_0 \overline{\mathbb{D}}$ can be identified to $H^p(\partial\mathbb{A})$ *via* the isomorphic isomorphism $f \in H^p(\mathbb{A}) \longmapsto \operatorname{tr} f \in H^p(\partial\mathbb{A})$:

$$H^p(\partial\mathbb{A}) = \left\{ h \in L^p(\partial\mathbb{A}), \widehat{h_{|\mathbb{T}}}(n) = r_0^n \, \widehat{h_{|r_0\mathbb{T}}}(n),\ n \in \mathbb{Z} \right\}.$$

From the fact that $|f|^p$ is a subharmonic function when f is analytic on any domain $\Omega \subset \mathbb{C}$, classes of analytic functions have been introduced using harmonic majorants (see [28]) extending the definition of Hardy spaces to general domains in $\mathbb{C}$. More precisely, for $1 \leq p < \infty$ and $z_0 \in \Omega$, $H^p(\Omega)$ is defined as the space of analytic functions f on Ω such that there exists a harmonic function $u : \Omega \longrightarrow [0, \infty)$ such that for $z \in \Omega$

$$|f(z)|^p \leq u(z).$$

The space is equipped with the norm

$$\inf\left\{u(z_0)^{1/p},\ |f|^p \leq u \text{ for } u \text{ harmonic function in } \Omega\right\}.$$

Remark 1 1. It follows from the Harnack inequality [4, 3.6,Ch.3], that different choices of z_0 give rise to equivalent norms in $H^p(\Omega)$.
2. If $\Omega = \mathbb{D}$ or $\mathbb{A}$, the two definitions of Hardy spaces coincide and the two previously defined norms on $H^p(\Omega)$ are equivalent.

3.2 Definitions of Generalized Hardy Spaces

Let $1 < p < \infty$ and $\nu \in W^{1,\infty}_{\mathbb{R}}(\mathbb{D})$ such that $\|\nu\|_{L^\infty(\mathbb{D})} \leq \kappa$ with $\kappa \in (0,1)$. The generalized Hardy space of the unit disc $H^p_\nu(\mathbb{D})$ was first defined in [26] and then in [7] as the collection of all measurable functions $f : \mathbb{D} \longrightarrow \mathbb{C}$ such that $\overline{\partial} f = \nu\, \overline{\partial f}$ in the sense of distributions in $\mathbb{D}$ and

$$\|f\|_{H^p_\nu(\mathbb{D})} := \left(\operatorname{ess\,sup}_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dt\right)^{1/p} < \infty. \tag{2}$$

The definition was extended, in [14], to the annulus $\mathbb{A}$: for $\nu \in W^{1,r}_{\mathbb{R}}(\mathbb{A})$, $r \in (2,\infty)$, $H^p_\nu(\mathbb{A})$ is the space of functions $f : \mathbb{A} \to \mathbb{C}$ such that $\overline{\partial} f = \nu\, \overline{\partial f}$ in the sense of distribution in $\mathbb{A}$ and satisfying

$$\|f\|_{H^p_\nu(\mathbb{A})} := \left(\operatorname{ess\,sup}_{r_0<r<1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p dt\right)^{1/p} < \infty. \tag{3}$$

Now, let $\Omega \subset \mathbb{C}$ be a Dini-smooth domain and ν such that

$$\nu \in W^{1,\infty}_{\mathbb{R}}(\Omega), \qquad \|\nu\|_{L^\infty(\Omega)} \leq \kappa, \quad \text{with } \kappa \in (0,1). \tag{4}$$

The definition of $H^p_\nu(\Omega)$ was further extended to this case in [6], where the norm is defined by

$$\|g\|_{H^p_\nu(\Omega)} := \sup_{n\in\mathbb{N}} \|g\|_{L^p(\partial\Delta_n)}, \tag{5}$$

where $(\Delta_n)_n$ is a fixed sequence of domains such that $\overline{\Delta_n} \subset \Omega$ and $\partial\Delta_n$ is a finite union of rectifiable Jordan curves of uniformly bounded length, such that each compact subset of Ω is eventually contained in Δ_n for n large enough. We refer to [6] for the existence of such sequence.

In parallel with Hardy spaces $H^p_\nu(\Omega)$ (with Ω equal to $\mathbb{D}$, $\mathbb{A}$ or more generally to a Dini-smooth domain), Hardy spaces $G^p_\alpha(\Omega)$ were defined in [6,7,14,26] for $\alpha \in L^\infty(\Omega)$ as the collection of measurable functions $w : \Omega \to \mathbb{C}$ such that $\overline{\partial} w = \alpha\, \overline{w}$ in $\mathcal{D}'(\Omega)$ and

$$\|w\|_{G^p_\alpha(\Omega)} = \left(\text{ess sup}_{\rho<r<1}\frac{1}{2\pi}\int_0^{2\pi}|w(re^{it})|^p dt\right)^{1/p} < \infty, \tag{6}$$

with $\rho = 0$ if $\Omega = \mathbb{D}$ and $\rho = r_0$ if $\Omega = \mathbb{A}$. More generally, if Ω is a Dini-smooth domain, the essential supremum is taken over all the $L^p(\partial\Delta_n)$ norm of w for $n \in \mathbb{N}$.

Remark 2 The generalized Hardy spaces $H^p_\nu(\Omega)$ and $G^p_\alpha(\Omega)$ are real Banach spaces (note that when $\nu = 0$ or $\alpha = 0$ respectively, they are complex Banach spaces).

Recall that if Ω is a bounded Dini-smooth domain, a function g lying in generalized Hardy spaces $H^p_\nu(\Omega)$ or $G^p_\alpha(\Omega)$ has a non-tangential limit *a.e.* on $\partial\Omega$ which is called the trace of g is denoted by $\operatorname{tr} g \in L^p(\partial\Omega)$ and

$$\|g\|_{H^p_\nu(\Omega)} \sim \|\operatorname{tr} g\|_{L^p(\partial\Omega)}, \tag{7}$$

(see [6,7,14]). We will denote by $\operatorname{tr}(H^p_\nu(\Omega))$ the space of traces of $H^p_\nu(\Omega)$-functions; it is a strict subspace of $L^p(\partial\Omega)$. Note also that $g \mapsto \|\operatorname{tr} g\|_{L^p(\partial\Omega)}$ is a norm on $H^p_\nu(\Omega)$, equivalent to the one given by (5). However, contrary to the case of Hardy spaces of analytic functions of the disk, $\|\cdot\|_{H^p_\nu(\mathbb{D})}$ and $\|\operatorname{tr}\cdot\|_{L^p(\mathbb{T})}$ are not equal in general (see (7)).

Finally, functions in $H^p_\nu(\Omega)$ and $G^p_\alpha(\Omega)$ are continuous in Ω:

Lemma 1 *Let $\Omega \subset \mathbb{C}$ be a bounded Dini-smooth domain, $\nu \in W^{1,\infty}_{\mathbb{R}}(\Omega)$ meeting (4) and $\alpha \in L^\infty(\Omega)$. Then, all functions in $G^p_\alpha(\Omega)$ and $H^p_\nu(\Omega)$ are continuous in Ω.*

Proof Indeed, let $\omega \in G^p_\alpha(\Omega)$. By [6, Prop. 3.2], $\omega = e^s F$ with $s \in C(\overline{\Omega})$ (since $s \in W^{1,r}(\Omega)$ for some $r > 2$) and $F \in H^p(\Omega)$. Thus, ω is continuous in Ω. If $f \in H^p_\nu(\Omega)$ and $\omega = \mathcal{J}^{-1}(g)$, then $\omega \in G^p_\alpha(\Omega)$ is continuous in Ω and since ν is continuous and (4) holds, $f \in C(\Omega)$. □

As in the analytic case [13, Ch.10], generalized Hardy classes $E^p_\nu(\Omega)$ (respectively $F^p_\alpha(\Omega)$) can be defined as the space of measurable functions f (respectively w) on Ω solving

$$\overline{\partial} f = \nu\,\overline{\partial f} \quad \text{in } \mathcal{D}'(\Omega), \tag{8}$$

respectively

$$\overline{\partial} w = \alpha\,\overline{w} \quad \text{in } \mathcal{D}'(\Omega), \tag{9}$$

and for which there exists a harmonic function $u : \Omega \to [0, +\infty)$ such that

$$|f(z)|^p \le u(z) \tag{10}$$

(respectively (10) holds for w) for almost every $z \in \Omega$. Fix a point $z_0 \in \Omega$. The space is equipped with the norm

$$\|f\|_{E^p_\nu(\Omega)} := \inf\left\{u^{1/p}(z_0),\, u : \Omega \to [0, +\infty) \text{ harmonic in } \Omega \text{ such that (10) holds}\right\}. \tag{11}$$

Observe that in the above definitions, different values of z_0 give rise to equivalent norms as in Remark 1. We first check:

Proposition 1 1. *The map $f \mapsto \|f\|_{E^p_\nu(\Omega)}$ is a norm on $E^p_\nu(\Omega)$.*
2. *The analogous conclusion holds for $F^p_\alpha(\Omega)$.*

Proof It is plain to see that $\|\cdot\|_{E^p_\nu(\Omega)}$ is positively homogeneous of degree 1 and subadditive. Assume now that $\|f\|_{E^p_\nu(\Omega)} = 0$. That $f = 0$ follows at once from the fact that, if $(u_j)_{j\geq 1}$ is a sequence of nonnegative harmonic functions on Ω such that $u_j(z_0) \to 0$, $j \to \infty$, for $z_0 \in \Omega$ from definition of $E^p_\nu(\Omega)$, then $u_j(z) \to 0$, $j \to \infty$, for all $z \in \Omega$. To check this fact, define

$$A := \left\{z \in \Omega;\ u_j(z) \to 0\right\}.$$

The Harnack inequality [4, 3.6,Ch.3] shows at once that A is open in Ω. If $B = \Omega \backslash A$, then the Harnack inequality also shows that B is open. Because $z_0 \in A \neq \emptyset$ and Ω is connected, then $A = \Omega$, which proves point 1 and, similarly, point 2. □

Remark 3 As [6, Thm 3.5, (ii)] shows, when Ω is a Dini-smooth domain, ν meets (4) and $\alpha \in L^\infty(\Omega)$, $H^p_\nu(\Omega) = E^p_\nu(\Omega)$ and $G^p_\alpha(\Omega) = F^p_\alpha(\Omega)$, with equivalent norms. In this case, if Ω is Dini-smooth, then, for $w \in G^p_\alpha(\Omega)$ we have that

$$\|w\|_{G^p_\alpha(\Omega)} \sim \inf u^{1/p}(z_0),$$

where u and the infimum are taken as in the definition of $F^p_\alpha(\Omega)$. The same stands for $f \in H^p_\nu(\Omega)$.

Let ν satisfying assumption (4) and $\alpha \in L^\infty(\Omega)$ associated with ν in the sense that

$$\alpha = \frac{-\overline{\partial}\nu}{1-\nu^2}, \tag{12}$$

Now, let us recall the link between H^p_ν and G^p_α functions [6,7].

Proposition 2 *A function $f : \Omega \to \mathbb{C}$ belongs to $H^p_\nu(\Omega)$ if and only if*

$$w = \mathcal{J}(f) := \frac{f - \nu\overline{f}}{\sqrt{1-\nu^2}} \tag{13}$$

belongs to $G^p_\alpha(\Omega)$. One has $\|f\|_{H^p_\nu(\Omega)} \sim \|w\|_{G^p_\alpha(\Omega)}$.

Proof That f solves (8) if and only if w solves (9) was checked in [6,7]. That $|f|^p$ has a harmonic majorant if and only if the same holds for $|w|^p$ and $\|f\|_{E^p_\nu(\Omega)} \sim \|w\|_{F^p_\alpha(\Omega)}$ are straightforward consequences of (13) and assumption (4). □

Proposition 2 immediately yields:

Lemma 2 *Let $\nu, \tilde{\nu}$ satisfying (4) and $\alpha, \tilde{\alpha}$ associated with ν (resp. $\tilde{\nu}$) as in Eq. (12). Then, $T \in \mathcal{L}(H^p_\nu(\Omega), H^p_{\tilde{\nu}}(\Omega))$ if and only if $\widetilde{T} \in \mathcal{L}(G^p_\alpha(\Omega), G^p_{\tilde{\alpha}}(\Omega))$ where $\widetilde{\mathcal{J}}T = \widetilde{T}\mathcal{J}$, and $\widetilde{\mathcal{J}}$ is the $\mathbb{R}$-linear isomorphism from $H^p_{\tilde{\nu}}(\mathbb{D})$ onto $G^p_{\tilde{\alpha}}(\Omega)$ defined by (13) with ν replaced by $\widetilde{\nu}$.*

4 Boundedness of Composition Operators on Generalized Hardy Spaces

Let Ω_1, Ω_2 be two bounded Dini-smooth domains in $\mathbb{C}$, ν defined on Ω_2 satisfying assumption (4), and ϕ satisfying:

$$\phi : \Omega_1 \to \Omega_2 \text{ analytic with } \phi \in W^{1,\infty}_{\Omega_2}(\Omega_1). \tag{14}$$

We consider the composition operator C_ϕ defined on $H^p_\nu(\Omega_2)$ by $C_\phi(f) = f \circ \phi$.

Observe first that $\nu \circ \phi \in W^{1,\infty}_{\mathbb{R}}(\Omega_1)$ since ν and ϕ are Lipschitz functions in Ω_2 and Ω_1 respectively and $\|\nu \circ \phi\|_{L^\infty(\Omega_1)} \le \kappa$; hence $\nu \circ \phi$ satisfies (4) on Ω_1.

Proposition 3 *The composition operator $C_\phi : H^p_\nu(\Omega_2) \to H^p_{\nu\circ\phi}(\Omega_1)$ is continuous.*

Proof Let $f \in H^p_\nu(\Omega_2)$. Observe that $f \circ \phi$ is a Lebesgue measurable function on Ω_1 and, since $\overline{\partial}\phi = 0$ in Ω_1,

$$\begin{aligned}\overline{\partial}(f \circ \phi) &= [(\overline{\partial} f) \circ \phi]\overline{\partial}(\overline{\phi}) = (\nu \circ \phi)[\overline{\partial f} \circ \phi]\overline{\partial}(\overline{\phi}) \\ &= (\nu \circ \phi)\overline{(\partial f \circ \phi)\partial\phi} = (\nu \circ \phi)\overline{\partial(f \circ \phi)},\end{aligned}$$

(equalities are considered in the sense of distributions). Now, if u is any harmonic majorant of $|f|^p$ in Ω_2, then $u \circ \phi$ is a harmonic majorant of $|f \circ \phi|^p$ in Ω_1, which proves that $C_\phi(f) \in H^p_{\nu\circ\phi}(\Omega_1)$. Moreover, by the Harnack inequality applied in Ω_2,

$$\left\|C_\phi(f)\right\|_{H^p_{\nu\circ\phi}(\Omega_1)} \le u(\phi(z_0))^{1/p} \le C\, u(z_0)^{1/p},$$

for $z_0 \in \Omega_2$ as in definition of E^p_ν, and where the constant C depends on Ω_2, z_0 and $\phi(z_0)$ but not on u, so that, taking the infimum over all harmonic functions $u \ge |f|^p$ in Ω_2, one concludes

$$\left\|C_\phi(f)\right\|_{H^p_{\nu\circ\phi}(\Omega_1)} \lesssim \|f\|_{H^p_\nu(\Omega_2)}.$$

□

Remark 4 In the case where $\Omega_1 = \Omega_2 = \mathbb{D}$, if $H^p_\nu(\mathbb{D})$ and $H^p_{\nu\circ\phi}(\mathbb{D})$ are equipped with the norms given by (11), the following upper bound for the operator norm of C_ϕ holds:

$$\|C_\phi\| \le \left(\frac{1 + |\phi(0)|}{1 - |\phi(0)|}\right)^{1/p}.$$

Indeed, if u is as before, one obtains

$$\begin{aligned}u \circ \phi(0) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - |\phi(0)|^2}{|e^{it} - \phi(0)|^2} u(e^{it})dt \\ &\le \frac{1 + |\phi(0)|}{1 - |\phi(0)|}\frac{1}{2\pi}\int_0^{2\pi} u(e^{it})dt = \frac{1 + |\phi(0)|}{1 - |\phi(0)|}u(0).\end{aligned}$$

In the doubly-connected case, assume that $\Omega = \mathbb{A}$. Let $z_0 \in \mathbb{A}$ and ψ be an analytic function from $\mathbb{D}$ onto $\mathbb{A}$ such that $\psi(0) = z_0$. Arguing as in [11], we obtain an "explicit" upper bound for $\|C_\phi\|$. Indeed, let u be as before. Using the harmonicity of $u \circ \psi$ in $\mathbb{D}$, for all s such that $\psi(s) = \phi(z_0)$, one has, for all $r \in (|s|, 1)$,

$$\begin{aligned} u(\phi(z_0)) = u(\psi(s)) &= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\left(\frac{r\,e^{it} + s}{r\,e^{it} - s}\right) u \circ \psi(r\,e^{it}) dt \\ &\leq \frac{r + |s|}{r - |s|} u(\psi(0)) = \frac{r + |s|}{r - |s|} u(z_0). \end{aligned}$$

Letting r tend to 1, we obtain

$$u(\phi(z_0)) \leq \inf_{s \in \psi^{-1}(\phi(z_0))} \frac{1 + |s|}{1 - |s|} . u(z_0),$$

which, with definition of $E_\nu^p(\Omega)$, yields $\|C_\phi\| \leq \left(\inf_{s \in \psi^{-1}(\phi(z_0))} \frac{1 + |s|}{1 - |s|} \right)^{1/p}$.

In the sequel, when necessary, we will consider the composition operator defined on G_α^p spaces instead of H_ν^p spaces. The next lemma shows that a composition operator defined on H_ν^p spaces is $\mathbb{R}$-isomorphic to a composition operator on G_α^p spaces.

Lemma 3 *Let ν (resp. ϕ) satisfying (4) (resp. (14)). The composition operator C_ϕ mapping $H_\nu^p(\Omega_2)$ to $H_{\widetilde{\nu}}^p(\Omega_1)$ with $\widetilde{\nu} = \nu \circ \phi$ is then equivalent to the composition operator $\widetilde{C_\phi}$ mapping $G_\alpha^p(\Omega_2)$ to $G_{\widetilde{\alpha}}^p(\Omega_1)$, where $\widetilde{\alpha}$ is associated with $\widetilde{\nu}$ through (12). Moreover,*

$$\widetilde{\alpha} = (\alpha \circ \phi)\overline{\partial \phi}. \tag{15}$$

In other words, for $\mathcal{J}$, $\widetilde{\mathcal{J}}$ defined as in Lemma 2, we have the following commutative diagram:

$$\begin{array}{ccc} H_\nu^p(\Omega_2) & \xrightarrow{C_\phi} & H_{\widetilde{\nu}}^p(\Omega_1) \\ \mathcal{J}\big\downarrow & & \widetilde{\mathcal{J}}\big\downarrow \\ G_\alpha^p(\Omega_2) & \xrightarrow{\widetilde{C}_\phi} & G_{\widetilde{\alpha}}^p(\Omega_1) \end{array}$$

Proof The inverse of $\mathcal{J}$ is given by (see [7]):

$$\mathcal{J}^{-1} : w \in G_\alpha^p(\Omega_2) \longmapsto f = \frac{w + \nu\,\overline{w}}{\sqrt{1 - \nu^2}} \in H_\nu^p(\Omega_2). \tag{16}$$

Note that

$$\widetilde{\alpha} = \frac{-\overline{\partial}\widetilde{\nu}}{1 - \widetilde{\nu}^2} = \frac{-\overline{\partial}(\nu \circ \phi)}{1 - \nu^2 \circ \phi} = \frac{-[(\overline{\partial}\nu) \circ \phi]\overline{\partial\phi}}{1 - \nu^2 \circ \phi} = (\alpha \circ \phi)\overline{\partial\phi},$$

and $\widetilde{\mathcal{J}}$ is also an $\mathbb{R}$-linear isomorphism from $H^p_{\widetilde{\nu}}(\Omega_1)$ onto $G^p_{\widetilde{\alpha}}(\Omega_1)$. Now, for any $f \in H^p_\nu(\Omega_2)$, we have that

$$\widetilde{\mathcal{J}}(C_\phi(f)) = \frac{f \circ \phi - (\nu \circ \phi)\overline{f \circ \phi}}{\sqrt{1 - \nu^2 \circ \phi}} = \left[\frac{f - \nu \overline{f}}{\sqrt{1 - \nu^2}}\right] \circ \phi = \widetilde{C_\phi}(\mathcal{J}(f)).$$

□

5 Invertibility of the Composition Operator on $H^p_\nu(\Omega)$

In this section, we characterize invertible composition operators between H^p_ν spaces.

We will need an observation on the extension of a function ν meeting condition (4). Before stating it, let us recall that, if Ω_1 and Ω_2 are open subsets of $\mathbb{C}$, the notation $\Omega_1 \subset\subset \Omega_2$ means that $\overline{\Omega_1}$ is a compact included in Ω_2.

Lemma 4 *Let $\Omega_1 \subset\subset \Omega_2 \subset \mathbb{C}$ be bounded domains and ν be a Lipschitz function on Ω_1 meeting condition* (4). *There exists a Lipschitz function $\widetilde{\nu}$ on $\mathbb{C}$ such that*:

1. $\widetilde{\nu}(z) = \nu(z)$ *for all $z \in \Omega_1$*,
2. *the support of $\widetilde{\nu}$ is a compact included in Ω_2*,
3. $\|\widetilde{\nu}\|_{L^\infty(\mathbb{C})} < 1$.

Proof Extend first ν to a compactly supported Lipschitz function on $\mathbb{C}$, denoted by ν_1. There exists an open set Ω_3 such that $\Omega_1 \subset\subset \Omega_3 \subset\subset \Omega_2$ and $\|\nu_1\|_{L^\infty(\Omega_3)} < 1$. Let $\chi \in \mathcal{D}(\mathbb{C})$ be such that $0 \leq \chi(z) \leq 1$ for all $z \in \mathbb{C}$, $\chi(z) = 1$ for all $z \in \Omega_1$ and $\chi(z) = 0$ for all $z \notin \Omega_3$. The function $\widetilde{\nu} := \chi \nu_1$ satisfies all the requirements. □

Let $1 < p < +\infty$, $\Omega \subset \mathbb{C}$ be a bounded Dini-smooth domain and ν meet (4). For $z \in \Omega$, let $\mathcal{E}^\nu_z$, $\mathcal{F}^\nu_z$ be the real-valued evaluation maps at z defined on $H^p_\nu(\Omega)$ and $G^p_\alpha(\Omega)$ by

$$\mathcal{E}^\nu_z(f) := \operatorname{Re} f(z) \quad \text{and} \quad \mathcal{F}^\nu_z(f) := \operatorname{Im} f(z) \quad \text{for all } f \in H^p_\nu(\Omega),\ f \in G^p_\alpha(\Omega).$$

Proposition 4 *For $z \in \Omega$, the evaluation maps $\mathcal{E}^\nu_z$ and $\mathcal{F}^\nu_z$ are continuous on $H^p_\nu(\Omega)$ and $G^p_\alpha(\Omega)$.*

Proof Let $f \in H^p_\nu(\Omega)$ and $z \in \Omega$. By definition of the norm in $H^p_\nu(\Omega)$, there exists a harmonic function u in Ω such that $|f|^p \leq u$ in Ω with $u^{1/p}(z_0) \lesssim \|f\|_{H^p_\nu(\Omega)}$ for a fixed $z_0 \in \Omega$. The Harnack inequality then yields

$$|f(z)| \leq u^{1/p}(z) \lesssim u^{1/p}(z_0) \lesssim \|f\|_{H^p_\nu(\Omega)}$$

and thus we have

$$|\operatorname{Re} f(z)| \lesssim \|f\|_{H^p_\nu(\Omega)} \quad \text{and} \quad |\operatorname{Im} f(z)| \lesssim \|f\|_{H^p_\nu(\Omega)},$$

which ends the proof. □

For the characterization of invertible composition operators on H^p_ν spaces, we will need the fact that $H^p_\nu(\Omega)$ separates points in Ω, when Ω is a Dini-smooth domain:

Lemma 5 *Assume that $\Omega \subset \mathbb{C}$ is a bounded Dini-smooth domain. Let $z_1 \neq z_2 \in \Omega$. Then, there exists $f \in H^p_\nu(\Omega)$ such that $f(z_1) \neq f(z_2)$.*

Proof There exists $F \in H^p(\Omega)$ such that $F(z_1) = 0$ and $F(z_2) \neq 0$ (take, for instance, $F(z) = z - z_1$). By Theorem 4, there exists $s \in W^{1,r}(\Omega)$, for some $r \in (2, +\infty)$ such that $w = e^s F \in G^p_\alpha(\Omega)$. One has $w(z_1) = 0$ and $w(z_2) \neq 0$. If $f := \mathcal{J}^{-1}(w) = \frac{w+\nu\overline{w}}{\sqrt{1-\nu^2}}$, $f \in H^p_\nu(\Omega)$ by Proposition 2, $f(z_1) = 0$ and $f(z_2) \neq 0$, since $\|\nu\|_{L^\infty(\Omega)} < 1$. $\square$

We will also use in the sequel a regularity result for a solution of a Dirichlet problem for Eq. (8), where the boundary data is C^1 and only prescribed on one curve of $\partial\Omega$:

Lemma 6 *Let $\Omega \subset \mathbb{C}$ be a bounded n-connected Dini-smooth domain. Write $\partial\Omega = \cup_{j=0}^n \Gamma_j$, where the Γ_j are pairwise disjoint Jordan curves. Fix $j \in \{0, \dots, n\}$. Let ν meet (4) and $\psi \in C^1_{\mathbb{R}}(\Gamma_j)$. There exists $f \in H^p_\nu(\Omega)$ such that Re tr $f = \psi$ on Γ_j and $\|f\|_{H^p_\nu(\Omega)} \lesssim \|\psi\|_{L^p(\Gamma_j)}$. Moreover, $f \in C(\overline{\Omega})$.*

Proof **Step 1:** Let us first assume that $\Omega = \mathbb{D}$. Since $\psi \in W^{1-1/q,q}_{\mathbb{R}}(\mathbb{T})$ for some $q > \max(2, p)$, the result [7, Thm 4.1.1] shows that there exists $f \in W^{1,q}(\mathbb{D})$ solving $\overline{\partial} f = \nu\overline{\partial f}$ in $\mathbb{D}$ with Re tr $f = \psi$ on $\mathbb{T}$. By [7, Prop. 4.3.3], $f \in H^q_\nu(\mathbb{D}) \subset H^p_\nu(\mathbb{D})$, and since $q > 2$, f is continuous on $\overline{\mathbb{D}}$.

Step 2: Assume that $\Omega = \mathbb{C}\backslash r_0\overline{\mathbb{D}}$ for some $r_0 \in (0, 1)$. Let $\psi \in C^1_{\mathbb{R}}(r_0\mathbb{T})$. For all $z \in \mathbb{T}$, define $\widetilde{\psi}(z) := \psi\left(\frac{r_0}{z}\right)$ and, for all $z \in \mathbb{D}$, define $\widetilde{\nu}(z) := \nu\left(\frac{r_0}{z}\right)$. Step 1 yields a function $\widetilde{f} \in H^p_{\widetilde{\nu}}(\mathbb{D})$, continuous on $\overline{\mathbb{D}}$, such that Re tr $\widetilde{f} = \widetilde{\psi}$ on $\mathbb{T}$. Define now $f(z) := \overline{\widetilde{f}\left(\frac{r_0}{z}\right)}$ for all $z \in \Omega$. Then, $f \in H^p_\nu(\Omega)$, f is continuous on $\overline{\Omega}$ and Re tr $f = \psi$ on $r_0\mathbb{T}$.

Step 3: Assume now that $\Omega = \mathbb{G}$ is a circular domain, as in (1). Extend ν to a function $\widetilde{\nu} \in W^{1,\infty}_{\mathbb{R}}(\mathbb{C})$ satisfying the properties of Lemma 4. If $\psi \in C^1_{\mathbb{R}}(\mathbb{T})$, step 1 provides a function $f \in H^p_{\widetilde{\nu}}(\mathbb{D})$, continuous on $\overline{\mathbb{D}}$, and such that Re tr $f = \psi$ on $\mathbb{T}$. The restriction of f to $\mathbb{G}$ belongs to $H^p_\nu(\mathbb{G})$ and satisfies all the requirements. If $\psi \in C^1_{\mathbb{R}}(a_j + r_j\mathbb{T})$, argue similarly using Step 2 instead of Step 1.

Step 4: Finally, in the general case where Ω is a Dini-smooth n-connected domain, Ω is conformally equivalent to a circular domain $\mathbb{G}$, via a conformal map which is C^1 up to the boundary of Ω, and we conclude the proof using Step 3. $\square$

Let Ω_1, Ω_2 be domains in $\mathbb{C}$ and $\phi : \Omega_1 \to \Omega_2$ be analytic with $\phi \in W^{1,\infty}_{\Omega_2}(\Omega_1)$. The adjoint of the operator C_ϕ will play an important role in the following arguments. Note first that, by Proposition 3, C^*_ϕ is a bounded linear operator from $(H^p_{\nu\circ\phi}(\Omega_1))'$ to $(H^p_\nu(\Omega_2))'$. Moreover:

Lemma 7 *For all $z \in \Omega_1$, $C^*_\phi(\mathcal{E}^{\nu\circ\phi}_z) = \mathcal{E}^\nu_{\phi(z)}$ and $C^*_\phi(\mathcal{F}^{\nu\circ\phi}_z) = \mathcal{F}^\nu_{\phi(z)}$.*

Proof Let $f \in H^p_\nu(\Omega_2)$. Then

$$\langle C^*_\phi(\mathcal{E}^{\nu\circ\phi}_z), f\rangle = \langle \mathcal{E}^{\nu\circ\phi}_z, C_\phi(f)\rangle = \langle \mathcal{E}^{\nu\circ\phi}_z, f\circ\phi\rangle = \operatorname{Re} f(\phi(z)) = \langle \mathcal{E}^{\nu}_{\phi(z)}, f\rangle,$$

and the argument is analogous for $\mathcal{F}^\nu_z$. □

Theorem 1 *Assume that* Ω_1, Ω_2 *are bounded Dini-smooth domains. Then, the composition operator* $C_\phi : H^p_\nu(\Omega_2) \to H^p_{\nu\circ\phi}(\Omega_1)$ *is invertible if, and only if,* ϕ *is a bijection from* Ω_1 *onto* Ω_2.

Proof Some ideas of this proof are inspired by [10, Thm 2.1]. If ϕ is invertible, then $C_{\phi^{-1}} = (C_\phi)^{-1}$.

Assume conversely that C_ϕ is invertible. Since C_ϕ is one-to-one with closed range, for all $L \in (H^p_{\nu\circ\phi}(\Omega_1))'$, one has

$$\left\| C^*_\phi L \right\|_{(H^p_\nu(\Omega_2))'} \gtrsim \|L\|_{(H^p_{\nu\circ\phi}(\Omega_1))'}. \tag{17}$$

Let $z_1, z_2 \in \Omega_1$ be such that $\phi(z_1) = \phi(z_2)$. Then, by Lemma 7,

$$C^*_\phi(\mathcal{E}^{\nu\circ\phi}_{z_1}) = \mathcal{E}^{\nu}_{\phi(z_1)} = \mathcal{E}^{\nu}_{\phi(z_2)} = C^*_\phi(\mathcal{E}^{\nu\circ\phi}_{z_2}).$$

Since C^*_ϕ is invertible, it follows that $\mathcal{E}^{\nu\circ\phi}_{z_1} = \mathcal{E}^{\nu\circ\phi}_{z_2}$. Similarly, $\mathcal{F}^{\nu\circ\phi}_{z_1} = \mathcal{F}^{\nu\circ\phi}_{z_2}$, so that $z_1 = z_2$ by Lemma 5, and ϕ is univalent.

Now, suppose that ϕ is not surjective. We claim that

$$\partial\phi(\Omega_1) \cap \Omega_2 \neq \varnothing. \tag{18}$$

Indeed, since ϕ is analytic and not constant in Ω_1, it is an open mapping, so that $\Omega_2 = \phi(\Omega_1) \cup (\Omega_2 \cap \partial\phi(\Omega_1)) \cup (\Omega_2 \backslash \overline{\phi(\Omega_1)})$, the union being disjoint. Assume now by contradiction that (18) is false. Then Ω_2 is the union of the two disjoints open sets in Ω_2, $\phi(\Omega_1)$ and $\Omega_2 \backslash \overline{\phi(\Omega_1)}$. One clearly has $\phi(\Omega_1) \neq \varnothing$. The connectedness of Ω_2 therefore yields that $\Omega_2 \backslash \overline{\phi(\Omega_1)} = \varnothing$. In other words,

$$\Omega_2 \subset \overline{\phi(\Omega_1)}. \tag{19}$$

But since ϕ is assumed not to be surjective, there exists $a \in \Omega_2 \backslash \phi(\Omega_1)$, and (19) shows that $a \in \Omega_2 \cap \partial\phi(\Omega_1)$, which gives a contradiction, since we assumed that (18) was false. Finally, (18) is proved.

Let $a \in \partial\phi(\Omega_1) \cap \Omega_2$ and $(z_n)_{n\in\mathbb{N}}$ be a sequence of Ω_1 such that

$$\phi(z_n) \underset{n\to\infty}{\longrightarrow} a.$$

Up to a subsequence, there exists $z \in \overline{\Omega_1}$ such that $z_n \underset{n\to\infty}{\longrightarrow} z$. Note that $z \in \partial\Omega_1$, otherwise $\phi(z) = a$ which is impossible (indeed, since $a \in \partial\phi(\Omega_1)$ and $\phi(\Omega_1)$ is open, thus $a \notin \phi(\Omega_1)$). Write $\partial\Omega_1 = \cup_{j=0}^{n}\Gamma_j$, where the Γ_j are pairwise disjoint Jordan curves, so that $z \in \Gamma_m$ for some $m \in \{0, \dots, n\}$.

Now, we claim that

$$\|\mathcal{E}_{z_n}^{\nu\circ\phi}\|_{(H^p_{\nu\circ\phi}(\Omega_1))'} \underset{n\to\infty}{\longrightarrow} +\infty.$$

Indeed, by the very definition of the norm in $(H^p_{\nu\circ\phi}(\Omega_1))'$,

$$\|\mathcal{E}_{z_n}^{\nu\circ\phi}\|_{(H^p_{\nu\circ\phi}(\Omega_1))'} = \sup_{\substack{g\in H^p_{\nu\circ\phi}(\Omega_1)\\ \|\mathrm{tr}\, g\|_p\le 1}} |\mathrm{Re}\, g(z_n)|. \tag{20}$$

For any $k \in \mathbb{N}$, there is $f_k \in H^p_{\nu\circ\phi}(\Omega_1)$ such that $|f_k(z_n)| \underset{n\to\infty}{\longrightarrow} k$ and $\|f_k\|_{H^p_{\nu\circ\phi}(\Omega_1)} \le 1$. Indeed, let $\psi_k \in C^1_{\mathbb{R}}(\Gamma_m)$ be such that $|\psi_k(z)| = 2k$ and $\|\psi_k\|_{L^p(\Gamma_m)} \le \frac{1}{C}$, where C is the implicit constant in Lemma 6. It follows from Lemma 6 that there is $f_k \in H^p_{\nu\circ\phi}(\Omega_1)$, continuous on $\overline{\Omega_1}$, such that $\mathrm{Re}\, tr(f_k) = \psi_k$ on Γ_m and $\|f_k\|_{H^p_{\nu\circ\phi}(\Omega_1)} \le 1$. Observe that, since f_k is continuous in $\overline{\Omega_1}$, $|\mathrm{Re}\ f_k(z_n)| \to |\psi_k(z)|$. As a consequence, there is $N_k \in \mathbb{N}$ such that

$$|\mathrm{Re}\ f_k(z_n)| \ge k$$

for all $n \ge N_k$. Therefore, by (20),

$$\|\mathcal{E}_{z_n}^{\nu\circ\phi}\|_{(H^p_{\nu\circ\phi}(\Omega_1))'} \ge k, \quad n \ge N_k.$$

Thus, $\|\mathcal{E}_{z_n}^{\nu\circ\phi}\|_{(H^p_{\nu\circ\phi}(\Omega_1))'} \to \infty$ as $n \to \infty$, as claimed.

Moreover, by Lemma 1, for all $g \in H^p_\nu(\Omega_2)$, $\mathcal{E}^\nu_{\phi(z_n)}(g) \underset{n\to\infty}{\longrightarrow} \mathcal{E}^\nu_a(g)$ which proves that

$$\sup_{n\in\mathbb{N}} |\mathcal{E}^\nu_{\phi(z_n)}(g)| < \infty.$$

It follows from the Banach–Steinhaus theorem that the $\|\mathcal{E}^\nu_{\phi(z_n)}\|_{(H^p_\nu(\Omega_2))'}$ are uniformly bounded. Thus, we have that

$$\frac{\|C^*_\phi(\mathcal{E}_{z_n}^{\nu\circ\phi})\|_{(H^p_\nu(\Omega_2))'}}{\|\mathcal{E}_{z_n}^{\nu\circ\phi}\|_{(H^p_{\nu\circ\phi}(\Omega_1))'}} = \frac{\left\|\mathcal{E}^\nu_{\phi(z_n)}\right\|_{(H^p_\nu(\Omega_2))'}}{\|\mathcal{E}_{z_n}^{\nu\circ\phi}\|_{(H^p_{\nu\circ\phi}(\Omega_1))'}} \underset{n\to\infty}{\longrightarrow} 0,$$

which contradicts (17). We conclude that ϕ is surjective. □

Remark 5 1. To our knowledge, the conclusion of Theorem 1 is new, even for Hardy spaces of analytic functions when Ω_1 or Ω_2 are multi-connected.

2. The proof of Theorem 1 does not use the explicit description of the dual of $H^p_\nu(\Omega_2)$ and $H^p_{\nu\circ\phi}(\Omega_1)$. Such a description exists when Ω_1 and Ω_2 are simply connected (see [7, Thm 4.6.1]).
3. The proof of Theorem 1 shows, in particular, that if C_ϕ is one-to-one, then so is ϕ.

It follows easily from Theorem 1 and Lemma 3 that:

Corollary 1 *Let Ω_1, Ω_2 be bounded Dini-smooth domains and $\phi \in W^{1,\infty}_{\Omega_2}(\Omega_1)$ be analytic in Ω_1. Let $\alpha \in L^\infty(\Omega_2)$. Then $C_\phi : G^p_\alpha(\Omega_2) \to G^p_{\tilde{\alpha}}(\Omega_1)$ is an isomorphism if and only if ϕ is a bijection from Ω_1 onto Ω_2.*

The characterizations given in Theorem 1 and Corollary 1 are the same as in the analytic case when $\Omega = \mathbb{D}$ (see [30]).

6 Isometries and Composition Operators on Generalized Hardy Spaces

Throughout this section, Ω will denote the unit disc $\mathbb{D}$ or the annulus $\mathbb{A}$ and $G^p_\alpha(\Omega)$ is equipped with the norm [see (7)]:

$$\|\omega\|_{G^p_\alpha(\Omega)} := \|\mathrm{tr}\,\omega\|_{L^p(\partial\Omega)}$$

The arguments below rely on the following observation:

Lemma 8 *Let Ω be the unit disc $\mathbb{D}$ or the annulus $\mathbb{A}$, $\phi : \Omega \to \Omega$ be a function in $W^{1,\infty}_\Omega(\Omega)$ analytic in Ω and $\alpha \in L^\infty(\Omega)$. Assume that C_ϕ is an isometry from $G^p_\alpha(\Omega)$ to $G^p_{\tilde{\alpha}}(\Omega)$. Then $\phi(\partial\Omega) \subset \partial\Omega$.*

Proof Assume by contradiction that the conclusion does not hold, so that there exists $B_0 \subset \partial\Omega$ with $m(B_0) > 0$ such that $\phi(B_0) \subset \Omega$.

For $\Omega = \mathbb{A}$, either B_0 is entirely contained in $\mathbb{T}$ or in $r_0\mathbb{T}$ or there exists a Borel set $B \subsetneq B_0$ of positive Lebesgue measure such that $B \subset \mathbb{T}$. For the last case, we still write B_0 instead of B and we can assume without loss of generality that $B_0 \subset \mathbb{T}$. Indeed, if $B_0 \subset r_0\mathbb{T}$, it is enough to use the composition with the inversion $\mathrm{Inv} : z \mapsto \frac{r_0}{z}$ since it is easy to check that the composition operator C_{Inv} is a unitary operator (invertible and isometric) on $G^p_\alpha(\Omega)$ using [6, Prop. 3.2].

The following argument is reminiscent of [8]. Let $\phi_1 := \phi$ and $\phi_{n+1} := \phi \circ \phi_n$ for all integer $n \geq 1$. Note that $\phi_k(B_0) \subset \Omega$ for all $k \geq 1$. For all integer $n \geq 1$, define

$$B_n := \{z \in \partial\Omega;\ \phi_n(z) \in B_0\}.$$

Observe that the B_n are pairwise disjoint. Indeed, if $z \in B_n \cap B_m \neq \emptyset$ with $n > m$, then

$$\phi_n(z) \in B_0 \quad \text{and} \quad \phi_m(z) \in B_0,$$

so that

$$\phi_{n-m}(\phi_m(z)) = \phi_n(z) \in B_0 \cap \phi_{n-m}(B_0) \subset B_0 \cap \Omega = \emptyset,$$

which is impossible.

Fix a function $F \in H^p(\Omega)$ such that

$$|\mathrm{tr}\ F| \begin{cases} =1 & \text{on } B_0 \\ \leq 1/2 & \text{on } \partial\Omega \backslash B_0. \end{cases} \tag{21}$$

We claim that such a function exists.

Indeed, if Ω is the unit disc $\mathbb{D}$, the outer function F defined as follows

$$F(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|g(e^{i\theta})| d\theta\right), \quad z \in \mathbb{D}, \tag{22}$$

with $g \in L^p(\mathbb{T})$ such that $|g| = 1$ on B_0 and $|g| = \frac{1}{2}$ on $\partial\Omega\backslash B_0$ satisfies the required conditions.

If $\Omega = \mathbb{A}$, we consider the function $f \in H^p(\mathbb{D})$ defined as in Eq. (22) and $g : \mathbb{A} \to \mathbb{C}$ is the restriction of f to $\mathbb{A}$. Observe that g is in $H^p(\mathbb{A})$ for each p, since $|g|^p = |f|^p \leq u$, where u is a harmonic function in $\mathbb{D}$. Set $M = \max_{\mathbb{T}_{r_0}} |g|$. Now let $\widetilde{g}_n(z) = z^n g(z)$, then for $z \in \mathbb{T}$ we have

$$|\widetilde{g}_n(z)| = |z^n g(z)| = |g(z)| = \begin{cases} 1 & \text{for } z \in B_0 \\ \frac{1}{2} & \text{for } z \in \mathbb{T}\backslash B_0 \end{cases}.$$

For $z \in \mathbb{T}_{r_0}$, we get

$$|\widetilde{g}_n(z)| = |z^n g(z)| = |r_0^n| \cdot |g(z)| \leq r_0^n M.$$

Now, $r_0 < 1$ so pick N large enough to ensure that $r_0^N M < 1/2$, and $F = \widetilde{g}_N$ has the requested properties.

Now, for all integer $j \geq 1$, $F^j \in H^p(\Omega)$ and

$$\lim_{j\to+\infty} \left\| F^j \right\|^p_{H^p(\Omega)} = m(B_0).$$

Moreover, by the maximum principle, since F is not constant in Ω,

$$|F(z)| < 1 \quad \text{for all } z \in \Omega. \tag{23}$$

By [5, Thm 1] and Theorem 4, for all $j \geq 1$, there exists a function $s_j \in C(\overline{\Omega})$ (indeed, $s_j \in W^{1,r}(\Omega)$ for some $r > 2$) with $\mathrm{Re}\ s_j = 0$ on $\partial\Omega$ such that

$$w_j := e^{s_j} F^j \in G^p_\alpha(\Omega) \quad \text{and} \quad \left\| s_j \right\|_{L^\infty(\Omega)} \leq 4 \left\| \alpha \right\|_{L^\infty(\Omega)}.$$

Thus, since Re $ps_j = 0$ on $\partial\Omega$,

$$\|w_j\|^p_{G^p_\alpha(\Omega)} = \int_{\partial\Omega} |\mathrm{tr}\, w_j|^p = \int_{\partial\Omega} \left|e^{ps_j}\right| |\mathrm{tr}\, F|^{jp}$$

$$= \int_{\partial\Omega} |\mathrm{tr}\, F|^{jp} = \left\|F^j\right\|^p_{H^p(\Omega)} \to m(B_0). \tag{24}$$

Let $\widetilde{\alpha}_0 := \widetilde{\alpha} = (\alpha \circ \phi)\overline{\partial\phi}$ and $\widetilde{\alpha}_{n+1} := (\widetilde{\alpha}_n \circ \phi)\overline{\partial\phi}$, $n \in \mathbb{N}$. Since C_ϕ is an isometry from $G^p_\alpha(\Omega)$ to $G^p_{\tilde{\alpha}}(\Omega)$, for all integers $n, j \geq 1$,

$$\left\|(C_\phi)^n w_j\right\|^p_{G^p_{\widetilde{\alpha}_n}(\Omega)} = \|w_j\|^p_{G^p_\alpha(\Omega)}. \tag{25}$$

But

$$(C_\phi)^n w_j = w_j \circ \phi_n. \tag{26}$$

For all $z \in B_n$, $\phi_n(z) \in B_0$ so that, for all $j, n \geq 1$,

$$\left|\mathrm{tr}\, w_j \circ \phi_n(z)\right| = |\mathrm{tr}\, F(\phi_n(z))|^j = 1. \tag{27}$$

For all $z \in \partial\Omega\backslash B_n$, $\phi_n(z) \in \overline{\Omega}\backslash B_0$, so that

$$\left|w_j \circ \phi_n(z)\right| \leq e^{4\|\alpha\|_{L^\infty(\Omega)}} |F(\phi_n(z))|^j \tag{28}$$

if $\phi_n(z) \in \Omega$ and

$$\left|\mathrm{tr}\, w_j \circ \phi_n(z)\right| \leq e^{4\|\alpha\|_{L^\infty(\Omega)}} |\mathrm{tr}\, F(\phi_n(z))|^j \tag{29}$$

if $\phi_n(z) \in \partial\Omega\backslash B_0$. Gathering (21), (23), (25), (26), (27), (28) and (29), one obtains, by the dominated convergence theorem,

$$\lim_{j\to+\infty} \|\omega_j\|^p_{G^p_\alpha(\Omega)} = m(B_n). \tag{30}$$

Comparing (24) and (30) yields $m(B_n) = m(B_0)$ for all integer $n \geq 1$. Since $m(B_0) > 0$ and the B_n are pairwise disjoint, we reach a contradiction. Finally, $\phi(\partial\Omega) \subset \partial\Omega$. □

6.1 Simply Connected Domains

We can now state:

Theorem 2 *Let $\phi : \mathbb{D} \to \mathbb{D}$ satisfying* (14), *let $\alpha \in L^\infty(\mathbb{D})$ and the associated $\tilde{\alpha}$ given by* (15). *Then the following assertions are equivalent*:

1. *C_ϕ is an isometry from $G^p_\alpha(\mathbb{D})$ to $G^p_{\tilde\alpha}(\mathbb{D})$,*
2. *C_ϕ is an isometry from $H^p(\mathbb{D})$ to $H^p(\mathbb{D})$,*
3. *$\phi(0) = 0$ and $\phi(\mathbb{T}) \subset \mathbb{T}$.*

Proof The equivalence between 2 and 3 is contained in [18, Thm 1]. We now prove that 1 and 2 are equivalent. Assume first that C_ϕ is an isometry from $G^p_\alpha(\mathbb{D})$ to $G^p_{\tilde\alpha}(\mathbb{D})$. Let $F \in H^p(\mathbb{D})$. By [5, Thm 1], there exists $s \in C(\overline{\mathbb{D}})$ such that $\operatorname{Re} s = 0$ on $\mathbb{T}$ and $w := e^s F \in G^p_\alpha(\mathbb{D})$. Then, since $\phi(\mathbb{T}) \subset \mathbb{T}$ by Lemma 8, one obtains

$$\left\|C_\phi F\right\|_{H^p(\mathbb{D})} = \left\|C_\phi w\right\|_{G^p_{\tilde\alpha}(\mathbb{D})} = \|w\|_{G^p_\alpha(\mathbb{D})} = \|F\|_{H^p(\mathbb{D})} .$$

Assume now that C_ϕ is an isometry on $H^p(\mathbb{D})$. Then 3 holds, so that $\phi(\mathbb{T}) \subset \mathbb{T}$. Let $w \in G^p_\alpha(\mathbb{D})$. Pick up $s \in C(\overline{\mathbb{D}})$ and $F \in H^p(\mathbb{D})$ such that $w = e^s F$, with $\operatorname{Re} s = 0$ on $\mathbb{T}$. Since $|e^s| = 1$ on $\mathbb{T}$ and $\phi(\mathbb{T}) \subset \mathbb{T}$,

$$\|w \circ \phi\|_{G^p_{\tilde\alpha}(\mathbb{D})} = \left\|e^{s\circ\phi} F \circ \phi\right\|_{G^p_\alpha(\mathbb{D})} = \|F \circ \phi\|_{H^p(\mathbb{D})} = \|F\|_{H^p(\mathbb{D})} = \|w\|_{G^p_\alpha(\mathbb{D})} .$$

□

Remark 6 The conclusion of Theorem 2 shows that C_ϕ is an isometry on $H^p(\mathbb{D})$ if and only if it is an isometry from $G^p_\alpha(\mathbb{D})$ to $G^p_{\tilde\alpha}(\mathbb{D})$ for *all* functions $\alpha \in L^\infty(\mathbb{D})$ and associated $\tilde\alpha$ given by (15).

Corollary 2 *Let $\phi : \mathbb{D} \to \mathbb{D}$ be a function satisfying (14), $\alpha \in L^\infty(\mathbb{D})$ and the associated $\tilde\alpha$ given by (15). Then C_ϕ is an isometry from $G^p_\alpha(\mathbb{D})$* **onto** *$G^p_{\tilde\alpha}(\mathbb{D})$ if and only if there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\phi(z) = \lambda z$ for all $z \in \mathbb{D}$.*

Proof Assume that C_ϕ is an isometry from G^p_α *onto* $G^p_{\tilde\alpha}$. Then C_ϕ is an isomorphism from G^p_α *onto* $G^p_{\tilde\alpha}$, and Theorem 1 shows that ϕ is bijective from $\mathbb{D}$ to $\mathbb{D}$. Moreover, Theorem 2 yields $\phi(0) = 0$ and $\phi(\mathbb{T}) \subset \mathbb{T}$. These conditions on ϕ imply that there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\phi(z) = \lambda z$. The converse is obvious. □

Remark 7 Note that the domain under consideration in Theorem 2 is the unit disk $\mathbb{D}$. One may wonder how to extend the conclusion of Theorem 2 to the case of simply-connected Dini-smooth domains. A natural way to try to do this is to use conformal maps. If ϕ is an analytic bijection from $\mathbb{D}$ to $\mathbb{D}$, it is a simple observation ([13, Ch. 10], [33, Prop. 1.2]) that $f \mapsto (f \circ \phi)(\phi')^{\frac{1}{p}}$, which is a *weighted* composition operator, is an isometry from $H^p(\mathbb{D})$ onto itself. This observation extends to the case where ϕ is an analytic bijection from $\mathbb{D}$ to any simply connected domain $\Omega \subsetneq \mathbb{C}$. But even in the Dini-smooth case and for Hardy spaces of analytic functions, this does not seem to provide a necessary and sufficient condition on ϕ which ensures that the *unweighted* composition operator $f \mapsto f \circ \phi$ is an isometry on $H^p(\Omega)$.

Note that the weighted composition operator $f \mapsto (f \circ \phi)(\phi')^{\frac{1}{p}}$ plays an important role in the study of composition operators on Hardy spaces of analytic functions on arbitrary simply connected domains of $\mathbb{C}$ [19,33], starting from the corresponding study on $\mathbb{D}$.

Let us now turn to the isometry property for the composition operator on $H^p_\nu(\mathbb{D})$.

Proposition 5 *Let $\phi : \mathbb{D} \to \mathbb{D}$ be a function in $W^{1,\infty}(\mathbb{D})$ analytic in $\mathbb{D}$. If C_ϕ is an isometry from $H^p_\nu(\mathbb{D})$ to $H^p_{\nu\circ\phi}(\mathbb{D})$, then $\phi(\mathbb{T}) \subset \mathbb{T}$ and $\phi(0) = 0$.*

Proof Assume that C_ϕ is an isometry from $H^p_\nu(\mathbb{D})$ to $H^p_{\nu\circ\phi}(\mathbb{D})$. We first claim that there exists $C > 0$ such that, for all $w \in G^p_\alpha(\mathbb{D})$ and all integer $n \geq 1$,

$$C^{-1} \|\mathrm{tr}\, w\|_{L^p(\mathbb{T})} \leq \|\mathrm{tr}\,(w \circ \phi_n)\|_{L^p(\mathbb{T})} \leq C \|\mathrm{tr}\, w\|_{L^p(\mathbb{T})}, \tag{31}$$

where, as in the proof of Lemma 8, $\phi_1 := \phi$ and $\phi_{n+1} := \phi \circ \phi_n$ for all integer $n \geq 1$. Indeed, let $w \in G^p_\alpha(\mathbb{D})$ and set $f := \frac{w+\nu\overline{w}}{\sqrt{1-\nu^2}}$. Then, since $\|\nu\|_{L^\infty(\mathbb{D})} < 1$, one has, for almost every $z \in \mathbb{T}$,

$$|\mathrm{tr}\, w(z)| \sim |\mathrm{tr}\, f(z)|. \tag{32}$$

As a consequence,

$$\|\mathrm{tr}\, w\|_{L^p(\mathbb{T})} \sim \|\mathrm{tr}\, f\|_{L^p(\mathbb{T})} \tag{33}$$

and, for all $n \geq 1$,

$$\|\mathrm{tr}\,(w \circ \phi_n)\|_{L^p(\mathbb{T})} \sim \|\mathrm{tr}\,(f \circ \phi_n)\|_{L^p(\mathbb{T})}, \tag{34}$$

where the implicit constant in (34) does not depend on n. Since C_ϕ is an isometry on $H^p_\nu(\mathbb{D})$, it follows that, for all $n \geq 1$,

$$\|\mathrm{tr}\,(f \circ \phi_n)\|_{L^p(\mathbb{T})} = \|\mathrm{tr}\, f\|_{L^p(\mathbb{T})}, \tag{35}$$

and (33), (34) and (35) yield (31).

Let us now establish that $\phi(\mathbb{T}) \subset \mathbb{T}$. Argue by contradiction and let B_n (for all $n \geq 0$) as in the proof of Lemma 8. Consider a function $F \in H^p(\mathbb{D})$ and define the functions s_j and w_j as in the proof of Lemma 8. By (31), for all integers $n, j \geq 1$,

$$\left\|\mathrm{tr}\, w_j \circ \phi_n\right\|^p_{L^p(\mathbb{T})} \sim \left\|\mathrm{tr}\, w_j\right\|^p_{L^p(\mathbb{T})}. \tag{36}$$

But, as already seen,

$$\left\|\mathrm{tr}\, w_j \circ \phi_n\right\|^p_{L^p(\mathbb{T})} \to m(B_n),$$

so that, by (36), $m(B_n) \gtrsim m(B_0)$ for all integer $n \geq 1$. Since $m(B_0) > 0$ and the B_n are pairwise disjoint, we reach a contradiction. Finally, $\phi(\mathbb{T}) \subset \mathbb{T}$.

Let us now prove that $\phi(0) = 0$. Recall now that, since C_ϕ is an isometry, for all functions $f, g \in H^p_\nu(\mathbb{D})$, see [25, Lem. 1.1]:

$$\int_{\mathbb{T}} (\mathrm{tr}\, f \circ \phi)\, |\mathrm{tr}\, g \circ \phi|^{p-2}\, \overline{\mathrm{tr}\, g \circ \phi} = \int_{\mathbb{T}} \mathrm{tr}\, f\, |\mathrm{tr}\, g|^{p-2}\, \overline{\mathrm{tr}\, g}. \tag{37}$$

Applying (37) with $g = 1$, one obtains, for all $f \in H^p_\nu(\mathbb{D})$,

$$\int_{\mathbb{T}} (\mathrm{tr}\, f \circ \phi) = \int_{\mathbb{T}} \mathrm{tr}\, f. \tag{38}$$

Let $u \in L^p_{\mathbb{R}}(\mathbb{T})$ and $f \in H^p_\nu(\mathbb{D})$ such that $\mathrm{Re}\,\mathrm{tr}\, f = u$. Taking the real part in the both sides of (38) yields

$$\int_{\mathbb{T}} u \circ \phi = \int_{\mathbb{T}} u. \tag{39}$$

Since this is true for all $u \in L^p_{\mathbb{R}}(\mathbb{T})$, one obtains that (38) holds for all $f \in H^p(\mathbb{D})$ (write $\mathrm{tr}\, f = u + iv$ and apply (39) with u and v), and this yields $\phi(0) = 0$ ($f(z) = z$ in (38)). $\square$

As a corollary of Proposition 5, we characterize isometries from $H^p_\nu(\mathbb{D})$ *onto* $H^p_{\nu\circ\phi}(\mathbb{D})$:

Corollary 3 *Let $\phi : \mathbb{D} \to \mathbb{D}$ be a function in $W^{1,\infty}(\mathbb{D})$ analytic in $\mathbb{D}$. Then C_ϕ is an isometry from $H^p_\nu(\mathbb{D})$* onto *$H^p_{\nu\circ\phi}(\mathbb{D})$ if and only if there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\phi(z) = \lambda z$ for all $z \in \mathbb{D}$.*

Proof Proposition 5 shows that $\phi(\mathbb{T}) \subset \mathbb{T}$ and $\phi(0) = 0$, Theorem 1 ensures that ϕ is a bijection from $\mathbb{D}$ onto $\mathbb{D}$, and the conclusion readily follows. $\square$

Note that we do not know how to characterize those composition operators which are isometries from $H^p_\nu(\mathbb{D})$ to $H^p_{\nu\circ\phi}(\mathbb{D})$.

6.2 Doubly-Connected Domains

In the annular case, we obtain a complete description of the composition operators which are isometries on generalized Hardy spaces on $\mathbb{A}$. Before stating this result, we check:

Lemma 9 *Let $\phi : \mathbb{A} \to \mathbb{A}$ be analytic with $\phi \in W^{1,\infty}_{\mathbb{A}}(\mathbb{A})$.*

1. *If C_ϕ is an isometry from $G^p_\alpha(\mathbb{A})$ into $G^p_{\tilde{\alpha}}(\mathbb{A})$, then $\phi(\partial\mathbb{A}) \subset \partial\mathbb{A}$.*
2. *If C_ϕ is an isometry from $H^p_\nu(\mathbb{A})$ into $H^p_{\tilde{\nu}}(\mathbb{A})$, then $\phi(\partial\mathbb{A}) \subset \partial\mathbb{A}$.*

Proof Item 1 is already stated in Lemma 8. For item 2, notice that, if C_ϕ is an isometry from $H^p_\nu(\mathbb{A})$ into $H^p_{\tilde{\nu}}(\mathbb{A})$, then there exists $C > 0$ such that, for all $w \in G^p_\alpha(\mathbb{A})$,

$$C^{-1} \|\mathrm{tr}\, w\|_{L^p(\partial\mathbb{A})} \le \|\mathrm{tr}\,(w \circ \phi_n)\|_{L^p(\partial\mathbb{A})} \le C \|\mathrm{tr}\, w\|_{L^p(\partial\mathbb{A})}. \tag{40}$$

Arguing as in the proof of Proposition 5, one concludes that $\phi(\partial\mathbb{A}) \subset \partial\mathbb{A}$. $\square$

We can now state and prove our description of the composition operators which are isometries on generalized Hardy spaces on $\mathbb{A}$:

Theorem 3 *Let $\phi : \mathbb{A} \to \mathbb{A}$ be analytic with $\phi \in W^{1,\infty}_{\mathbb{A}}(\mathbb{A})$, $\alpha \in L^\infty(\mathbb{A})$ and ν meeting (4). The following conditions are equivalent:*

1. *C_ϕ is an isometry from $H^p(\mathbb{A})$ into $H^p(\mathbb{A})$,*
2. *C_ϕ is an isometry from $G^p_\alpha(\mathbb{A})$ into $G^p_{\tilde{\alpha}}(\mathbb{A})$,*
3. *C_ϕ is an isometry from $H^p_\nu(\mathbb{A})$ into $H^p_{\tilde{\nu}}(\mathbb{A})$,*
4. *either there exists $\lambda \in \mathbb{C}$ of unit modulus such that $\phi(z) = \lambda z$ for all $z \in \mathbb{A}$, or there exists $\mu \in \mathbb{C}$ of unit modulus such that $\phi(z) = \mu \frac{r_0}{z}$ for all $z \in \mathbb{A}$.*

Note that, even for Hardy spaces of analytic functions on $\mathbb{A}$, the characterization of isometries on H^p given in Theorem 3, items 1 and 4, is new.

The proof of Theorem 3 uses:

Proposition 6 *Let $\phi : \overline{\mathbb{A}} \to \overline{\mathbb{A}}$ be a non-constant continuous function holomorphic in $\mathbb{A}$. Assume that $\phi(\partial\mathbb{A}) \subset \partial\mathbb{A}$. Then:*

1. *either there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\phi(z) = \lambda z$ for all $z \in \overline{\mathbb{A}}$,*
2. *or there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\phi(z) = \lambda \frac{r_0}{z}$ for all $z \in \overline{\mathbb{A}}$.*

Proof Observe first that $\phi(r_0\mathbb{T}) \subset r_0\mathbb{T}$ or $\phi(r_0\mathbb{T}) \subset \mathbb{T}$. Indeed, $\phi(r_0\mathbb{T}) = (\phi(r_0\mathbb{T}) \cap r_0\mathbb{T}) \cup (\phi(r_0\mathbb{T}) \cap \mathbb{T})$. Since $r_0\mathbb{T}$ is connected and ϕ is continuous on $r_0\mathbb{T}$, $\phi(r_0\mathbb{T})$ is also connected and the conclusion readily follows, since $\phi(r_0\mathbb{T}) \cap r_0\mathbb{T}$ and $\phi(r_0\mathbb{T}) \cap \mathbb{T}$ are disjoint closed subsets of $\phi(r_0\mathbb{T})$. Replacing ϕ by $\frac{r_0}{\phi}$, we may and do assume that $\phi(r_0\mathbb{T}) \subset r_0\mathbb{T}$.

Arguing similarly, one obtains that $\phi(\mathbb{T}) \subset \mathbb{T}$ or $\phi(\mathbb{T}) \subset r_0\mathbb{T}$. If $\phi(\mathbb{T}) \subset r_0\mathbb{T}$, one has $|\phi| = r_0$ on $\partial\mathbb{A}$, and the maximum principle, applied to ϕ and $\frac{1}{\phi}$, entails $|\phi| = r_0$ in $\mathbb{A}$. Therefore, by the strong maximum principle, ϕ is constant in $\mathbb{A}$, which is impossible by the assumptions on ϕ. Thus, $\phi(\mathbb{T}) \subset \mathbb{T}$.

Now, if $h(z) := \frac{\phi(z)}{z}$ for all $z \in \overline{\mathbb{A}}$, the function h is continuous in $\overline{\mathbb{A}}$, holomorphic in $\mathbb{A}$ and satisfies $|h(z)| = 1$ for all $z \in \partial\mathbb{A}$. As before, the maximum principle applied with h and $\frac{1}{h}$ ensures that h is constant in $\mathbb{A}$, which ends the proof. □

Let us now turn to the proof of Theorem 3.

Proof Assume that 1 holds. Since $\phi(\partial\mathbb{A}) \subset \partial\mathbb{A}$ (by Lemma 9) and ϕ is one-to-one (by item 3 in Remark 5), Proposition 6 shows that 4 also holds. Conversely, it is clear that 4 implies 1, 2 and 3.

Assume now that 2 holds. An analogous argument ensures that 4 holds, so that 1 is satisfied. Similarly, if 3 holds, then 4 and therefore 1 are satisfied. □

7 Conclusion

7.1 Some Results on Arbitrary Domains

We discuss below how to extend the definition of generalized Hardy spaces over arbitrary domains and which results on composition operators on such spaces remain true.

Let $\Omega \subset \mathbb{C}$ be a connected open subset of the complex plane. There are two natural ways to define generalized Hardy spaces on Ω: the first one by means of harmonic majorants and the second one by introducing Smirnov type generalized Hardy spaces. The introduction of Smirnov classes for pseudo-analytic functions immediately leads to many questions on the property of functions of such spaces (the existence of a trace on the boundary, for example). Moreover, the boundedness of composition operators on such spaces is not guaranteed (see [33] for the analytic case). We intend to deal with composition operators on generalized Hardy spaces of Smirnov type on arbitrary domains in a future work. Note that when Ω is a Dini-smooth domain, as already pointed out in Remark 3, all these spaces coincide.

However, one can easily extend some properties of the composition operator defined on the first type of generalized Hardy spaces on an arbitrary connected domain Ω. Let $\nu \in W^{1,\infty}_{\mathbb{R}}(\Omega)$ meeting condition (4). As in Sect. 3.2, we define $E^p_\nu(\Omega)$ and $F^p_\alpha(\Omega)$ by (8), (9), (10) and their associated norm by (11) and

$$\|w\|_{F^p_\alpha(\Omega)} := \inf\left\{u^{1/p}(z_0), u : \Omega \to [0,+\infty) \text{ harmonic in } \Omega \text{ such that (10) holds}\right\}.$$

Remark 8 The arguments of the proof of Proposition 1 still show that $\|.\|_{E^p_\nu(\Omega)}$ and $\|.\|_{F^p_\alpha(\Omega)}$ are norms on $E^p_\nu(\Omega)$ and $F^p_\alpha(\Omega)$ respectively.

As in the analytic case [13], one can give a characterization of $E^p_\nu(\Omega)$ and $F^p_\alpha(\Omega)$ in the finitely-connected case. Recall [21, Chapter 5, Paragraph 6, Theorem 2] that a bounded, finitely-connected domain Ω, such that $\partial\Omega$ is the union of a finite number of disjoint Jordan loops, is conformally equivalent to a circular domain.

Lemma 10 *Assume that $\Omega \subsetneq \mathbb{C}$ is a bounded, finitely-connected domain, such that $\partial\Omega$ is the union of a finite number of disjoint Jordan loops. Let $\mathbb{G}$ be a circular domain and $\phi : \mathbb{D} \to \Omega$ be a holomorphic bijection. Let $\nu \in W^{1,\infty}(\Omega)$ meet (4), $\alpha \in L^\infty(\Omega)$ and $\widetilde{\alpha} = (\alpha \circ \phi)\overline{\partial\phi}$. Then:*

1. *$f \in E^p_\nu(\Omega)$ if and only if $f \circ \phi \in H^p_{\nu\circ\phi}(\mathbb{G})$,*
2. *$w \in F^p_\alpha(\Omega)$ if and only if $w \circ \phi \in G^p_{\widetilde{\alpha}}(\mathbb{G})$.*

Proof Indeed, if $f \in E^p_\nu(\Omega)$, then $f \circ \phi$ solves (8) in $\mathbb{G}$ with $\nu \circ \phi$ instead of ν (see the proof of Proposition 3 above). Moreover, if u is a harmonic function in Ω such that $|f|^p \le u$, then $u \circ \phi$ is a harmonic function in $\mathbb{G}$ and $|f \circ \phi|^p \le u \circ \phi$, so that $f \circ \phi \in H^p_{\nu\circ\phi}(\mathbb{G})$. The converse, as well as the case of F^p_α, are analogous. □

It is obvious that the composition operator $C_\phi : E^p_\nu(\Omega_2) \longrightarrow E^p_{\nu\circ\phi}(\Omega_1)$ (respectively $F^p_\alpha(\Omega_2) \to F^p_{\widetilde{\alpha}}(\Omega_1)$) is bounded when $\phi : \Omega_1 \to \Omega_2$ is analytic, $\phi \in W^{1,\infty}_{\Omega_1}(\Omega_2)$ and Ω_1, Ω_2 are connected open subsets of $\mathbb{C}$.

Necessary and sufficient conditions for the invertibility of the composition operator on $E^p_\nu(\Omega)$ and $F^p_\alpha(\Omega)$ when Ω is a simply connected domain can be obtained. In this case, Theorem 1 and Corollary 1 are expressed as follows.

Corollary 4 *Let $\Omega_1, \Omega_2 \subsetneq \mathbb{C}$. Assume that, for $i = 1, 2$, Ω_i is a bounded, finitely-connected domain, such that $\partial\Omega$ is the union of a finite number of disjoint Jordan loops. Let $\phi \in W^{1,\infty}_{\Omega_2}(\Omega_1)$ be analytic in Ω_1.*

1. *Let $\alpha \in L^\infty(\Omega_2)$. Then $C_\phi : F^p_\alpha(\Omega_2) \to F^p_{\tilde{\alpha}}(\Omega_1)$ is an isomorphism if and only if ϕ is a bijection from Ω_1 onto Ω_2.*
2. *Let $\nu \in W^{1,\infty}(\Omega_2)$ meet (4). Then the composition operator $C_\phi : E^p_\nu(\Omega_2) \to E^p_{\nu\circ\phi}(\Omega_1)$ is invertible if, and only if, ϕ is a bijection from Ω_1 onto Ω_2.*

Proof Let $\psi_1 : \mathbb{G}_1 \to \Omega_1$ and $\psi_2 : \mathbb{G}_2 \to \Omega_2$ be analytic bijections where $\mathbb{G}_1$ and $\mathbb{G}_2$ are appropriate circular domains. Assume that $C_\phi : F^p_\alpha(\Omega_2) \to F^p_{\tilde{\alpha}}(\Omega_1)$ is an isomorphism. Then $C_{\psi_2^{-1}\circ\phi\circ\psi_1} : G^p_{\alpha\circ\psi_1}(\mathbb{G}_1) \to G^p_{\tilde{\alpha}\circ\psi_2}(\mathbb{G}_2)$ is an isomorphism. Corollary 2 entails that $\psi_2^{-1} \circ \phi \circ \psi_1$ is a bijection from $\mathbb{G}_1$ onto $\mathbb{G}_2$, which shows that ϕ is a bijection from Ω_1 onto Ω_2. The proof of 2 is similar. □

The extension of our results on isometries (Sect. 6) to more general domains is more delicate (recall that, for instance, the results in Sect. 6 are limited to the case where $\Omega = \mathbb{D}$ or $\Omega = \mathbb{A}$).

7.2 Related Issues

We extended to the case of generalized Hardy spaces some well-known properties of composition operators on classical Hardy spaces of analytic functions H^p for $1 < p < \infty$. Some questions are still open. As mentioned before, it would be interesting to give a complete characterization of isometries among those composition operators on $H^p_\nu(\mathbb{D})$ spaces.

We intend to tackle compactness issues in a forthcoming work. In particular, we proved that the compactness of C_ϕ on generalized Hardy spaces is equivalent to the same property on H^p, in smooth simply-connected domains [12, Thm3.12], [31]. The case of multiply-connected situations deserves further investigation. Finally, Sect. 7.1 points out many questions related to the definition of generalized Hardy spaces over general domains, the introduction of Smirnov classes for pseudo-analytic functions and the extension of some results on composition operators for arbitrary domains on which we will focus in a future work.

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Appendix: Factorization Results

We extend below [5, Thm 1] to the case of n-connected Dini smooth domains. Theorem 4 may be seen as a converse to the factorization result [6, Prop. 3.2], see [23] for more details. It is a straightforward generalization of a factorization result on generalized Hardy spaces on simply connected domains; however, the authors could not locate such a factorization for multiply connected Dini-smooth domains in the literature. For that reason, we give a short proof of this extension.

Theorem 4 *Let $\Omega \subset \mathbb{C}$ be a n-connected Dini smooth domain. Let $F \in H^p(\Omega)$, $\alpha \in L^\infty(\Omega)$. There exists a function $s \in W^{1,r}(\Omega)$ for all $r \in (2, +\infty)$ such that $tr\, Re\, s = 0$ on $\partial\Omega$, $w = e^s F$ and $\|s\|_{W^{1,r}(\Omega)} \lesssim \|\alpha\|_{L^\infty(\Omega)}$.*

The proof is inspired by the one of [5, Thm 1]. By conformal invariance, it is enough to deal with the case where $\Omega = \mathbb{G}$ is a circular domain. We first assume that $\alpha \in W^{1,2}(\mathbb{G}) \cap L^\infty(\mathbb{G})$. For all $\varphi \in W^{1,2}_{\mathbb{R}}(\mathbb{G})$, let $G(\varphi) \in W^{1,2}_{0,\mathbb{R}}(\mathbb{G})$ be the unique solution of

$$\Delta(G(\varphi)) = \mathrm{Im}\left(\partial(\alpha e^{-2i\varphi})\right).$$

We claim:

Lemma 11 *The operator G is bounded from $W^{1,2}_{\mathbb{R}}(\mathbb{G})$ from $W^{2,2}_{\mathbb{R}}(\mathbb{G})$ and compact from $W^{1,2}_{\mathbb{R}}(\mathbb{G})$ to $W^{1,2}_{\mathbb{R}}(\mathbb{G})$.*

Proof Let $\varphi \in W^{1,2}_{\mathbb{R}}(\mathbb{G})$. As in [5], $\partial(\alpha e^{-2i\varphi}) \in L^2(\mathbb{G})$ and $\left\|\partial(\alpha e^{-2i\varphi})\right\|_{L^2(\mathbb{G})} \lesssim \|\varphi\|_{W^{1,2}(\mathbb{G})}$. It is therefore enough to show that the operator T, which, to any function $\psi \in L^2_{\mathbb{R}}(\mathbb{G})$, associates the solution $h \in W^{1,2}_{0,\mathbb{R}}(\mathbb{G})$ of $\Delta\psi = h$ is continuous from $L^2(\mathbb{G})$ to $W^{2,2}(\mathbb{G})$, which is nothing but the standard $W^{2,2}$ regularity estimate for second order elliptic equations (see [15, Sec. 6.3,Thm 4] and note that $\mathbb{G}$ is C^2). This shows that G is bounded from $W^{1,2}_{\mathbb{R}}(\mathbb{G})$ from $W^{2,2}_{\mathbb{R}}(\mathbb{G})$, and its compactness on $W^{1,2}_{\mathbb{R}}(\mathbb{G})$ follows then from the Rellich–Kondrachov theorem. □

Proof of Theorem 4 As in the proof of [5, Thm 1], Lemma 11 entails that G has a fixed point in $W^{1,2}_{\mathbb{R}}(\Omega)$, which yields the conclusion of Theorem 4 when $\alpha \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, and a limiting procedure ends the proof. □

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