comparison of a Boundary Element and a Bounded Extremal Problem

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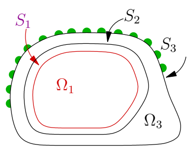
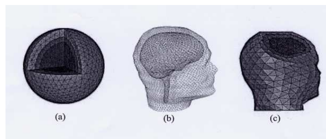
1. Cortical Imaging

The head is modeled as 3 nested volumes Ω_i of constant conductivity.





The potential V inside the head satisfies the Poisson equation $\boxed{-\nabla\cdot(\sigma\nabla V) = \mu}$ where μ represents the primary sources localized inside the brain Ω_1.

Data: pointwise values of V measured by electrodes on the scalp S_3, denoted $\mathbf{v}$.

Goal: to recover V and $\partial_\mathbf{n} V$ on S_1, representing the surface of the cortex.

Cortical imaging enables Source Localization by Rational Approximation.

Data propagation

$\mathbf{v}$ on electrodes, $\partial_\mathbf{n} V = 0$ on S_3	$\xrightarrow{\Delta V = 0 \text{ in } \Omega_3}$ Cauchy problem	V and $\partial_\mathbf{n} V$ on S_2	$\xrightarrow{\Delta V = 0 \text{ in } \Omega_2}$ Cauchy problem	V and $\partial_\mathbf{n} V$ on S_1

If V and $\partial_\mathbf{n} V$ are known on a **dense subset** of S_3, then the Cauchy problem has a **unique solution**. In practice, $\mathbf{v}$ is only known at the electrode positions, and is subject to noise. Cortical imaging is **ill-posed**, making regularization necessary.

2. Green identity and Representation Theorems

$$\int_\Omega (V\,\Delta W - W\,\Delta V)\,dv = \int_{\partial\Omega} (V\,\partial_\mathbf{n} W - W\,\partial_\mathbf{n} V)\,ds$$

The above identity allows to represent harmonic functions ($\Delta V = 0$), using only their values on the boundary $\partial\Omega$. The auxiliary function W can be chosen equal to a **fundamental solution** of the Laplacian $W(X,Y) = \frac{1}{4\pi\|X-Y\|}$. Then:

(i) for the Boundary Element Method, using the associated **boundary integral operators**, $\mathcal{S}$ single-layer, $\mathcal{D}$ double-layer, $\mathcal{N}$ hyper-singular, if V is harmonic inside and outside $\partial\Omega$, and if $[V]$, $[\partial_\mathbf{n} V]$ denote the jumps across $\partial\Omega$,

then, in Ω, $$\boxed{\begin{cases} -\partial_\mathbf{n} V = \mathcal{N}[V] - \mathcal{D}^*[\partial_\mathbf{n} V], \\ V = -\mathcal{D}[V] + \mathcal{S}[\partial_\mathbf{n} V]; \end{cases}} \qquad (1)$$

(ii) for the Bounded Extremal Problem, in the case (a) where Ω is an annular region, using the associated **Poisson kernel** $\mathcal{P}_\Omega$,

then, for X in Ω, $\boxed{V(X) = \int_{\partial\Omega} V(Y)\,\partial_\mathbf{n}\mathcal{P}_\Omega(X,Y)\,ds(Y).}$ (2)

In the ball Ω_1 of radius R,

$\mathcal{P}_{\Omega_1}(X,Y) = \frac{R^2-\|X\|^2}{4\pi R\|X-Y\|^3} = \partial_\mathbf{n} W(X,Y) - \frac{R}{\|X\|}\partial_\mathbf{n} W(\frac{R^2}{\|X\|^2}X, Y).$

3. Boundary Element Method (BEM)

The surfaces S_i are represented by a triangular mesh.

The variables $\begin{cases} V \\ \sigma\partial_\mathbf{n} V \end{cases}$ are discretized using piecewise $\begin{cases} \text{linear} \\ \text{constant} \end{cases}$ functions,

and represented by a vector $\mathbf{Z}$. Discretizing the boundary integral equations (1) leads to an underconstrained linear system $H\,\mathbf{Z} = 0$ [2].

Minimization scheme

A linear measurement operator M extracts from $\mathbf{Z}$ the values of the potential V at the electrode positions. Due to measurement noise, one will minimize

$$\mathbf{M}(\mathbf{Z}) = \|M\,\mathbf{Z} - \mathbf{v}\|^2.$$

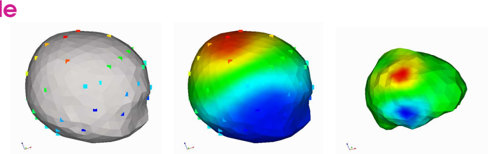
A regularization is performed by controlling $\mathbf{R}(\mathbf{Z})$ representing

$$\int_{S_1\cup S_2\cup S_3} \|\nabla_S V\|^2 + \|V(r+\alpha\mathbf{n}) - V(r) - \alpha\partial_\mathbf{n} V\|^2.$$

The goal is to minimize $\mathbf{M}(\mathbf{Z}) + \lambda\mathbf{R}(\mathbf{Z})$ with $\mathbf{Z}$ belonging to $\operatorname{Ker} H$.

By imposing $\mathbf{Z} = P^\perp_{\operatorname{Ker} H}\,\mathbf{Y}$ and seeking $\mathbf{Y}$, the minimization becomes unconstrained.

Example



4. Bounded Extremal Problem (BEP)

BEP refers to best approximation issues in Hardy classes of analytic functions in $\Omega \subset \mathbb{R}^3$:

$H^2(\Omega) = \{\nabla U$ for functions U such that $\Delta U = 0$ in Ω and $\int_{\partial\Omega} \|\nabla_\partial U\|^2 < \infty\}$,

with the notation $\nabla_\partial U = (\partial_\mathbf{n} U, \nabla_S U)$ on $\partial\Omega$.

Minimization problem

From **interpolated data** $(v, \partial_\mathbf{n} v)$ on S_i, minimize $\int_{S_i} \|\nabla_\partial v - \mathbf{g}\|^2$ among the functions $\mathbf{g} \in H^2(\Omega_i)$ constrained by $\int_{S_{i-1}} \|\mathbf{g}\|^2 \le \rho$.

With an appropriate choice of the regularization parameter ρ, the solution $\mathbf{g} \simeq \nabla V$ in Ω_i provides us with Cauchy data $(V, \partial_\mathbf{n} V)$ on S_{i-1} [1].

Algorithm

In the case (a) of spherical layers Ω_i, for a Lagrange parameter $\lambda > 0$ such that $\int_{S_{i-1}} \|\mathbf{g}\|^2 = \rho$, the unconstrained minimization of:

$$\int_{S_i} \|\nabla_\partial v - \mathbf{g}\|^2 + \lambda \int_{S_{i-1}} \|\mathbf{g}\|^2$$

is performed using (2) and Toeplitz operators, directly from **data** $(v, \partial_\mathbf{n} v)$ on S_i.

5. Source Localization by Rational Approximation

Consider an **unknown** pointwise dipolar source $\mathbf{C}$ in the ball Ω_1 (a): $\boxed{\mu = \mathbf{p}\cdot\nabla\delta_\mathbf{C}}$

Data: the values of $(V, \partial_\mathbf{n} V)$ propagated (from S_3) to S_1,

Goal: to recover the source location $\mathbf{C} = (\mathbf{x_C}, \mathbf{y_C}, \mathbf{z_C})$ in Ω_1.

Filtering outside sources with spherical harmonics provides a function

$$V_-(X) = \frac{<\mathbf{p}, X-\mathbf{C}>}{4\pi\sigma_1\|X-\mathbf{C}\|^3} \text{ on } S_1. \qquad (3)$$

Considering $X = (x,y,z)$, the disks $D_m = B \cap \{z = z_m\} \subset \mathbb{R}^2 \simeq \mathbb{C}$, the complex variable $\xi = x + i\,y$, the function $V^2_m(\xi) = V^2_-(x,y,z_m)$ is, for each m, **rational** with a triple pole at $\xi_m \in D_m$:

- (ξ_m) are aligned together and also with $\xi_\mathbf{C} = \mathbf{x_C} + i\mathbf{y_C}$,
- $|\xi_m|$ is maximum for m^* such that $z_{m^*} = \mathbf{z_C}$.

For each m, (ξ_m) is approximated by the poles of the best $L^2(\partial D_m)$ rational approximant of degree 3 to V^2_m: this **localizes the source position** $\mathbf{C}^* \simeq \mathbf{C}$.

Further: the **dipolar moment** $\mathbf{p}$ can be estimated with the computation of the residues; this scheme has been extended to situations with several sources [3].

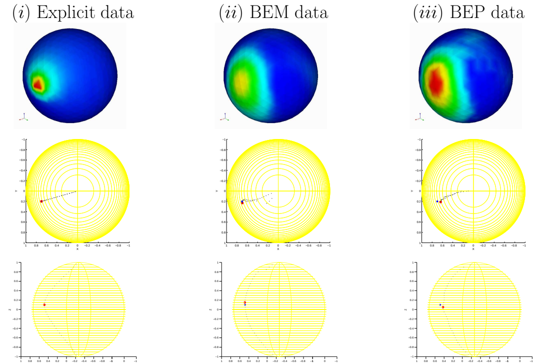
6. Evaluation of Results

In a 3-sphere model of radii $(.87; .92; 1)$, and conductivities $(1; 1/30; 1)$, a dipolar source is placed at $\mathbf{C} = [.7\ \ .2\ \ .1]$ and $\mathbf{v}$ is generated on S_3. The cortical potential is:

(i) computed explicitly from (3); *(ii)* propagated by the Boundary Element Method, *(iii)* propagated by solving two Bounded Extremal Problems.

The source position $\mathbf{C}$ (•) is compared with $\mathbf{C}^*$ (•) estimated by Rational Approximation.

(i) Explicit data — *(ii)* BEM data — *(iii)* BEP data



$\|\mathbf{C}^* - \mathbf{C}\| = .0002$ — $\|\mathbf{C}^* - \mathbf{C}\| = .06$ — $\|\mathbf{C}^* - \mathbf{C}\| = .08$

References

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