Hybrid and Subexponential Linear Logics

Joëlle Despeyroux
INRIA & CNRS (I3S), Sophia-Antipolis, France

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Joint work with Carlos Olarte & Elaine Pimentel
Universidade Federal do Rio Grande do Norte, Brazil

Motivation : Comparing HyLL and SELL

- HyLL (Hybrid Linear Logic)
- SELL (Subexponential Linear Logic)
- HyLL and SELL: two extensions of Linear Logic
- used for specifying systems with temporal or spatial modalities
- In particular modeling and reasoning about biological systems

$\hookrightarrow$ *Relative expressiveness power of HyLL and SELL*

Outline

Linear Logic

- Terms:

$$\begin{array}{rl} t ::= & c \mid x \mid f(\vec{t}) \qquad \textit{Ex: } \texttt{gene}(a) \\ A, B, \ldots ::= & p(\vec{t}) \mid A \otimes B \mid \mathbf{1} \mid A \multimap B \mid A \,\&\, B \mid \top \mid A \oplus B \mid \mathbf{0} \\ & !A \mid \forall x.\ A \mid \exists x.\ A \qquad \textit{Ex: } \texttt{pres}(x) \otimes \texttt{abs}(y) \end{array}$$

- Judgements are of the form: $\Gamma; \Delta \vdash C$, where
  Γ is the *unrestricted context*
    its hypotheses can be consumed any number of times.
  Δ (a *multiset*) is a *linear context*
    every hypothesis in it must be consumed singly in the proof.
  C is true assuming the hypotheses $\Gamma; A_1 \cdots A_n$ are true
  Ex: $\texttt{bio\_system};\ \texttt{pres}(\mathtt{x}), \texttt{abs}(\mathtt{y}) \vdash \texttt{pres}(\mathtt{z})$
- Judgemental rules:

$$\Gamma; p(\vec{t}) \vdash p(\vec{t})\ [init] \qquad \frac{\Gamma, A; \Delta, A \vdash C}{\Gamma, A; \Delta \vdash C}\ copy$$

Sequent Calculus for Linear Logic [1]

- Exponentials:

$$\frac{\Gamma; . \vdash A}{\Gamma; . \vdash !A}\ !R \qquad \frac{\Gamma, A; \Delta \vdash C}{\Gamma; \Delta, !A \vdash C}\ !L$$

- Multiplicatives:

$$\frac{\Gamma; \Delta, A \vdash B}{\Gamma; \Delta \vdash A \multimap B}\ [\multimap_R] \qquad \frac{\Gamma; \Delta \vdash A \quad \Gamma; \Delta', B \vdash C}{\Gamma; \Delta, \Delta', A \multimap B \vdash C}\ [\multimap_L]$$

$$\Gamma; . \vdash \mathbf{1}\ [\mathbf{1}R] \qquad \frac{\Gamma; \Delta \vdash C}{\Gamma; \Delta, 1 \vdash C}\ 1\,L$$

$$\frac{\Gamma; \Delta \vdash A \quad \Gamma; \Delta' \vdash B}{\Gamma; \Delta, \Delta' \vdash A \otimes B} \otimes R \qquad \frac{\Gamma; \Delta, A, B \vdash C}{\Gamma; \Delta, A \otimes B \vdash C} \otimes L$$

Sequent Calculus for Linear Logic [2]

- Additives:

$$\Gamma; \Delta \vdash \top \; [\top R] \qquad \Gamma; \Delta, \mathbf{0} \vdash C \; [\mathbf{0} L]$$

$$\frac{\Gamma; \Delta \vdash A \qquad \Gamma; \Delta \vdash B}{\Gamma; \Delta \vdash A \,\&\, B} \,\& R \qquad \frac{\Gamma; \Delta, A_i \vdash C}{\Gamma; \Delta, A_1 \,\&\, A_2 \vdash C} \,\& L_i$$

$$\frac{\Gamma; \Delta \vdash A_i}{\Gamma; \Delta \vdash A_1 \oplus A_2} \oplus R_i \qquad \frac{\Gamma; \Delta, A \vdash C \quad \Gamma; \Delta, B \vdash C \,@\, w}{\Gamma; \Delta, A \oplus B \,@\, u \vdash C} \oplus L$$

Example

- *Activation*:

$$\mathtt{Active}(a,b) \stackrel{\text{def}}{=} \mathtt{pres}(a) \multimap \delta_1(\mathtt{pres}(a) \otimes \mathtt{pres}(b)).$$

- *Inhibition*

$$\mathtt{Inhib}(a,b) \stackrel{\text{def}}{=} \mathtt{pres}(a) \multimap \delta_1(\mathtt{pres}(a) \otimes \mathtt{abs}(b)).$$

Hybrid Logic

- A form of modal logic that allows *naming of worlds*.
- Very general idea. Can be applied for
  - Almost all known modal and temporal logics
  - Many substructural logics (eg. linear logic)
- Ideas go back to Prior (1960s) and Allen (1980s)
  - but still active and recently energized area

Hybrid Linear Logic

- Add a new metasyntactic class of *worlds*, written "w":

Definition

A *constraint domain* $\mathcal{W}$ is a monoid structure $\langle W, ., \iota \rangle$.
The elements of W are called worlds, and
the partial order $\preceq : W \times W$—defined as $u \preceq w$ if there exists $v \in W$ such that $u.v = w$—is the *reachability relation* in $\mathcal{W}$.

- The identity world ι, $\preceq$-initial, represents the lack of any constraints: ILL $\subseteq$ HyLL$[\iota]$ $\subset$ HyLL[W].
- Ex: Time: $\mathcal{T} = \langle \boldsymbol{N}, +, 0 \rangle$ or $\langle \mathbb{R}^+, +, 0 \rangle$.

Hybrid Linear Logic

- Make all judgements situated *at a world*: $A @ w$

  A is true at world w

- Judgements are of the form:

$$\Gamma ; \Delta \vdash C @ w,$$

  where Γ and Δ are sets of judgements of the form $A @ w$

- All ordinary rules continue essentially unchanged.
- Judgemental rules

$$\Gamma ; p(\vec{t}) @ w \vdash p(\vec{t}) @ w \; [init] \qquad \frac{\Gamma, A @ w ; \Delta, A @ w \vdash C @ w}{\Gamma, A @ w ; \Delta \vdash C @ w} \; copy$$

Sequent Calculus for HyLL [2]

- Exponentials rules

$$\frac{\Gamma; . \vdash A \ @ \ w}{\Gamma; . \vdash !A \ @ \ w} \ !R \qquad \frac{\Gamma, A \ @ \ u; \Delta \vdash C \ @ \ w}{\Gamma; \Delta, !A \ @ \ u \vdash C \ @ \ w} \ !L$$

- Multiplicatives

$$\frac{\Gamma; \Delta, A \ @ \ w \vdash B \ @ \ w}{\Gamma; \Delta \vdash A \multimap B \ @ \ w} \ [\multimap_R] \qquad \frac{\Gamma; \Delta \vdash A \ @ \ w \quad \Gamma; \Delta', B \ @ \ w \vdash C \ @ \ w}{\Gamma; \Delta, \Delta', A \multimap B \ @ \ w \vdash C \ @ \ w} \ [\multimap_L]$$

$$\frac{\Gamma; \Delta \vdash A \ @ \ w \quad \Gamma; \Delta' \vdash B \ @ \ w}{\Gamma; \Delta, \Delta' \vdash A \otimes B \ @ \ w} \otimes R \qquad \frac{\Gamma; \Delta, A \ @ \ u, B \ @ \ u \vdash C \ @ \ w}{\Gamma; \Delta, A \otimes B \ @ \ u \vdash C \ @ \ w} \otimes L$$

Sequent Calculus for HyLL [3]

- Additives

$$\frac{\Gamma; \Delta \vdash A \mathbin{@} w \qquad \Gamma; \Delta \vdash B \mathbin{@} w}{\Gamma; \Delta \vdash A \mathbin{\&} B \mathbin{@} w} \;\& R$$

$$\frac{\Gamma; \Delta, A_i \mathbin{@} u \vdash C \mathbin{@} w}{\Gamma; \Delta, A_1 \mathbin{\&} A_2 \mathbin{@} u \vdash C \mathbin{@} w} \;\& L_i \qquad \frac{\Gamma; \Delta \vdash A_i \mathbin{@} w}{\Gamma; \Delta \vdash A_1 \oplus A_2 \mathbin{@} w} \;\oplus R_i$$

$$\frac{\Gamma; \Delta, A \mathbin{@} u \vdash C \mathbin{@} w \quad \Gamma; \Delta, B \mathbin{@} u \vdash C \mathbin{@} w}{\Gamma; \Delta, A \oplus B \mathbin{@} u \vdash C \mathbin{@} w} \;\oplus L$$

Hybrid Connectives

- Make the claim that "A is true at world w"
  a *mobile* proposition in terms of a *satisfaction* connective:
- Terms:

$$
\begin{aligned}
t &::= \quad c \mid x \mid f(\vec{t}) \\
A, B, \ldots &::= \quad \ldots \mid A \ \mathtt{at}\ w \mid \downarrow u.\ A \mid \forall u.\ A \mid \exists u.\ A
\end{aligned}
$$

Satisfaction

- To introduce the *satisfaction* proposition $(A \ \mathtt{at} \ u)$ (at any world v), the proposition A must be true in the world u:

$$\frac{\Gamma; \Delta \vdash A \ @ \ u}{\Gamma; \Delta \vdash (A \ \mathtt{at} \ u) \ @ \ v} \ \mathtt{at} \ R$$

- The proposition $(A \ \mathtt{at} \ u)$ itself is then true at any world, not just in the world u.
- i.e. $(A \ \mathtt{at} \ u)$ carries with it the world at which it is true. Therefore, suppose we know that $(A \ \mathtt{at} \ u)$ is true (at any world v); then, we also know that $A \ @ \ u$:

$$\frac{\Gamma; \Delta, A \ @ \ u \vdash C \ @ \ w}{\Gamma; \Delta, (A \ \mathtt{at} \ u) \ @ \ v \vdash C \ @ \ w} \ \mathtt{at} \ L$$

Localisation

- The other hybrid connective of *localisation*, $\downarrow u.\ A$, is intended to be able to name the current world:
- If $\downarrow u.\ A$ is true at world w, then the variable u stands for w in the body A:

$$\frac{\Gamma ; \Delta \vdash [w/u]A \ @\ w}{\Gamma ; \Delta \vdash \downarrow u.A \ @\ w} \downarrow R$$

- Suppose we have a proof of $\downarrow u.A \ @\ v$ for some world v; Then, we also know $[v/u]A \ @\ v$:

$$\frac{\Gamma ; \Delta, [v/u]A \ @\ v \vdash C \ @\ w}{\Gamma ; \Delta, \downarrow u.A \ @\ v \vdash C \ @\ w} \downarrow L$$

Properties of the Sequent Calculus System [1]

Theorem

1. *If* $\Gamma; \Delta \vdash C \mathbin{@} w$, *then* $\Gamma, \Gamma'; \Delta \vdash C \mathbin{@} w$ *(weakening)*
2. *If* $\Gamma, A \mathbin{@} u, A \mathbin{@} u; \Delta \vdash C \mathbin{@} w$, *then* $\Gamma, A \mathbin{@} u; \Delta \vdash C \mathbin{@} w$ *(contraction)*
3. $\Gamma; A \mathbin{@} w \vdash A \mathbin{@} w$ *(identity)*

Theorem (cut)

1. *If* $\Gamma; \Delta \vdash A \mathbin{@} u$ *and* $\Gamma; \Delta', A \mathbin{@} u \vdash C \mathbin{@} w$, *then* $\Gamma; \Delta, \Delta' \vdash C \mathbin{@} w$.
2. *If* $\Gamma; . \vdash A \mathbin{@} u$ *and* $\Gamma, A \mathbin{@} u; \Delta \vdash C \mathbin{@} w$, *then* $\Gamma; \Delta \vdash C \mathbin{@} w$.

Properties of the Sequent Calculus System [2]

Theorem (invertibility)

- *On the right:* $\&R$, $\top R$, $\multimap R$, $\forall R$, $\downarrow R$ *and* $\mathtt{at}\ R$;
- *On the left:* $\otimes L$, $\mathbf{1}L$, $\oplus L$, $\mathbf{0}L$, $\exists L$, $!L$, $\downarrow L$ *and* $\mathtt{at}\ L$

Theorem

1. *(consistency) There is no proof of* $.;. \vdash \mathbf{0} \mathbin{@} w$.
2. *(conservativity) For "pure" contexts* Γ *and* Δ *and "pure" proposition* A*:* $\quad \Gamma; \Delta \vdash_{ILL} A$.

Theorem (HyLL is -at least as powerful as- S5)

$.; \Diamond A \mathbin{@} w \vdash \Box\Diamond A \mathbin{@} w$.

Defined Modal Connectives - delay

- Defined modal connectives:

$$\Box A \stackrel{\text{def}}{=} \downarrow u.\ \forall w.\ (A \text{ at } u.w) \qquad \Diamond A \stackrel{\text{def}}{=} \downarrow u.\ \exists w.\ (A \text{ at } u.w)$$

$$\delta_v\, A \stackrel{\text{def}}{=} \downarrow u.\ (A \text{ at } u.v) \qquad \dagger\, A \stackrel{\text{def}}{=} \forall u.\ (A \text{ at } u)$$

- The connective δ represents a form of *delay*:
  Derived right rule:

$$\frac{\Gamma; \Delta \vdash A \ @\ w.v}{\Gamma; \Delta \vdash \delta_v\, A \ @\ w}\ \delta R$$

Example

- *Activation*:

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- *Inhibition*

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Oscillation

$A \wedge \mathrm{EF}(B \wedge \mathrm{EF} A)$

Definition (one oscillation)

$$\mathrm{oscillate}_1\,(A, B, u, v) \stackrel{\mathrm{def}}{=} A \ \& \ \ \delta_u(B \ \& \ \ \delta_v\, A) \ \& \ (A \,\&\, B \multimap 0).$$

Definition (oscillation - object)

$$\mathrm{oscillate}_h\,(A, B, u, v) \stackrel{\mathrm{def}}{=} \dagger[(A \multimap \delta_u\, B) \ \& \ (B \multimap \delta_v\, A)] \ \& \ (A \,\&\, B \multimap 0).$$

Definition (oscillation - meta)

$\mathrm{oscillate}\,(A, B, u, v)$
$\stackrel{\mathrm{def}}{=}$ for any w, $(A \ @ \ w \vdash B \ @ \ w.u)$, $(B \ @ \ w.u \vdash A \ @ \ w.u.v)$,
and $(\vdash A \,\&\, B \multimap 0 \ \ @ \ w)$.

Subexponentials in Linear Logic

Subexponential Signature

$\Sigma = \langle I, \preceq, U \rangle$ where I is a set of labels, $U \subseteq I$ set of unbounded subexp and $\preceq$ is a pre-order among the elements of I.

$$\frac{\Gamma, F \longrightarrow G}{\Gamma, !^a F \longrightarrow G}\ !^a{}_L \qquad \frac{!^{a_1} F_1, \ldots, !^{a_n} F_n \longrightarrow F}{!^{a_1} F_1, \ldots, !^{a_n} F_n \longrightarrow !^a F}\ !^a{}_R, \text{ provided } a \preceq a_i$$

$$\frac{\Gamma \longrightarrow G}{\Gamma, !^b F \longrightarrow G}\ W \qquad \frac{\Gamma, !^b F, !^b F \longrightarrow G}{\Gamma, !^b F \longrightarrow G}\ C \qquad \text{provided } b \in U$$

Assume two independent spatial domains a and b ($a \not\preceq b$). Then,

$$(!^a C \multimap !^b D), !^b C \not\vdash !^b D$$

Quantification on Subexponentials

$$\frac{\mathcal{A}; \mathcal{L}; \Gamma, P[l/x] \vdash G}{\mathcal{A}; \mathcal{L}; \Gamma, \forall x : a.P \vdash G} \; \forall_L \qquad \frac{\mathcal{A}, l_e : a; \mathcal{L}; \Gamma \vdash P[l_e/x]}{\mathcal{A}; \mathcal{L}; \Gamma \vdash \forall x : a.P} \; \forall_R$$

$$\frac{\mathcal{A}, l_e : a; \mathcal{L}; \Gamma, P[l_e/x] \vdash G}{\mathcal{A}; \mathcal{L}; \Gamma, \exists x : a.P \vdash G} \; \exists_L \qquad \frac{\mathcal{A}; \mathcal{L}; \Gamma \vdash P[l/x]}{\mathcal{A}; \mathcal{L}; \Gamma \vdash \exists x : a.P} \; \exists_R$$

- Creating "new" locations: $\Gamma, \exists l.(F) \vdash G$
- Asserting something about all locations: $\Gamma, \forall l.(F) \vdash G$
- Proving that all locations satisfies G: $\Gamma \vdash \forall l.(G)$
- Proving that G holds in some location: $\Gamma \vdash \exists l.(G)$

Theorem (Cut-elimination)

For any signature Σ, the proof system $\mathrm{SELL}^{\forall}$ admits cut-elimination.

How to use the subexponentials ?

The intuition

Connective	Meaning
$\nabla_s = !^s$	$!^s P$ is located at s.
$\nabla_s = !^s ?^s$	$!^s ?^s P$ is confined to s.
$\forall l : a\ P$	P can move to locations below (outside) a

Moving/Translocating components

$\preceq$	Meaning
$a \preceq a.b$	Components may "float" $[[P, Q]_b]_a \longrightarrow [P, [Q]_b]_a$

Spatial Modalities

$\preceq$	Meaning
$a \not\preceq b$	P and Q does not interact: $[P]_a, [Q]_b$
$a \not\preceq a.b$	Components are confined: $[[P, Q]_b]_a \not\longrightarrow [P, [Q]_b]_a$

Defined Modal Connectives

- Defined modal connectives in SELL:

$$\begin{array}{llll} \Box_u A & \stackrel{\mathrm{def}}{=} \forall l : u.\ !^l A & \Diamond_u A & \stackrel{\mathrm{def}}{=} \exists l : u.\ !^l A \\ \Box A & \stackrel{\mathrm{def}}{=} \forall t : \infty.\ !^t A & \Diamond A & \stackrel{\mathrm{def}}{=} \exists t : \infty.\ !^t A \\ [\![\delta_v\, A]\!]_u & \stackrel{\mathrm{def}}{=} [\![A]\!]_{u.v} & [\![\dagger\, A]\!]_u & \stackrel{\mathrm{def}}{=} \forall u : \infty.[\![A]\!]_u \end{array}$$

- In HyLL:

$$\begin{array}{llll} \Box A & \stackrel{\mathrm{def}}{=} \downarrow u.\ \forall w.\ (A \text{ at } u.w) & \Diamond A & \stackrel{\mathrm{def}}{=} \downarrow u.\ \exists w.\ (A \text{ at } u.w) \\ \delta_v\, A & \stackrel{\mathrm{def}}{=} \downarrow u.\ (A \text{ at } u.v) & \dagger\, A & \stackrel{\mathrm{def}}{=} \forall u.\ (A \text{ at } u) \end{array}$$

HyLL and Linear Logic

- Linear logic defines two kind of contexts: classical (unbounded) and linear.
- SELL generalizes this idea by slitting the context in as many parts as needed.
- Subexponentials are not canonical: $!^a F \not\Leftrightarrow\, !^b F$, thus SELL as a logical framework is more expressive than LL.
- What about HyLL? Do the worlds in HyLL add more expressive power?

Bio example

- *Inhibition* in HyLL

$$\mathtt{Inhib}(a,b) \stackrel{\text{def}}{=} \mathtt{pres}(a) \multimap \delta_1(\mathtt{pres}(a) \otimes \mathtt{abs}(b))$$

- *Inhibition* in classical SELL

$$\mathtt{Inhib}(a,b) \stackrel{\text{def}}{=} \forall t : \infty.\ !^{t}\mathtt{a} \multimap !^{t+1}(\mathtt{a} \otimes \mathtt{b}^{\perp})$$

$$\begin{aligned}\mathtt{Inhib}(a,b,c) \stackrel{\text{def}}{=}\ & \forall t : \infty. \\ & \left[!^{t}\mathtt{a} \otimes (\mathtt{b} \oplus \mathtt{b}^{\perp}) \otimes \mathtt{c} \multimap !^{t+1}(\mathtt{a} \otimes \mathtt{b}^{\perp}) \otimes \mathtt{c}\right] \mathbin{\&} \\ & \left[!^{t}\mathtt{a} \otimes (\mathtt{b} \oplus \mathtt{b}^{\perp}) \otimes \mathtt{c}^{\perp} \multimap !^{t+1}(\mathtt{a} \otimes \mathtt{b}^{\perp}) \otimes \mathtt{c}^{\perp}\right]\end{aligned}$$

- *Inhibition* in SELL

$$\mathtt{Inhib}(x,y,z) \stackrel{\text{def}}{=} \forall t : \infty.\ !^{t}\mathtt{count}(1,y,z) \multimap !^{t+1}\mathtt{count}(1,0,z)$$

More examples

- HyLL has been used to
  encode transition systems (Sπ calculus) and
  to specify/verify biological interacting systems.
  Biological example with formal proofs in Coq.
- SELL has been used to
  represent contexts of proof systems
  to specify systems with temporal, epistemic
  and spatial modalities
  and soft-constraints or preferences;
  to specify bigraphs and
  to specify/verify biological/multimedia interacting systems.

Encodings in Linear Logic

Two meta-level predicates $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ for identifying objects that appear on the left or right side of the sequents in the object logic. Rules

$$\frac{\Delta, A \longrightarrow \Gamma}{\Delta, A \wedge B \longrightarrow \Gamma}\wedge_{L1} \quad \frac{\Delta, B \longrightarrow \Gamma}{\Delta, A \wedge B \longrightarrow \Gamma}\wedge_{L2} \quad \frac{\Delta \longrightarrow \Gamma, A \quad \Delta \longrightarrow \Gamma, B}{\Delta \longrightarrow \Gamma, A \wedge B}\wedge_{R}$$

are specified in LL as

$$\wedge_L : \exists A, B.(\lfloor A \wedge B\rfloor^{\perp} \otimes (\lfloor A\rfloor \oplus \lfloor B\rfloor))$$

$$\wedge_R : \exists A, B.(\lceil A \wedge B\rceil^{\perp} \otimes (\lceil A\rceil \,\&\, \lceil B\rceil))$$

The linear logic connectives indicate how these object level formulas are connected: contexts are copied ($\&$) or split ($\otimes$), in different inference rules ($\oplus$) or in the same sequent ($\parr$).

HyLL and Linear Logic

HyLL rules can be encoded in LL as:

$$
\begin{array}{lcl}
\otimes R & : & \exists C, C', w.(\lceil (C \otimes C')@w \rceil^{\perp} \otimes \lceil C@w \rceil \otimes \lceil C'@w \rceil) \\
\otimes L & : & \exists C, C', w.(\lfloor (C \otimes C')@w \rfloor^{\perp} \otimes (\lfloor C@w \rfloor \parr \lfloor C'@w \rfloor)) \\
\mathtt{at}\ R & : & \exists C, u, w.(\lceil (C\ \mathtt{at}\ u)@w \rceil^{\perp} \otimes \lceil C@u \rceil) \\
\mathtt{at}\ L & : & \exists C, u, w.(\lfloor (C\ \mathtt{at}\ u)@w \rfloor^{\perp} \otimes \lfloor C@u \rfloor) \\
\downarrow R & : & \exists A, u, w.(\lceil \downarrow u.A@w \rceil^{\perp} \otimes \lceil (A\ w)@w \rceil) \\
\downarrow L & : & \exists A, u, w.(\lfloor \downarrow u.A@w \rfloor^{\perp} \otimes \lfloor (A\ w)@w \rfloor)
\end{array}
$$

Theorem (Adequacy)

Let Υ be the set of above clauses. The sequent $\Gamma; \Delta \vdash F@w$ is provable in HyLL iff $\vdash ?\Upsilon, ?\lfloor \Gamma \rfloor, \lfloor \Delta \rfloor, \lceil F@w \rceil$ is provable in LL.
The adequacy of the encodings is on the level of derivations *[i.e. when focusing on a LL specification clause, the (bipole) derivation corresponds exactly to applying the introduction rule at the object level].*

HyLL and SELL

HyLL rules into SELL$^{\forall}$:

$$
\begin{array}{lcl}
\otimes\ R & : & \exists C, C'.\exists w:\infty.(!^{w}\lceil (C \otimes C')@w \rceil^{\perp} \otimes ?^{w}\lceil C@w \rceil \otimes ?^{w}\lceil C'@w \rceil \\
\mathtt{at}\ R & : & \exists A.\exists u:\infty, w:\infty.(!^{w}\lceil (A\ \mathtt{at}\ u)@w \rceil^{\perp} \otimes ?^{u}\lceil A@u \rceil) \\
\mathtt{at}\ L & : & \exists A.\exists u:\infty, w:\infty.(!^{w}\lfloor (A\ \mathtt{at}\ u)@w \rfloor^{\perp} \otimes ?^{u}\lfloor A@u \rfloor) \\
\downarrow R & : & \exists A.\exists u:\infty, w:\infty.(!^{w}\lceil \downarrow u.A@w \rceil^{\perp} \otimes ?^{w}\lceil (A\ w)@w \rceil) \\
\downarrow L & : & \exists A.\exists u:\infty, w:\infty.(!^{w}\lfloor \downarrow u.A@w \rfloor^{\perp} \otimes ?^{w}\lfloor (A\ w)@w \rfloor)
\end{array}
$$

Theorem (Adequacy)

Let Υ be the set of formulas resulting from the encoding in the above definition. The sequent $\Gamma;\Delta \vdash F@w$ is provable in HyLL iff $\vdash ?^{c}\Upsilon, ?^{c}\lfloor\Gamma\rfloor, \lfloor\Delta\rfloor, ?^{w}\lceil F@w\rceil$ is provable in $SELL^{\forall}$. Moreover, the adequacy of the encodings is on the level of derivations.

Information Confinement

- Information confinement in SELL:
  inconsistency is local: $!^w ?^w \mathbf{0} \nvdash \mathbf{0}$
  inconsistency is not propagated: $!^w ?^w \mathbf{0} \nvdash\, !^v ?^v \mathbf{0}$
- In HyLL it is not possible to confine inconsistency:
  even if we exchange the rule $\mathbf{0}L$ by

$$\Gamma; \Delta, \mathbf{0}@w \vdash F@w \ [0_L]$$

  the rule $\mathbf{0}L$ would still be admissible:

$$\dfrac{\Gamma; \Delta, \mathbf{0}@w \vdash (\mathbf{0} \text{ at } v)@w \;\; 0_L \qquad \dfrac{\Gamma; \Delta, \mathbf{0}@v \vdash F@v \;\; 0_L}{\Gamma; \Delta, (\mathbf{0} \text{ at } v)@w \vdash F@v}\, at_L}{\Gamma; \Delta, \mathbf{0}@w \vdash F@v}\,\mathrm{cut}$$

CTL in HyLL [1]

Encoding of temporal logic operators in $\mathrm{HyLL}[\mathcal{T}]$, where $\mathcal{T} = \langle \mathbb{N}, +, 0\rangle$, representing instants of time:

- State quantifiers
$$\mathsf{F} \Leftrightarrow \Diamond, \quad \mathsf{G} \Leftrightarrow \Box \quad \text{and} \quad \mathsf{X}P \Leftrightarrow \delta_1 P$$
$$P_1 \mathsf{U} P_2 \Leftrightarrow \downarrow u.\ \exists v.\ P_2 \texttt{ at } u.v \otimes \forall w \prec v.\ P_1 \texttt{ at } u.w$$
- Path quantifiers
  E corresponds to the existence of a proof: $\mathsf{EF} \Leftrightarrow \Diamond$, $\mathsf{EG} \Leftrightarrow \Box$
  A: consider all the possible rules to be applied at each step.
  Let R be the set of rules of our transition system.
  - $\mathsf{AX}P$ is encoded as $\text{forall } r \text{ in } R\ \delta_1 P$. More precisely:
    $\mathsf{AX}P \Leftrightarrow \text{forall } r \text{ in } R\ (\texttt{fireable}(r) \,\&\, \delta_1 P) \oplus \texttt{not\_fireable}(r)$
  - $\mathsf{AG}P \leftrightarrow P \wedge \mathsf{AG}(P \multimap \mathsf{AX}(P))$.
    $\mathsf{AG}P \Leftrightarrow P \otimes \forall n.\ (P\ at\ n) \multimap \text{forall } r \text{ in } R\ (P \texttt{ at } n+1)$.
  - $\mathsf{AF}P \leftrightarrow P \vee \mathsf{AX}(\mathsf{AF}P)$.
    for a bound k on the number of steps needed.

CTL in HyLL [2]

Let $\mathtt{V} = \{a_1, ..., a_n\}$ propositional variables and $\mathtt{s} = \mathtt{p}_1(a_1) \wedge \cdots \wedge \mathtt{p}_n(a_n)$ represent a state where $\mathtt{p}_i \in \{\mathtt{pres}, \mathtt{abs}\}$ and $r : \mathtt{s} \rightarrow \mathtt{s}'$ be a state transition.
Encoding $[\![\cdot]\!]$ from CTL states and state transitions to HyLL:

$$[\![\mathtt{pres}(a_i)]\!] = \mathtt{pres}(a_i) \quad [\![\mathtt{abs}(a_i)]\!] = \mathtt{abs}(a_i)$$
$$[\![\mathtt{s}]\!] = \bigotimes_{i \in 1..n} [\![\mathtt{p}_i(a_i)]\!] \quad [\![r : \mathtt{s} \rightarrow \mathtt{s}']\!] = \forall w. (([\![\mathtt{s}]\!] \;\mathtt{at}\; w) \multimap \delta_1([\![\mathtt{s}']\!]) \;\mathtt{at}\; w$$

Let F, G be CTL formulas built from states and $\wedge, \vee, \mathrm{U}, \mathrm{EX}, \mathrm{EF}$.

$$\begin{array}{lclclcl} \mathcal{C}[\![\mathtt{s}]\!] & = & [\![\mathtt{s}]\!] & \quad & \mathcal{C}[\![F \wedge G]\!] & = & \mathcal{C}[\![F]\!] \,\&\, \mathcal{C}[\![G]\!] \\ \mathcal{C}[\![F \vee G]\!] & = & \mathcal{C}[\![F]\!] \oplus \mathcal{C}[\![G]\!] & & \mathcal{C}[\![\mathrm{E}[F \mathrm{U} G]]\!] & = & \mathcal{C}[\![F]\!] \mathrm{U}\, \mathcal{C}[\![G]\!] \\ \mathcal{C}[\![\mathrm{EX} F]\!] & = & \delta_1\, \mathcal{C}[\![F]\!] & & \mathcal{C}[\![\mathrm{EF} F]\!] & = & \Diamond \mathcal{C}[\![F]\!] \end{array}$$

Such encodings are *faithful*, i.e. a CTL formula F holds at state $\mathtt{s}$ in $\mathcal{R}$ iff $[\![\mathcal{R}]\!]@0; [\![\mathtt{s}]\!]@w \vdash \mathcal{C}[\![F]\!]@w$ is provable in HyLL.

CTL in μMALL

$$\frac{\Sigma \vdash \Delta, S\vec{t} \qquad \vec{x} \vdash B\ S\vec{x}, (S\vec{x})^{\perp}}{\Sigma \vdash \Delta, \nu B\vec{t}}\ \nu \qquad \frac{\Sigma \vdash \Delta, B(\mu B)\vec{t}}{\Sigma \vdash \Delta, \mu B\vec{t}}\ \mu$$

where S is the (co)inductive invariant. The μ rule corresponds to unfolding while ν allows for (co)induction. Σ represents the (first-order) signature.

CTL in μMALL (con't)

Path quantifiers as fixpoints:

$$\begin{array}{rcl} \mathsf{EF}F &=& \mu Y.F \vee \mathsf{EX}Y \\ \mathsf{EG}F &=& \nu Y.F \wedge \mathsf{EX}Y \end{array} \qquad \begin{array}{rcl} \mathsf{E}[F \mathsf{U} G] &=& \mu Y.G \vee (F \wedge \mathsf{EX}Y) \end{array}$$

$$\begin{array}{rcl} \mathsf{AF}F &=& \mu Y.F \vee \mathsf{AX}Y \\ \mathsf{AG}F &=& \nu Y.F \wedge \mathsf{AX}Y \end{array} \qquad \begin{array}{rcl} \mathsf{A}[F \mathsf{U} G] &=& \mu Y.G \vee (F \wedge \mathsf{AX}Y) \end{array}$$

CTL in μMALL (con't)

Definition (CTL into μMALL)

Let $\mathcal{R}$ be of transition rules and a state $\mathrm{s} = \mathrm{p}_1(a_1) \wedge \cdots \wedge \mathrm{p}_n(a_n)$.

$$[\![\mathtt{pres}(a_i)]\!] = a_i \qquad [\![\mathtt{abs}(a_i)]\!] = a_i^\perp \qquad [\![p]\!] = \mathtt{pos}(p)$$

$$\begin{array}{lcl} [\![s]\!] & = & [\![\mathrm{p}_1(a_1)]\!]^\perp \parr \cdots \parr [\![\mathrm{p}_n(a_n)]\!]^\perp \\ \mathtt{pos}(\mathrm{s}) & = & [\![\mathrm{p}_1(a_1)]\!] \otimes \cdots \otimes [\![\mathrm{p}_n(a_n)]\!] \\ \mathtt{neg}(\mathrm{s}) & = & ([\![\mathrm{p}_1(a_1)]\!]^\perp \otimes \top) \oplus \cdots \oplus ([\![\mathrm{p}_n(a_n)]\!]^\perp \otimes \top) \end{array}$$

p is a state formula. $\mathtt{pos}(\mathrm{s})$ (resp. $\mathtt{neg}(\mathrm{s})$) tests if r can (resp. cannot) be fired at the current state.
We map CTL $\wedge$ [resp. $\vee$] into & [resp. $\oplus$].

$$\begin{array}{lcl} \mathcal{C}[\![\mathrm{AX}F]\!]_\mathcal{R} & = & \mathop{\&}\limits_{\mathrm{s}\to\mathrm{s}'\in\mathcal{R}} (\mathtt{neg}(\mathrm{s}) \oplus (\mathtt{pos}(\mathrm{s}) \otimes ([\![\mathrm{s}']\!] \parr \phi))) \\ \mathcal{C}[\![\mathrm{EX}F]\!]_\mathcal{R} & = & \bigoplus\limits_{\mathrm{s}\to\mathrm{s}'\in\mathcal{R}} (\mathtt{pos}(\mathrm{s}) \otimes ([\![\mathrm{s}']\!] \parr \phi)) \end{array}$$

CTL in μMALL (con't)

Definition (CTL into μMALL (con't))

$$
\begin{aligned}
\mathcal{C}[\![\mathsf{A}F F]\!]_{\mathcal{R}} &= \mu Y.\ \phi \oplus \mathop{\&}_{s \to s' \in \mathcal{R}} (\mathrm{neg}(s) \oplus (\mathrm{pos}(s) \otimes ([\![s']\!] \parr Y))) \\
\mathcal{C}[\![\mathsf{E}F F]\!]_{\mathcal{R}} &= \mu Y.\ \phi \oplus \bigoplus_{s \to s' \in \mathcal{R}} (\mathrm{pos}(s) \otimes ([\![s']\!] \parr Y)) \\
\mathcal{C}[\![\mathsf{A}G F]\!]_{\mathcal{R}} &= \nu Y.\ \phi \mathbin{\&} \mathop{\&}_{s \to s' \in \mathcal{R}} (\mathrm{neg}(s) \oplus (\mathrm{pos}(s) \otimes ([\![s']\!] \parr Y))) \\
\mathcal{C}[\![\mathsf{E}G F]\!]_{\mathcal{R}} &= \nu Y.\ \phi \mathbin{\&} \bigoplus_{s \to s' \in \mathcal{R}} (\mathrm{pos}(s) \otimes ([\![s']\!] \parr Y)) \\
\mathcal{C}[\![\mathsf{A}[F U G]]\!]_{\mathcal{R}} &= \mu Y. \psi \oplus \left(\phi \mathbin{\&} \mathop{\&}_{s \to s' \in \mathcal{R}} (\mathrm{neg}(s) \oplus (\mathrm{pos}(s) \otimes ([\![s']\!] \parr Y)) \right) \\
\mathcal{C}[\![\mathsf{E}[F U G]]\!]_{\mathcal{R}} &= \mu Y. \psi \oplus \left(\phi \mathbin{\&} \bigoplus_{s \to s' \in \mathcal{R}} (\mathrm{pos}(s) \otimes ([\![s']\!] \parr Y)) \right)
\end{aligned}
$$

CTL in μMALL (con't)

Let s $\models^{\mathcal{R}}_{CTL} F$ denote "the CTL formula F holds at state s in $\mathcal{R}$".

Theorem (Adequacy)

Let $V = \{a_1, \ldots, a_n\}$ be a set of propositional variables, $\mathcal{R}$ be a set of transition rules on V and F be a CTL formula. Then, $s \models^{\mathcal{R}}_{CTL} F$ iff the sequent $\vdash [\![s]\!], \mathcal{C}[\![F]\!]_{\mathcal{R}}$ is provable in μMALL.

Conclusion and Future Work

- Done: HyLL into LL, HyLL into SELL$^{\forall}$, Information confinement, CTL into μMALL.
- Claim: Logical Frameworks are safe and general frameworks, for specifying and verifying properties of a large number of systems.
- To do: automatic proofs for SELL$^{\forall}$ for biology, biomedicine (diagnosis), neuroscience, ...

Thanks for your attention