A Riemannian Framework for Orientation Distribution Function Computing

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Abstract

Compared with Diffusion Tensor Imaging (DTI), High Angular Resolution Imaging (HARDI) can better explore the complex microstructure of white matter. Orientation Distribution Function (ODF) is used to describe the probability of the fiber direction. Fisher information metric has been constructed for probability density family in Information Geometry theory and it has been successfully applied for tensor computing in DTI [1,2,3]. In this paper, we present a state of the art Riemannian framework for ODF computing based on Information Geometry and sparse representation of orthonormal bases [4]. In this Riemannian framework, the exponential map, logarithmic map and geodesic have closed forms. And the weighted Frechet mean exists uniquely on this manifold. We also propose a novel scalar measurement, named Geometric Anisotropy (GA), which is the Riemannian geodesic distance between the ODF and the isotropic ODF. The Renyi entropy $H_{1/2}$ of the ODF can be computed from the GA. Moreover, we present an Affine-Euclidean framework and a Log-Euclidean framework so that we can work in an Euclidean space. As an application, Lagrange interpolation on ODF field is proposed based on weighted Frechet mean. We validate our methods on synthetic and real data experiments. Compared with existing Riemannian frameworks on ODF, our framework is model-free. The estimation of the parameters, i.e. Riemannian coordinates, is robust and linear. Moreover it should be noted that our theoretical results can be used for any probability density function (PDF) under an orthonormal basis representation.

Riemannian framework for ODFs

ODF could be seen as a PDF family. Riemannian framework is derived from the PDF family based on Information Geometry theory.

I. Von Mises-Fisher Mixture model [5]:

$$p(\mathbf{x}) = \sum_{i=1}^{m} w_i M(\mathbf{x} \mid \boldsymbol{\mu}_i, k_i)$$

$$w_i > 0 \quad \sum_{i=1}^{m} w_i = 1 \quad k_i > 0 \quad \boldsymbol{\mu}_i \in S^2$$

- Model based (mixture model)
- Can not represent all ODFs since it does not form a basis.
- Parameters are harder to estimate
- Metric is defined in a multiplicative space

II. Orthonormal basis representation of the square root of ODF

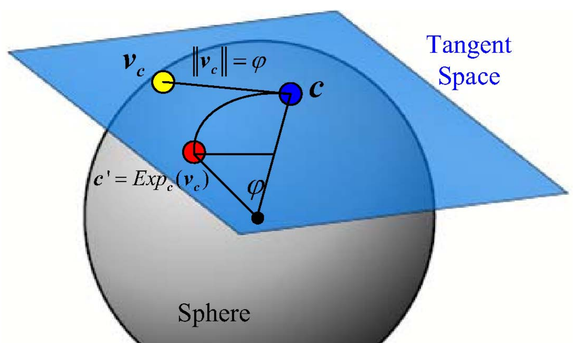
$$p(\mathbf{x} \mid \mathbf{c}) = \left(\sum_{i=1}^{K} c_i B_i(\mathbf{x}) \right)^2$$

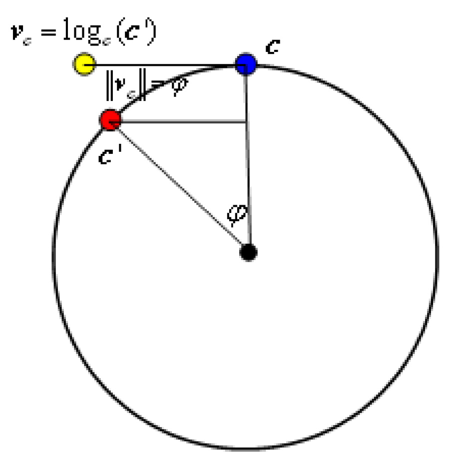
$$\sum_{i=1}^{K} c_i^2 = 1 \qquad \sum_{i=1}^{K} c_i B_i(\mathbf{x}) \ge 0$$

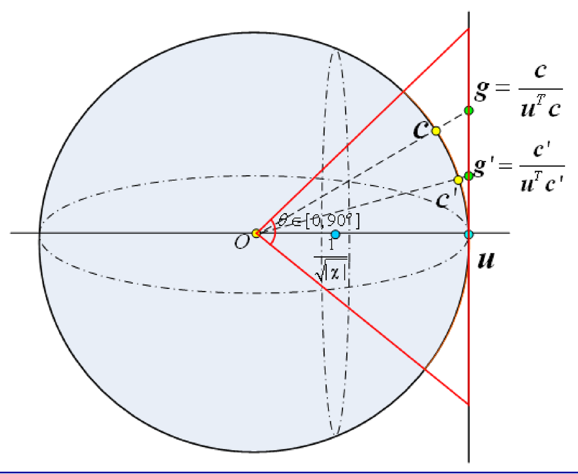
- Model free (basis representation)
- Can represent all ODFs
- Easy to estimate
- Metric is defined in a simple and well defined manifold (a small part of S^{k-1})

The intrinsic Riemannian framework for ODF computing

- Fisher metric [6]: $g_{ij} = 4\int_{S^{K-1}} \partial_i \sqrt{p(\mathbf{x} \mid \mathbf{c})}\, \partial_j \sqrt{p(\mathbf{x} \mid \mathbf{c})}\, d\mathbf{x} = 4\delta_{ij}$
- Geodesic: $d_{g_{ij}}(p(\cdot \mid \mathbf{c}), p(\cdot \mid \mathbf{c}')) = d_{\delta_{ij}}(\mathbf{c}, \mathbf{c}') = \arccos(\mathbf{c}^T \mathbf{c}')$
- Exponential map: $Exp_{\mathbf{c}}(\mathbf{v}_c) = \mathbf{c}' = \mathbf{c} \cos\varphi + \frac{\mathbf{v}_c}{\|\mathbf{v}_c\|} \sin\varphi$, where $\varphi = \|\mathbf{v}_c\|$
- Logarithmic map: $Log_{\mathbf{c}}(\mathbf{c}') = \mathbf{v}_c = \frac{\mathbf{c}' - \mathbf{c}\cos\varphi}{\|\mathbf{c}' - \mathbf{c}\cos\varphi\|}\varphi$, where $\varphi = \arccos(\mathbf{c}^T \mathbf{c}')$







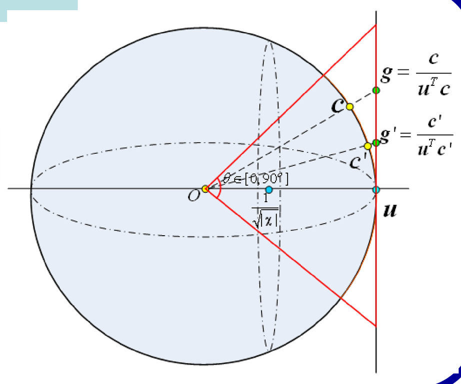
Properties and results of the manifold (parameter space PS)

- PS is closed
- PS is convex
- PS is contained in a convex cone with 90⁰
- The projection of any $\mathbf{c}$ on the uniform ODF $\mathbf{u}$ is more than $\frac{1}{\sqrt{4\pi}}$

- Geometric Anisotropy: $GA(p(\mathbf{x} \mid \mathbf{c})) = d(\mathbf{c}, \mathbf{u}) = \arccos(\mathbf{c}^T \mathbf{u})$
- Renyi Entropy: $H_{1/2} = 2\log\left(\int_{S^{K-1}} \sqrt{p(\mathbf{x} \mid \mathbf{c})}\, d\mathbf{x}\right) = \log(4\pi(\mathbf{c}^T \mathbf{u})^2)$
- Affine-Euclidean framework and Log-Euclidean framework

Affine-Euclidean and Log-Euclidean frameworks

- Diffeomorphism: (AE) $F(c) = \frac{\mathbf{c}}{\mathbf{u}^T \mathbf{c}} - \mathbf{u}$ (LE) $F(c) = Log_{\mathbf{u}}(\mathbf{c})$
- Geodesic: $d(p(\cdot \mid \mathbf{c}), p(\cdot \mid \mathbf{c}')) = d_{Euc}(F(\mathbf{c}), F(\mathbf{c}')) = \|F(\mathbf{c}) - F(\mathbf{c}')\|$
- Exponential map: $Exp_c(\mathbf{v}_c) = \mathbf{c}' = F^{-1}(F(\mathbf{c}) + F_* \mathbf{v}_c)$
- Logarithmic map: $Log_c(\mathbf{c}') = \mathbf{v}_c = F^*(F(\mathbf{c}') - F(\mathbf{c}))$



Weighted Fréchet Mean

- Definition: $\mu_w = \arg\min_{f \in PS} \sum_{i=1}^{N} w_i d(f, f_i)^2$ uniquely exists on the PS [7]
- No close form. Numerical algorithm [7]:

Algorithm 1: Weighted Frechet Mean
Input: $f_1, ..., f_N \in PS^K$, $\mathbf{w} = (w_1, ..., w_N)'$, $w_i \ge 0, i = 1, 2, ..., N$, $\sum_{i=1}^{N} w_i = 1$
Output: $\mu_{\mathbf{w}}$, the *Weighted Frechet Mean*.
Initialization: $\mu_{\mathbf{w}}^{(0)} = \frac{\sum_{i=1}^{N} w_i f_i}{\|\sum_{i=1}^{N} w_i f_i\|}$, $k = 0$
Do
$v_{\mu_{\mathbf{w}}^{(k)}} = \sum_{i=1}^{N} w_i Log_{\mu_{\mathbf{w}}^{(k)}}(f_i)$
$\mu_{\mathbf{w}}^{(k+1)} = Exp_{\mu_{\mathbf{w}}^{(k)}}(v_{\mu_{\mathbf{w}}^{(k)}})$
$k = k + 1$
while $\|v_{\mu_w^{(k)}}\| > \varepsilon$

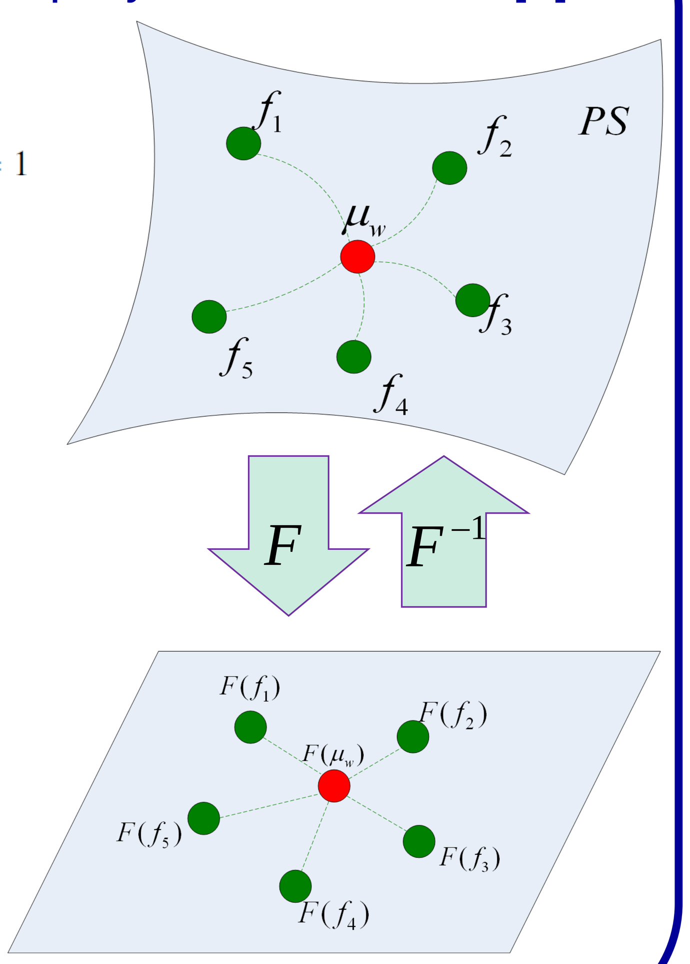
- There is a close form for AE and LE frameworks

$$\mu_{\mathbf{w}} = F^{-1}\left(\sum_{i=1}^{N} w_i F(f_i) \right)$$

- Interpolation in ODF field

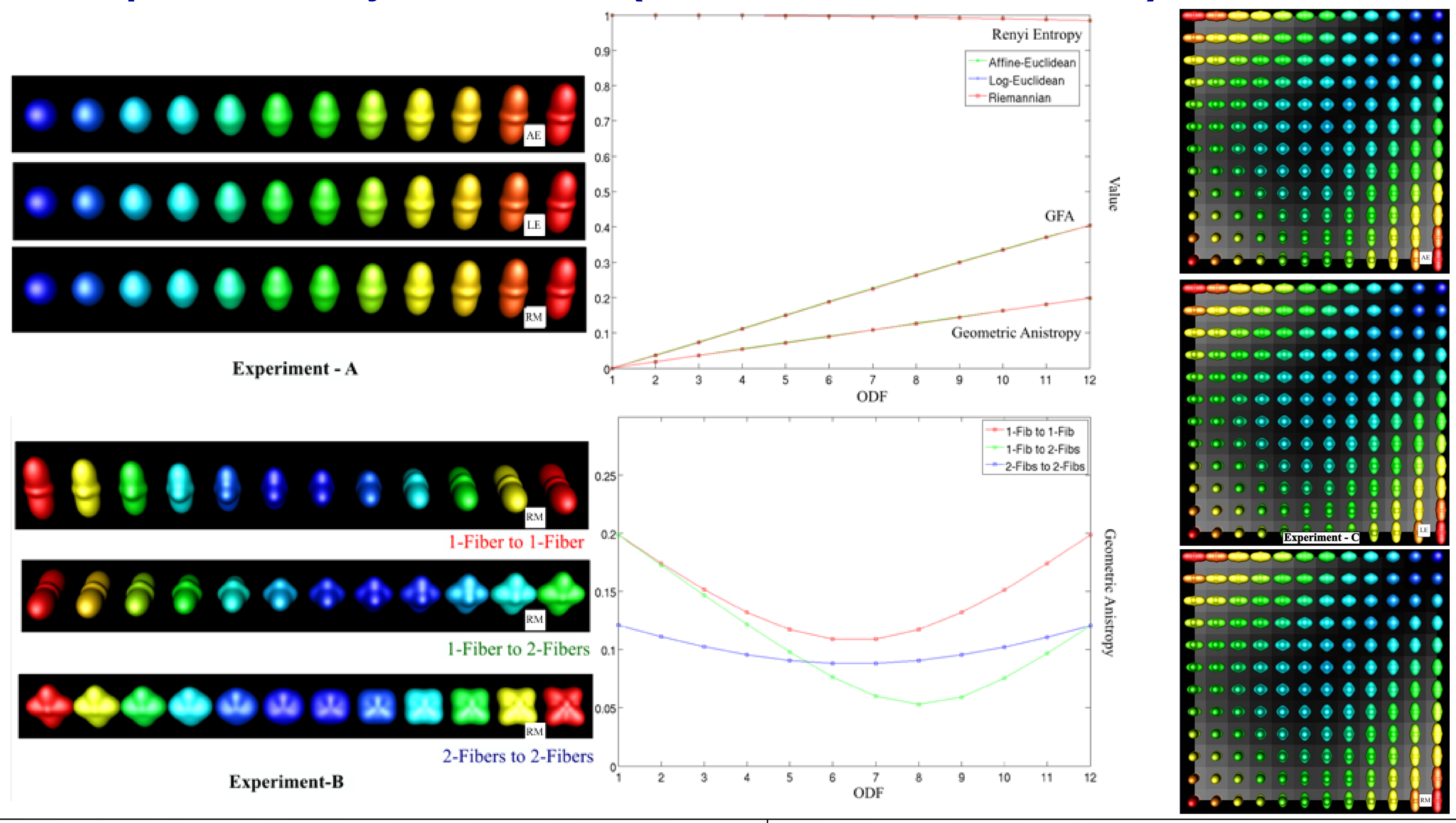
$$f(x, y, z) = \widetilde{\sum}_{i=1}^{N_1} \widetilde{\sum}_{j=1}^{N_2} \widetilde{\sum}_{k=1}^{N_3} w_i^{(1)}(x) w_j^{(2)}(y) w_k^{(3)}(z) f(x_i, y_j, z_k)$$

$$w_i^{(1)}(x) = \prod_{l=1, l\neq i}^{N_1} \frac{x - x_l}{x_i - x_l}, \quad w_j^{(2)}(y) = \prod_{m=1, m\neq j}^{N_2} \frac{y - y_m}{y_j - y_m}, \quad w_k^{(3)}(z) = \prod_{n=1, n\neq k}^{N_3} \frac{z - z_n}{z_k - z_n}$$

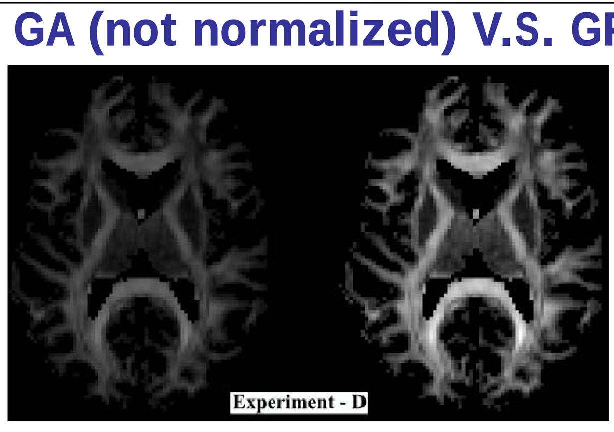


Experiments & Results

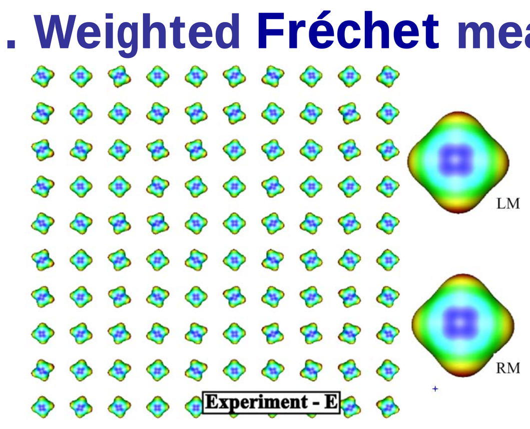
I. Interpolation on synthetic data (2 ODFs in 1D, 4 ODFs in 2D)



II. GA (not normalized) V.S. GFA [8]



III. Weighted Fréchet mean



Contributions & Conclusions

- **Only several coefficients are needed because ODFs are sparse enough**
- **Basis representation can represent all ODFs in brain.**
- **Parameter space is well studied in mathematics.**
- **Riemannian, Affine-Euclidean and Log-Eucliedan frameworks.**
- **Three frameworks guarantee a Positive ODF profile.**
- **Weighted Fréchet mean uniquely exists.**
- **Geometric Anisotropy & Renyi Entropy.**

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