Some algebraic tools

The SIROPA Maple library

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Using Computer algebra in applications

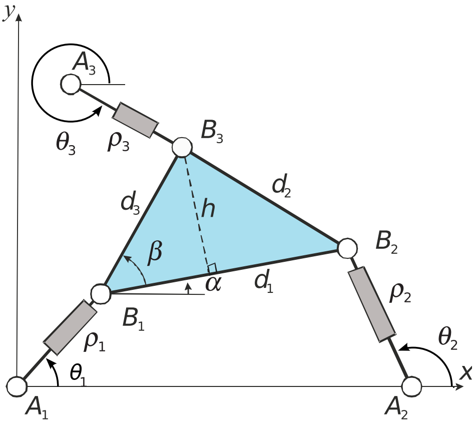
Warning : exact computations ! A result is wrong or true, never close to be true or close to be true. For example, a real root is NOT a complex root with a small imaginary part.

- sensitive to the modelization : not for stability reasons but for efficiency reasons.

- several algebraic objects/algorithms with precise (and sometimes complicated) geometrical meanings.

The SIROPA library (G. Moroz), makes default choices for you for some selected problems in robotics.

Algebraic Modeling : From Robots To Polynomials



- Conception parameters : d_1, d_2, d_3
- Control parameters : ρ_1, ρ_2, ρ_3
- Pose variables : B_{1x}, B_{1y}, α
- Passive variables : $\theta_1, \theta_2, \theta_3$
- 3 degrees of freedom

$$\begin{cases} {B_{1x}}^2 + {B_{1y}}^2 - {\rho_1}^2 & = 0 \\ (B_{1x} + 17.04\cos(\alpha) - 15.91)^2 + (B_{1y} + 17.04\sin(\alpha))^2 - {\rho_2}^2 & = 0 \\ (B_{1x} + 10.82\cos(\alpha) - 13.16\sin(\alpha) - 2)^2 + & \\ \qquad (B_{1y} + 13.16\cos(\alpha) + 10.82\sin(\alpha) - 5)^2 - {\rho_3}^2 & = 0 \end{cases}$$

Algebraic Modeling : Trigonometric Functions

2 possible strategies :

A $(cos(\theta), sin(\theta)) \mapsto (\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2})$ where $t = tan(\frac{\theta}{2})$

B $(cos(\theta), sin(\theta)) \mapsto (c, s)$ with $c^2 + s^2 = 1$

Case A

- $\theta = \pi$ for $t = \infty$
- Spurious complex component

Case B

- 2 variables
- 1 additional equation

Siropa toolbox :

- Provides automatic trigonometric/algebraic conversion
- Extends solving functions from polynomials to trigonometric expressions

Algebraic Modeling : Spatial Rotation

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \underbrace{\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}}_{R} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- Euler angles

$$R = \begin{pmatrix} c_\phi \, c_\theta \, c_\psi - s_\phi \, s_\psi & -c_\phi \, c_\theta \, s_\psi - s_\phi \, c_\psi & c_\phi \, s_\theta \\ s_\phi \, c_\theta \, c_\psi + c_\phi \, s_\psi & -s_\phi \, c_\theta \, s_\psi + c_\phi \, c_\psi & s_\phi \, s_\theta \\ -s_\theta \, c_\psi & s_\theta \, s_\psi & c_\theta \end{pmatrix}$$

- Quaternions

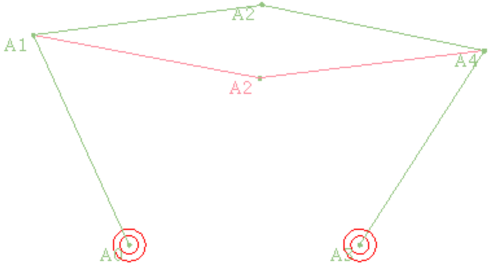
$$R = \begin{pmatrix} q_1^2 + q_2^2 & q_2 q_3 - q_1 q_4 & q_2 q_4 + q_1 q_3 \\ q_2 q_3 + q_1 q_4 & q_1^2 + q_3^2 & q_3 q_4 - q_1 q_2 \\ q_2 q_4 - q_1 q_3 & q_3 q_4 + q_1 q_2 & q_1^2 + q_4^2 \end{pmatrix}$$

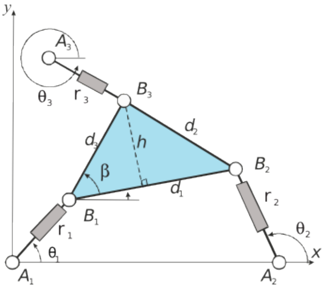
and $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$

Some Mechanisms in Siropa Toolbox

Mechanisms

- Parallel_3RPR
- Parallel_3RPR_full
- Parallel_3PRR
- ParallelPRP2PRR
- Parallel_RPRRP
- Parallel_RR_RRR
- Parallel_PRRP
- Orthoglide
- ParallelRPR2PRR





Solving ???

$$\mathcal{E} = \{p_1, \ldots, p_r\}, \mathcal{F} = \{f_1, \ldots, f_l\}, \text{ with } p_i, f_i \in \mathbb{Q}[U, X]$$

$$U = U_1, \ldots, U_d \Rightarrow \textbf{parameters}$$

$$X = X_{d+1}, \ldots, X_n \Rightarrow \textbf{ indeterminates}$$

$$\mathcal{V} = \{y \in \mathbb{C}^n, p_1(y) = 0, \ldots, p_r(y) = 0\}$$

$$\mathcal{C} = \{y \in \mathbb{C}^n, p_1(y) = 0, \ldots, p_r(y) = 0, f_1(y) \neq 0, \ldots, f_s(y) \neq 0\}$$

$$\mathcal{S} = \{y \in \mathbb{R}^n, p_1(y) = 0, \ldots, p_r(y) = 0, f_1(y) > 0, \ldots, f_s(y) > 0\}$$

Definition : given $\mathcal{A} \in \mathbb{C}^n$, the Zariski Closure of $\mathcal{A}$ denoted by $\bar{\mathcal{A}}$ is the smallest algebraic set (complex solutions of a system of equations) s.t. $\mathcal{A} \subset \overline{\mathcal{A}}$.

Définition : $\langle p_1, \ldots, p_r \rangle = \left\{ \sum_{i=1}^{r} q_i\, p_i, q_i \in \mathbb{Q}[U, X] \right\}$is the ideal of $\mathbb{Q}[U, X]$ generated by $\{p_1, \ldots, p_r\}$.

Gröbner Bases

Given a monomial ordering $<$ (a total order on the monomials $U_1^{e_1}...U_d^{e_d}X_{d+1}^{e_{d+1}}...X_n^{e_n}$ which is compatible with the multiplication : $m_1 < m_2 \Rightarrow m\,m_1 < m\,m_2, \forall m$), one can extend the classical univariate euclidean division to the division of a multivariate polynomial by a set of multivariate polynomials.

THE Gröbner basis, for a given ordering, of an ideal $< p_1, ..., p_r >$ is a set of polynomials $g_1, ..., g_{r'}$ such that $\langle p_1, ..., p_r \rangle = \langle g_1, ..., g_{r'} \rangle$ and such that the division of a polynomial by $\{g_1, ..., g_r'\}$ is “canonical”. The division is then named the NormalForm modulo the Gröbner basis.

In particular : $\{y \in \mathbb{C}^n, \, g_1(y) = 0, \, ..., \, g_{r'}(y) = 0\} = \{y \in \mathbb{C}^n, \, p_1(y) = 0, \, ..., \, p_r(y) = 0\}$

Elimination Orderings

Let $G=\{g_1,\ldots,g_{r'}\}$ be a Gröbner basis of $\langle\mathcal{E}\rangle=\langle p_1,\ldots,p_r\rangle\subset\mathbb{Q}[U,X]$.

Any monomial ordering such that $U_i < X_j$, $\forall i=1\ldots d$, $\forall j=d+1\ldots n$ is an ordering that "eliminates" the variables X_j.

For example, the lexicographic ordering $U_1<\ldots<U_d<X_{d+1}<\ldots<X_n$

Given an ordering that eliminates X_j, $\langle\mathcal{E}\rangle\cap\mathbb{Q}[U]=\langle g_{i_1},\ldots,g_{i_r}\rangle$ where the g_{i_j} are elements of G that do not depends on the variables $X_i, i=d+1\ldots n$.

$\langle\mathcal{E}\rangle\cap\mathbb{Q}[U]$???? What's this ?

$V(\langle\mathcal{E}\rangle)=\{(y)\in\mathbb{C}^n, p_1(y)=0,\ldots,p_r(y)=0\}$

$V(\langle\mathcal{E}\rangle\cap\mathbb{Q}[U])=\overline{\Pi_U(V(\langle\mathcal{E}\rangle))}\subset\mathbb{C}^d$ is **the Zariski closure $(\bar{.})$ of the projection onto the parameter's space (Π_U) of the zeroes of $\langle\mathcal{E}\rangle$.**

Interpretation :

For almost all the $u\in V(\langle\mathcal{E}\rangle\cap\mathbb{Q}[U])$, there exists (at least) one x such that $(u,x)\in V(\langle\mathcal{E}\rangle)$.

An Example

$\mathcal{E}=\{U_1\,X_2-1\}$, $V(\langle\mathcal{E}\rangle)=\{(x,y)\in\mathbb{C}^n, x\,y-1=0\}$ is an hyperbola.

$\Pi_{U_1}(\langle\mathcal{E}\rangle)=\mathbb{C}\setminus\{0\}$ (almost all the complex values)

$G=\{U_1X_2-1\}$ is a Gröbner basis of $\langle\mathcal{E}\rangle$ for any ordering.

Also $\langle\mathcal{E}\cap\mathbb{Q}[U_1]\rangle=\{0\}$ and thus $V(\langle\mathcal{E}\cap\mathbb{Q}[U_1]\rangle)=\mathbb{C}$ (all the complex values)

Remark :

When $n = d + 1$, say $U = U_1$, ..., U_d and $X = X_{d+1}$, the resultant of 2 polynomials "eliminates" 1 variable.

$r_{1,2}=\text{Resultant}(p_1,p_2,X_{d+1})$ we then have $V(r_{1,2})=\overline{\Pi_U(V(\langle p_1,p_2\rangle))}$

Over the reals

One example : $p = X_2^2 + U_1^2$

The Zariski closure of the projection of $\{(u, x) \in \mathbb{C}^2, p(u, x) = 0\}$onto the U_1 axis ($\mathbb{C}$) is the U_1 axis itself.

This means that for almost (in the present case all) all the complex values of U_1, the "system" $p\big(u, X_2^2\big) = 0$ has (at least) one solution.

BUT the system $\{(u, x) \in \mathbb{R}^2, p(u, x) = 0\}$ has only one solution in $\mathbb{R}^2$ ((0,0)) so that the Zariski closure of the projection of $\{(u, x) \in \mathbb{R}^2, p(u, x) = 0\}$onto the U_1 axis is reduced to 0.

So it is wrong that for almost all the real values of U_1, the "system" $p\big(u, X_2^2\big) = 0$ has (at least) one solution.

In the general case : $\overline{\Pi_U(V(\mathcal{E}))} \cap \mathbb{R}^d \neq \overline{\Pi_U(V(\mathcal{E}) \cap \mathbb{R}^n)} \cap \mathbb{R}^d$ and thus, the elimination of variables (Zariski closure of some projection) is not sufficient for "solving" a system over the reals, whatever "solving" means.

Over the reals

One example : $p = X_2^2 + U_1$

The Zariski closure of the projection of $\{(u, x) \in \mathbb{C}^2, p(u, x) = 0\}$onto the U_1 axis ($\mathbb{C}$) is the U_1 axis itself. Also $\overline{\Pi_U(V(\mathcal{E}))} = \mathbb{C}$.

This means that for almost (in the present case all) all the complex values of U_1, the "system" $p(u, X_2^2) = 0$ has (at least) one solution.

BUT the system $\{(u, x) \in \mathbb{R}^2, p(u, x) = 0\}$ has solutions in $\mathbb{R}^2$ if and only if $U_1 \leqslant 0$. Also, $\Pi_U(V(\mathcal{E}) \cap \mathbb{R}^2) = \mathbb{R}^-$ and $\overline{\Pi_U(V(\mathcal{E}) \cap \mathbb{R}^2)} \cap \mathbb{R} = \mathbb{R}$.

In the general case : even if $\overline{\Pi_U(V(\mathcal{E}))} \cap \mathbb{R}^d = \overline{\Pi_U(V(\mathcal{E}) \cap \mathbb{R}^n)} \cap \mathbb{R}^d$, the elimination of variables (Zariski closure of some projection) is absolutely not sufficient for "solving" a system over the reals, whatever "solving" means.

In short : in the general case, eliminating variables does not provide neither sufficient nor necessary conditions for "solving" systems over the reals. One has to work a little bit more ....

Degree and Dimension

Dimension : “number of free complex variables”.

Geometrical Degree : “maximum number of complex solutions once all the free variables are set to generic values”

Algebraic degree : “maximum number of complex solutions counted with multiplicities once all the free variables are set to generic values”

Dimension and Algebraic degree can be computed through the Hilbert function from a Gröbner basis for any monomial ordering.

Systems of dimension 0 are systems without “free” variables. Their complex solutions define a finite set of points.

Remark : a system can be of dimension >0 but may have a finite number of solutions. Example : $X^2+Y^2=0$ has dimension 1 in $\mathbb{C}$.

Real dimension ? Real degree ? : too difficult to compute.

Zero-dimensional Systems

$\mathcal{E} \subset \mathbb{Q}[Y_1, ..., Y_n]$ and G a Gröbner basis of $\langle\mathcal{E}\rangle$ for any monomial ordering $<$.

$\langle\mathcal{E}\rangle$ is zero-dimensional if and only if

$\forall i, \exists g_i \in G$ such that the leading coefficient of g_i for $<$ is a pure power of Y_i..

Degree=number of monomials that are not reducible modulo G (you can "read" then on a Gröbner basis for any ordering)=number of complex roots counted with multiplicities.

(In fact : $\frac{\mathbb{Q}[Y_1, ..., Y_n]}{I}$ is a $\mathbb{Q}$-vector space of dimension D (the degree of $\langle\mathcal{E}\rangle$) and the set of irreducible monomials is a basis).

Zero-dimensional Systems

Performing some linear algebra in $\frac{\mathbb{Q}[Y_1,...,Y_n]}{I}$, one can then compute a so called Rational Univariate Representation [Rouillier 1999] :

A linear form t, $n+2$ univariate polynomials with rational coefficients f_t, $g_{t,1}$, $g_{t,Y_1}, \ldots g_{t,Y_n}$ such that the following bijection :

$$\begin{array}{lcl} V(\langle \mathcal{E} \rangle) & \approx & V(\langle f_t \rangle) \\ \alpha = (\alpha_1, ..., \alpha_n) & \rightarrow & t(\alpha) \\ (\frac{g_{t,X_1}(\beta)}{g_{t,1}(\beta)}, ..., \frac{g_{t,X_n}(\beta)}{g_{t,1}(\beta)}) & \leftarrow & \beta \end{array}$$

preserves also the multiplicities and the real roots.

So, solving a zero-dimensional system remains to solving a univariate problem.

Zero-dimensional Systems

Using the so called Descartes' rule of signs together with interval arithmetic, one can isolate efficiently all the real roots of a univariate polynomial with rational coefficients [Rouillier, Zimmermann 2003].

Input : a polynomial with rational coefficients

Output : intervals with rational bounds that isolate all the real roots of the polynomial.

(this algorithm is used in Maple 14 instead of numerical algorithms in the function fsolve).

Zero-dimensional Systems

The (default) implementation :

1 - compute a Gröbner basis (F4 algorithm by JC Faugere, available in Maple version > 11)

2 - compute a rational univariate representation $(f_t, g_{t,1}, g_{t,Y_1}, \dots g_{t,Y_n})$ (RUR algorithm by F. Rouillier, available in Maple version > 11)

3 - Isolate (and eventually refine) the real roots of f_t (USPENSKY algorithm by F. Rouillier and P. Zimmermann, available in Maple version > 11)

4 - For each real root (interval) β of f_t, compute the box $(\frac{g_{t,X_1}(\beta)}{g_{t,1}(\beta)}, \dots, \frac{g_{t,X_n}(\beta)}{g_{t,1}(\beta)})$ using multiprecision interval arithmetic, and you get isolating boxed for all the real solutions of the system.

Note : for refining the roots up to an arbitrary precision, it is sufficient to refine the isolating intervals obtaines at step 3 and run again step 4.

Zero-dimensional Systems

Efficiency and influence of the modelization :

“On solving the direct kinematics problem for parallel robots”

JC Faugère JP Merlet, F. Rouillier - INRIA Research report RR-5923 (2006).

From a 3-point “classical” modelization to an ad-hoc quaternion based modelization the speed up is about 1000.

The later allows to verify the 40 real positions of Dietmaier’s general parallel platform as well as well as slight deformations in few seconds.

In fact we could observe that one can cut to 18 digits the floating point numbers used by Dietmaier for discribing the geometry of his robot without loosing positions, and that the number of positions falls to 38 when using 16 digits... very unstable ...

Systems depending on parameters

$$\mathcal{E} = \{p_1, \ldots, p_r\}, \mathcal{F} = \{f_1, \ldots, f_l\}, \text{ with } p_i, f_i \in \mathbb{Q}[U, X]$$

$$U = U_1, \ldots, U_d \Rightarrow \textbf{parameters}$$

$$X = X_{d+1}, \ldots, X_n \Rightarrow \textbf{ indeterminates}$$

$$\mathcal{C} = \{x \in \mathbb{C}^n, p_1(x) = 0, \ldots, p_r(x) = 0, f_1(x) \neq 0, \ldots, f_s(x) \neq 0\}$$

$$\mathcal{S} = \{x \in \mathbb{R}^n, p_1(x) = 0, \ldots, p_r(x) = 0, f_1(x) > 0, \ldots, f_s(x) > 0\}$$

$\Pi_U \colon \mathbb{C}^n \longrightarrow \mathbb{C}^d$ the canonical projection onto the parameters' space

- Exists parameters values u s.t. $\mathcal{C}_{|_{U=u}} \neq \emptyset$ or $\mathcal{S}_{|_{U=u}} \neq \emptyset$?

- Number of complex (resp. real) points of $\mathcal{C}_{|_{U=u}}$ or $\mathcal{S}_{|_{U=u}}$?

- "Simple" description of $\mathcal{C}$ or $\mathcal{S}$ wrt Π_U ?

Well-behaved systems

Most systems comming from applications (outside mathematics) are s.t. :

- for almost all $u \in \mathbb{R}^d$, the real roots of $\mathcal{E}_{|U=u} = 0$ can be computed (real roots) by a basic version of Newton's method

Most of them verify the following conditions (well-behaved systems)

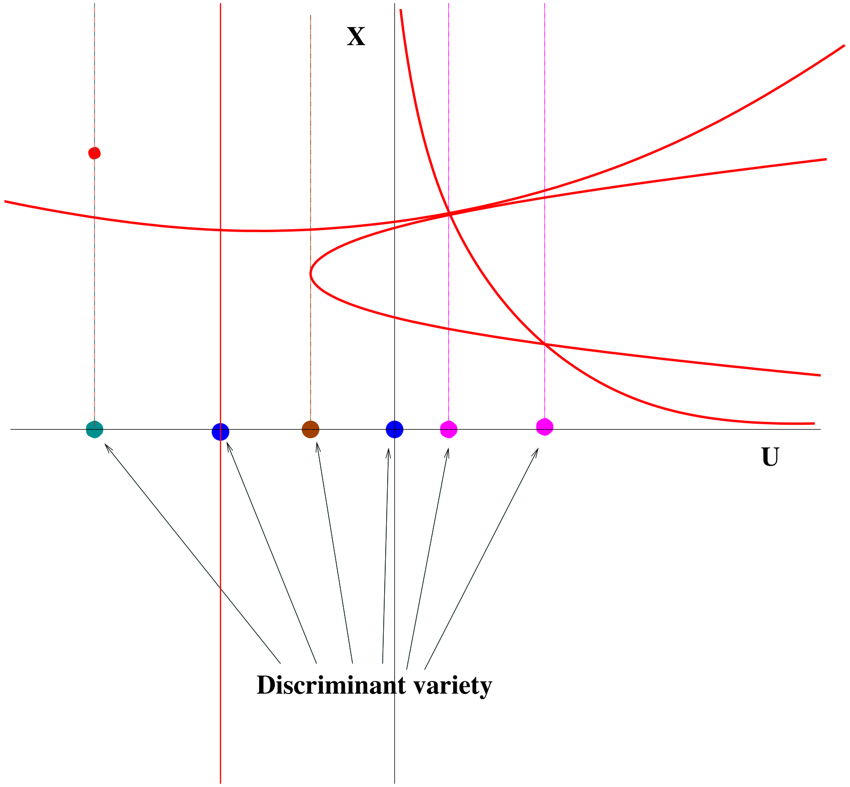
- $\#\mathcal{E} = n - d$;
- $\overline{\Pi_U(V(\langle\mathcal{E}\rangle))} = \mathbb{C}^d = \overline{\Pi_U(\mathcal{C})}$;
- $\langle\mathcal{E}_{|U=u}\rangle \subset \mathbb{C}[X]$ is radical and zero-dimensional for allmost all $u \in \mathbb{C}^d$.

We suppose from now that the systems are well behaved (this can be checked during the computations).

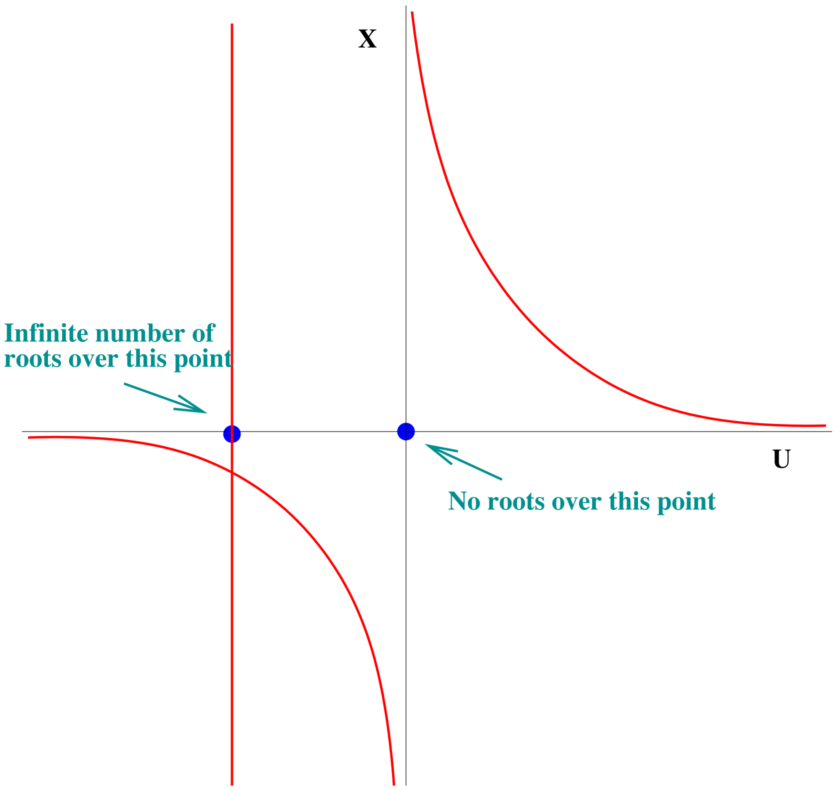
The study of $\mathcal{C} = \{x \in \mathbb{C}^n, p = 0, f \neq 0, p \in \mathcal{E}, f \in \mathcal{F}\}$

If one wants (at least) to discuss the number of roots, one needs to characterize parameter's subsets $\mathcal{U} \subset \Pi_U(\mathcal{C})$ st $\#\left(\Pi_U^{-1}(u) \cap \mathcal{C}\right)$ is constant on $\mathcal{U}$.

"bad" parameters $(\mathcal{U} \cap \{\text{bad parameters}\} = \emptyset)$:



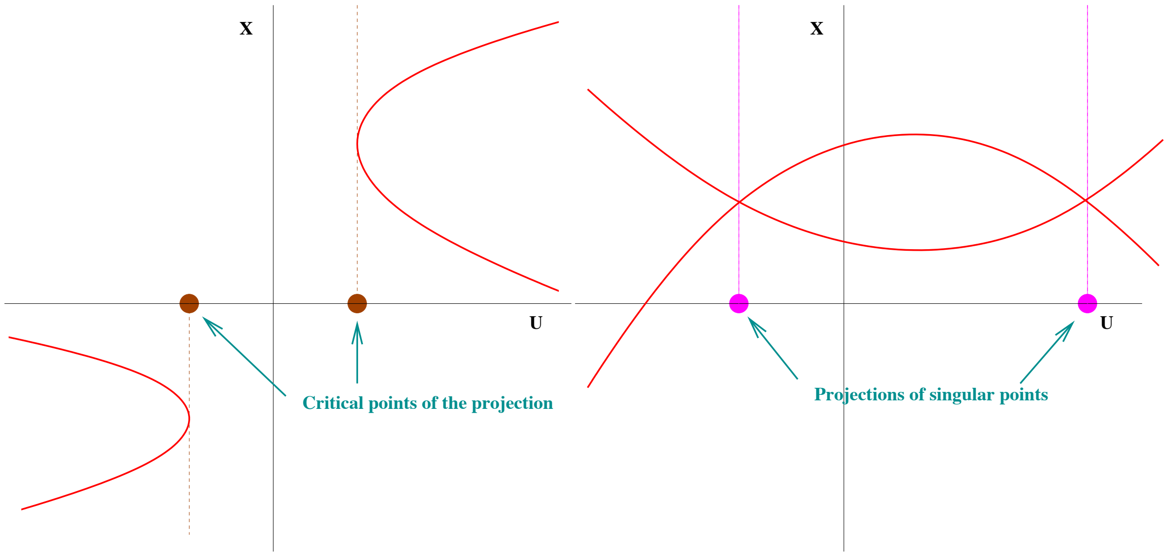
Points “going” to infinity



$O_\infty = \{u \in \mathbb{C}^d$, for any compact neighborhood $\mathcal{V} \ni u$, $\Pi_U^{-1}(\mathcal{V}) \cap \mathcal{C}$ is not compact$\}$.

$\mathcal{U}$ can not intersect properly O_∞ : $\mathcal{U} \cap O_\infty = \emptyset$ or $\mathcal{U}$

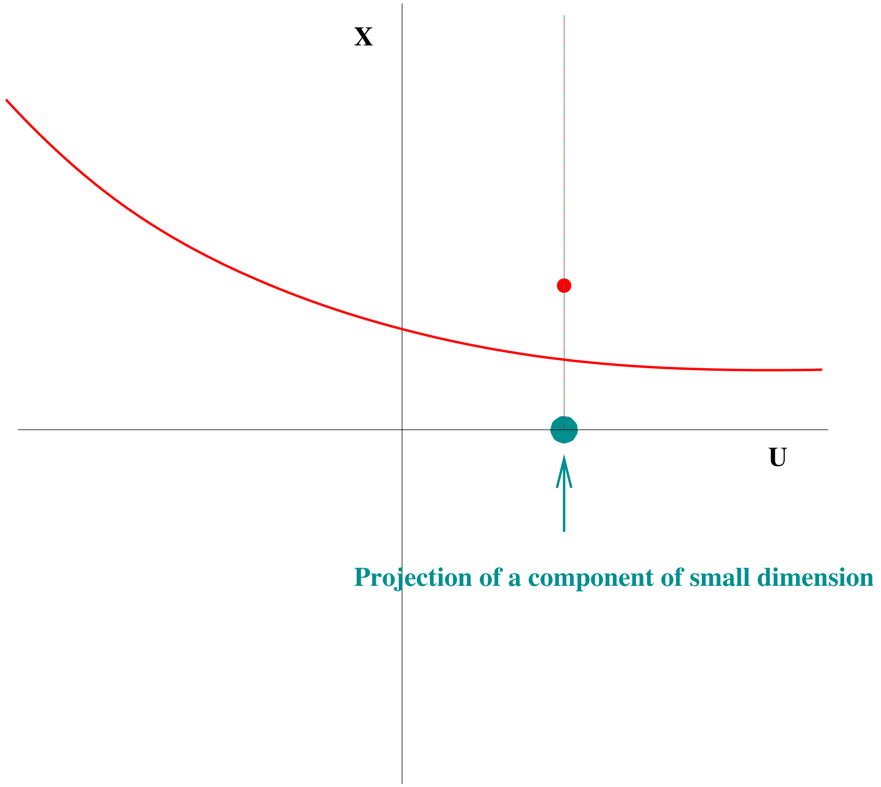
Critical values of the projection and projection of singular points :


X
U
Critical points of the projection
X
U
Projections of singular points

$O_c = \{\text{critical values of } \Pi_U \text{ on } \mathrm{Reg}(\mathcal{C})\} \cup \{\text{projections of singular points of } \mathcal{C}\}$

$\mathcal{U}$ can not intersect O_c : $\mathcal{U} \cap O_c = \emptyset$

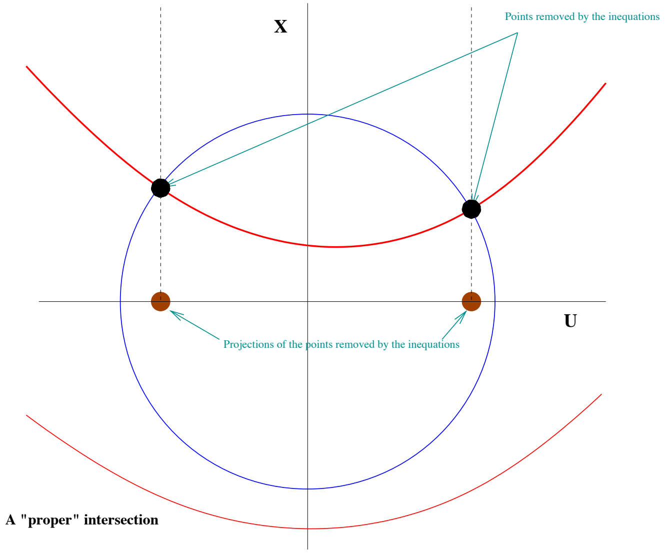
Components of small dimension

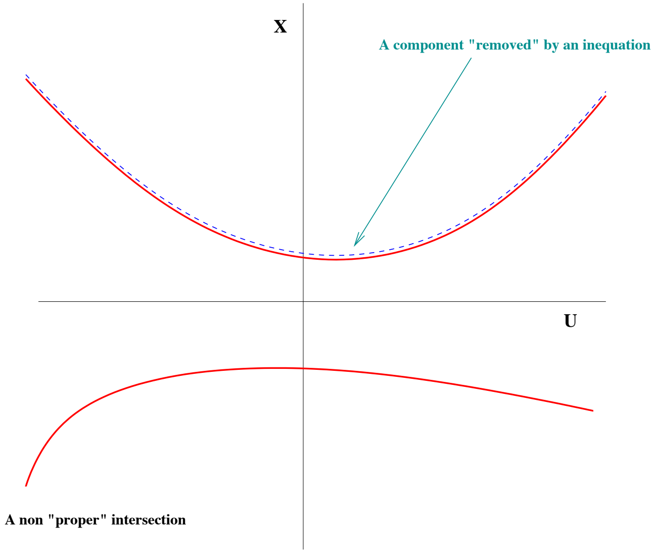


$O_{\text{sd}} = \{$projection - by Π_U - of the components of ”small” dimension$\}$

$\mathcal{U}$ can not intersect O_{sd} : $\mathcal{U} \cap O_{\text{sd}} = \emptyset$

Inequations !


X
Points removed by the inequations
U
Projections of the points removed by the inequations
A "proper" intersection


X
A component "removed" by an inequation
U
A non "proper" intersection

$$\text{Study}\ \bar{\mathcal{C}} = V\left(\langle\mathcal{E}\rangle : \left(\prod_{f\in\mathcal{F}} f\right)^{\infty}\right) = \overline{V(\langle\mathcal{E}\rangle)\setminus V\left(\prod_{f\in\mathcal{F}} f\right)}.$$

$$O_{\mathcal{F}} = \Pi_U\left(V\left(\prod_{f\in\mathcal{F}} f\right)\cap\bar{\mathcal{C}}\right)$$

$\mathcal{U}$ can not intersect properly O_f : $\mathcal{U}\cap O_{\mathcal{F}}=\emptyset$ or $\mathcal{U}$

Solving ???

Summary : if $\mathcal{U}$ is s.t. $u \longrightarrow \#\Pi_U^{-1}(u)$ is constant on $\mathcal{U}$, then $\mathcal{U}$ can not intersect properly $O_\infty \cup O_c \cup O_{\text{sd}} \cup O_\mathcal{F}$ with

$O_\infty = \{u \in \overline{\Pi_\mathcal{U}(\bar{\mathcal{C}})}$, for any compact neighborhood $\mathcal{V} \ni u$, $\Pi_U^{-1}(\mathcal{V}) \cap \bar{\mathcal{C}}$ is not compact$\}$

$O_c = \{$critical values of $\Pi_U\} \cup \{$singular points of $\mathcal{C}\}$

$O_{\text{sd}}=$ projection of the components of $\mathcal{C}$ of dimension less than $\dim(\Pi_U(\mathcal{C}))$

$O_\mathcal{F}=\Pi_U\Big(\bar{\mathcal{C}} \cap V\Big(\prod_{f \in \mathcal{F}} f\Big)\Big)$

Proposition [Lazard,Rouillier 2007] : If $\mathcal{U} \subset \Pi_U(\mathcal{C})$ is any submanifold which does not meet $O_\infty \cup O_c \cup O_{\text{sd}} \cup O_\mathcal{F}$, then $\Pi_U\colon \mathcal{C} \cap \Pi_U^{-1}(\mathcal{U}) \longrightarrow \mathcal{U}$ is a (analytic) covering.

In particular, the number of roots of $\mathcal{C}$ is constant over $\mathcal{U}$ and we have a "simple" description of $\mathcal{C}$ over $\mathcal{U}$.

$\Rightarrow$ Definition of "solving" a parametric system independently from any computational strategy.

Complex Discriminant Varieties

Definition and theorem [Lazard, Rouillier 2007] : $W_D = O_\infty \cup O_c \cup O_{\mathrm{sd}} \cup O_{\mathcal{F}}$ is an algebraic variety named **the** minimal discriminant variety of $\mathcal{C}$ w.r.t. Π_U.

Definition 1. *An algebraic variety W is a (large) discriminant variety of $\mathcal{C}$ w.r.t. Π_U iff:*

- $W_D \subset W \subsetneqq \overline{\Pi_U(\mathcal{C})}$
- $W = \overline{\Pi_U(\mathcal{C})}$ *iff* $\mathcal{C}_{|U=u}$ *is infinite or empty for almost all* $u \in \overline{\Pi_U(\mathcal{C})}$;

A D.V. is an algebraic variety W such that :

- $\Pi_U(\mathcal{C}) \setminus W = \cup_{i=1}^{k} \mathcal{U}_i$ is a finite union of submanifolds of dimension $\dim\left(\overline{\Pi_U(\mathcal{C})}\right)$.
- $\Pi_U \colon \Pi_{\mathcal{U}}^{-1}(\mathcal{U}_i) \cap \mathcal{C} \longrightarrow \mathcal{U}_i$ is a (analytic) cover $\forall i$.

Discriminant Varieties in the Real case

If W_D is the minimal discriminant variety for $\mathcal{C}$ wrt Π_U, then either $W_D \cap \mathbb{R}^d$ is a (non necessarilly minimal) discriminant variety for $\mathcal{S}$ wrt Π_U or W_D contains $\Pi_U(\mathcal{S})$.

In the second case, we simply replace $\mathcal{S}$ by $\mathcal{S} \cap W_D$ and compute again (the dimension of the projection then decreases).

Note that this correspond tho the case where the dimension of the real counterpart of the main components (those of dimension δ whose projection is not contained in W_D) differ from the "complex" dimension.

To detect this : $\mathcal{S}_{|U=u}$ has no solutions $\forall u \in \overline{\Pi_U(\mathcal{C})} \cap \mathbb{R}^d \setminus W_D$

The "real" version of the minimal discriminant variety is a semi-algebraic set.

Discriminant Varieties in the Real case

Over each connected component of $\Pi_U(\mathcal{S}) \setminus W_D$:

- the number of real roots is constant;
- the sheets are locally diffeomorphic to the connected components;

For “solving“ the initial problem, one needs to describe the connected components of $\Pi_U(\mathcal{C}) \cap \mathbb{R}^d \setminus W_D$ (we “eliminated“ $n - d$ variables).

- Compute one point on each C.C. + solving a zero-dimensional system : qualitative information.
- Compute a Cylindrical Algebraic Decomposition adapted to the polynomials defining the discriminant variety : full description.
- In practice : we use a “partial” CAD - avoid most projections as well as computations with algebraic numbers. In short : do not decompose W_D.

[optional] : For a full description : apply the algorithm on $\mathcal{S} \cap W_D$.

$\overline{O_{\mathcal{F}}}$

- $\langle\mathcal{E}\rangle : (\prod_{f\in\mathcal{F}} f)^{\infty} = \left(\langle\mathcal{E}\rangle + \langle T\left(\prod_{f\in\mathcal{F}} f\right) - 1\rangle\right) \cap \mathbb{Q}[U, X]$

- $\bar{\mathcal{C}} = V\left(\langle\mathcal{E}\rangle : (\prod_{f\in\mathcal{F}} f)^{\infty}\right) = \overline{V(\langle\mathcal{E}\rangle) \setminus (\cup_{f\in\mathcal{F}} V(\langle f\rangle))}$

Known result : If G is a Gröbner basis for any product of orderings $<_{U,X} = (<_U, <_X)$ with $U_i <_{U,X} X_j, \forall i, j$, then $G \cap \mathbb{Q}[U]$ is a Gröbner basis for $<_U$ of $\langle G\rangle \cap \mathbb{Q}[U]$.

In particular $\overline{\Pi_U(V(G))} = V(\langle G\rangle \cap \mathbb{Q}[U])$ so that we can "efficiently" compute an ideal $I_{\mathcal{F}}$ such that $V(I_{\mathcal{F}}) = \overline{O_{\mathcal{F}}}$.

O_∞

G a Gröbner basis of $\langle\mathcal{E}\rangle$ wrt a DRL-block ordering $<_{U,X}$.

Theorem 2. *if* $\mathcal{E}_0 = G \bigcap \mathbb{Q}[U]$. *Then:*

- $\mathcal{E}_0$ *is a Gröbner basis of* $I \bigcap \mathbb{Q}[U]$ *w.r.t.* $<_U$ *;*
- *Set* $\mathcal{E}_i^\infty = \mathcal{E}_0 \cup \mathcal{E}_i^\infty$ *for* $i = d+1 \ldots n$
- $\mathcal{E}_i^\infty$ *is a Gröbner basis of some ideal* $I_i^\infty \subset \mathbb{Q}[U]$ *w.r.t.* $<_U$ *;*
- $W_\infty = \bigcup_{i=d+1}^{n} V(I_i^\infty)$

Nothing to “compute“ when G is known !

Remark : valid for any parametric systems

Computing $\overline{O_c}$ and $\overline{O_{\mathrm{sd}}}$

The main computational problems :

- Jacobian criteria are independant from the equations in the case of radical and equi-dimensional ideals.

  In such cases $\overline{O_c} = V\Big(\Big(\langle\mathcal{E}\rangle + \mathrm{Jac}_X^{n-\delta}(\mathcal{E})\Big) \cap \mathbb{Q}[U]\Big)$.

- In the general case, $V\Big(\Big(\langle\mathcal{E}\rangle + \mathrm{Jac}_X^{n-\delta}(\mathcal{E})\Big) \cap \mathbb{Q}[U]\Big)$

  → may give too much points (non radical ideals, embeeded components)

  → may miss some points (components of small dimension)

  → may be of same dimension than $\overline{\Pi_U(\mathcal{C})}$ (non radical ideals)

- We want to avoid as most as possible to compute a decomposition of the ideal into radical and/or equi-dimensional components (avoid also primary decompositions)

In the case of well-behaved systems

One can not suppose, even in practice, that $\langle\mathcal{E}\rangle$ is radical or equidimensional

artefacts from modelizations (from fractions to polynomials, changes of coordinates like $t = \tan(\alpha/2)$, etc.) often introduce primary but not prime components of arbitrary dimensions.

BUT in the case of well-behaved systems :

- the components of dimension $< n-d$ are ”embeeded” components since $\#\mathcal{E} = n-d$ (in particular $O_{\mathrm{sd}} = \emptyset$).
- the projection of the zero set of components of dimension $> n-d$ are in O_∞

Theorem 3. *(Well-behaved systems- Lazard,Rouillier 2007)*

$O_{\mathrm{sd}} = \emptyset$ *and :*

$$W_D = O_\infty \cup \overline{O_{\mathcal{F}}} \cup V\Big(\Big(\langle\mathcal{E}\rangle + \mathrm{Jac}_X^{n-\delta}(\mathcal{E})\Big) \cap \mathbb{Q}[U]\Big) = O_\infty \cup O_{\mathcal{F}} \cup O_c$$

Discriminant Varieties

By computing few Gröbner bases for a well chosen monomial block ordering, one gets a Discriminant Variety that defines a partition of the parameter's space into "cells".

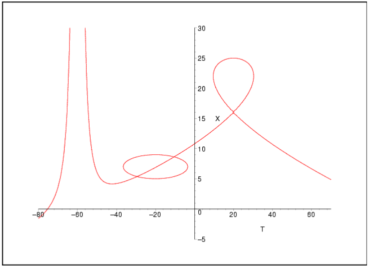
Over each cell, the number of complex (resp. real) roots is constant and, the "leafs" of solution never meet.

We have "eliminated" all the indeterminates without loosing important information.

Now, we have to describe this partition in practice ....

Robotic Singularities and Jacobien

$$f_1 = 0, \ldots, f_k = 0$$
$$\subset$$
$$\mathbb{Q}[T_1, \ldots, T_m, X_1, \ldots, X_n]$$



Parallel Singularities

$$\textit{Maximal minors of} \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial X_1} & \cdots & \frac{\partial f_k}{\partial X_n} \end{pmatrix}$$

Robotic Singularities and Jacobien

$$f_1 = 0, \ldots, f_k = 0$$
$$\subset$$
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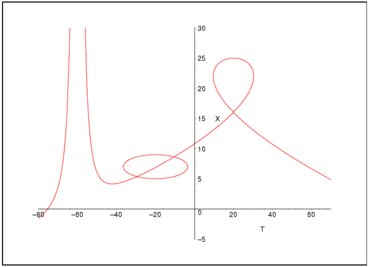
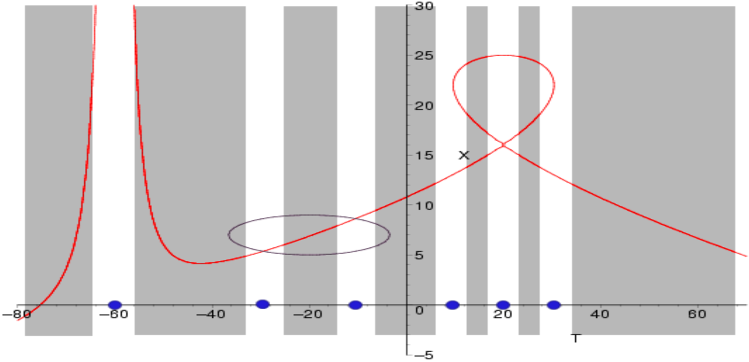


Parallel Singularities

$$\textit{Maximal minors of} \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial X_1} & \cdots & \frac{\partial f_k}{\partial X_n} \end{pmatrix}$$

Discriminant Variety

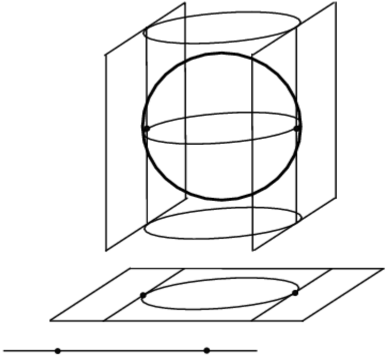


Discriminant Variety V of a parametric system S

- V : variety in the parameter space
- Path not crossing V $\Longrightarrow$ constant number of solutions for S

CAD (Cylindrical Algebraic Decomposition)

CAD adapted to a polynomial P in n variables



- Partition of $\mathbb{R}^n$ in cells
- The projections of 2 cell are either :
  - disjoint
  - equal
- The sign of P is constant on each cell

If P discriminant variety of S

Then S has a constant number of solutions on each n-dimensional cell

The Cylindrical Algebraic Decomposition

Given a set of polynomials $p_1, ..., p_r$ in $\mathbb{R}[U_1, ..., U_d]$, decompose $\mathbb{R}^d$ in "cells" where the polynomials have all a constant sign.

Remark : it computes more than required, but this "old" algorithm (Collins 1975) is unfortunately the only one able to provide us a description of the partition of $\mathbb{R}^d$ defined by a discriminant variety.

As we are only interested by the cells of maximal dimension, the CAD can be optimized and simplified.

The Cylindrical Algebraic Decomposition

For each polynomial : $f \in \mathcal{S}_1 \subset \mathbb{Q}[X_2, \ldots, X_n][X_1]$, find conditions over $[X_2, \ldots, X_n]$ such that the number of real roots may change :

the projection of points at infinity (zeroes of $\mathrm{LC}(f, X_1)$)

the critical values of the projection w.r.t. X_1 (discriminant of f w.r.t X_1)

For each couple $(f, g) \in \mathcal{S}_1 \subset \mathbb{Q}[X_1][X_2, \ldots, X_n]$ compute the projection of the intersection $V(f) \cap V(g)$: resultant(f, g) w.r.t. X_1.

$\Rightarrow$ this generates a set $\mathcal{S}_2$ of polynomials in $\mathbb{Q}[X_2, \ldots, X_n]$

Then apply the same projection recursively to $\mathcal{S}_2, \mathcal{S}_3, \ldots, \mathcal{S}_{n-1}$.

At the end of the projection step, you have :

- $\mathcal{S}_i \subset \mathbb{Q}[X_i, \ldots, X_n]$ induces a partition of $\mathbb{R}^i$ if we consider $V(\mathcal{S}_i)$ and the union of cells (simply connected components) that do not meet any $V(f)$, $f \in \mathcal{S}_i$.

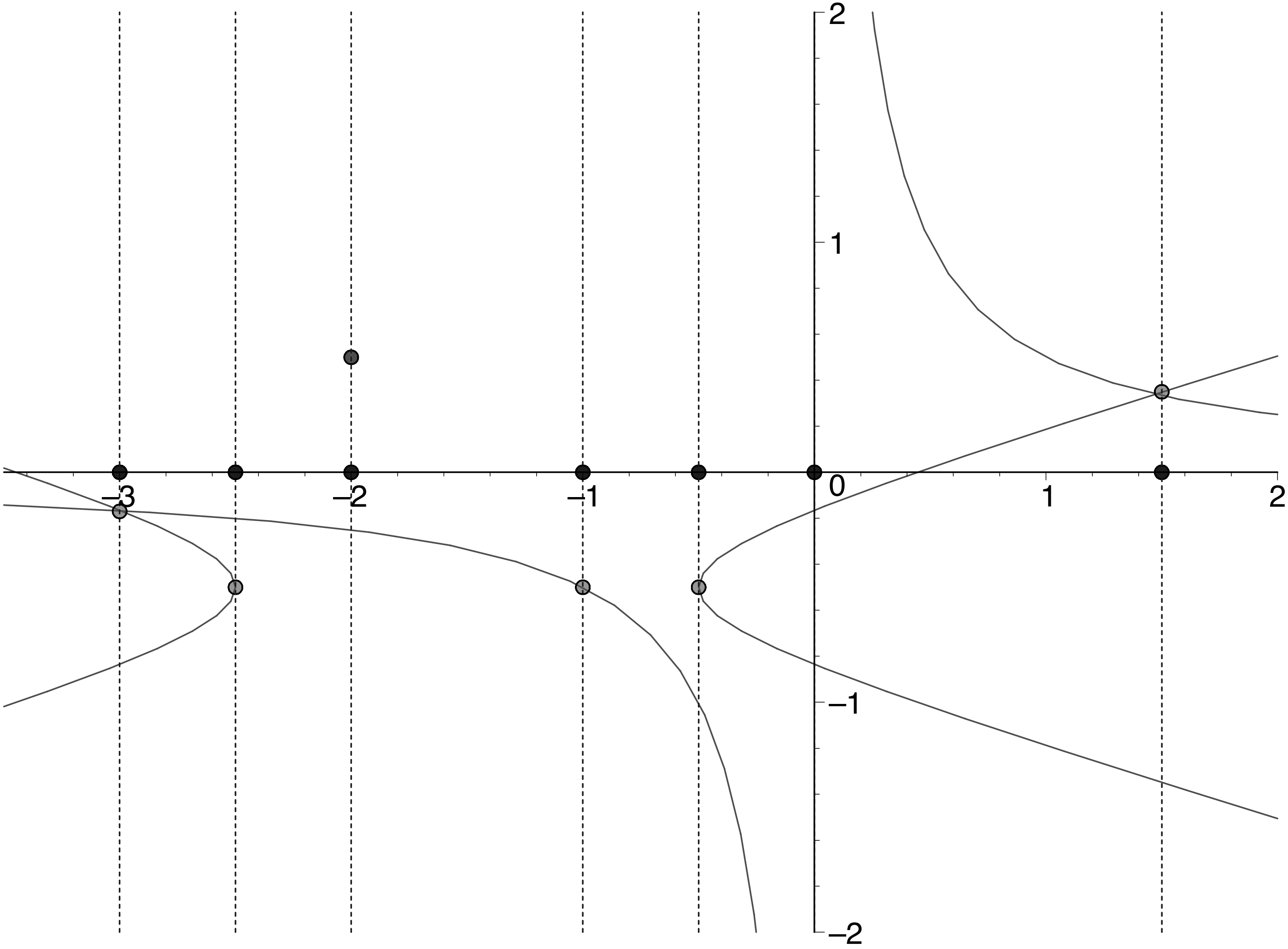
- Over each element of the above partition, the polynomials of S_{i-1} have a constant sign.

CAD : the lifting step

Using the CAD, the cells are described recursively by the polynomials sets $\mathcal{S}_i$, and the CAD computes one point on each cell by the following process : by specializing the variables by the coordinates of the simple points, one then computes the sign condition described by the cell.

Start with $S_n \subset \mathbb{Q}[X_n]$: the cells of $\mathbb{R}$ “adapted“ to $\mathcal{S}_{n-1}$ are the points of $V(S_n)$ and the intervals between them. We define the set of sample points as $V(S_n)$ and one point in each interval.

For each sample point, we specialize the X_n coordinate of the polynomials of $\mathcal{S}_{n-1}$, and do the same : we then obtain sample points in $\mathbb{R}^2$.



2
1
−1
−2
−3
−2
−1
0
1
2

Conclusion

Computer algebra $\Rightarrow$ exact computations $=>$ exhaustive classifications, certifications of results, etc.

BUT

- global resolution and thus strong limitations in terms of degree/number of varaibles.

- adding constraints and/or restrictions such as inequations makes the problem more complicated in general

- Many black boxes / objects to consider for having a full algorithm, and many possible ..... mistakes