Singularities of parallel manipulators from the viewpoint of differential geometry

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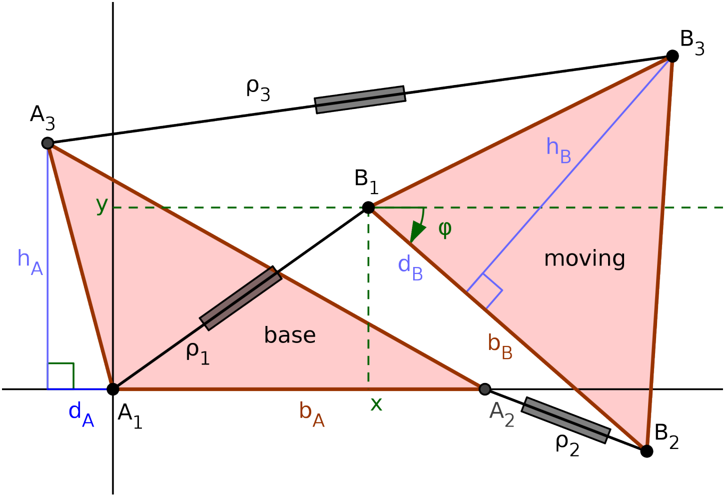
Summary

Planar parallel 3-RPR manipulators



Spaces and mapping

- *Workspace*: space of planar motions (3 degrees of freedom). Elements: poses. Coordinates : φ, x, y.
- Actuated *joint space*: lengths of the 3 legs ρ_1, ρ_2, ρ_3.
- *Inverse Kinematic Mapping*: Workspace $\longrightarrow$ Joint space.

$$
\begin{aligned}
\rho_1 &= \left(x^2 + y^2\right)^{1/2} \\
\rho_2 &= \left((x + b_B \cos\varphi - b_A)^2 + (y + b_B \sin\varphi)^2\right)^{1/2} \\
\rho_3 &= \Big((x + d_B \cos\varphi - h_B \sin\varphi - d_A)^2 \\
&\qquad + (y + d_B \sin\varphi + h_B \cos\varphi - h_A)^2\Big)^{1/2}
\end{aligned}
$$

- *Direct Kinematic Problem*: given ρ_1, ρ_2, ρ_3, solve for φ, x, y. This is a sixth degree problem: up to six solutions.

Jacobian of the IKM and singularities

- *Jacobian determinant* of the IKM: 3×3 determinant Jac with i-th row $\frac{\partial \rho_i}{\partial \varphi} \quad \frac{\partial \rho_i}{\partial x} \quad \frac{\partial \rho_i}{\partial y}$
- The IKM can be locally inverted (meaning: a solution (φ_0, x_0, y_0) of the DKP can be smoothly followed under small modifications of ρ_1, ρ_2, ρ_3) if and only if $\mathrm{Jac}(\varphi_0, x_0, y_0) \neq 0$.
- *Singular poses*: (φ, x, y) such that $\mathrm{Jac}(\varphi, x, y) = 0$. They form the singularity surface in the workspace. Image by IKM: image singularity surface in the joint space.
- *Aspect*: connected component of the complement of the singularity surface in the workspace. Any two poses in the same aspect can be joined by a continuous path avoiding singularities (the image of this path may cross the image singularity surface).

Framework

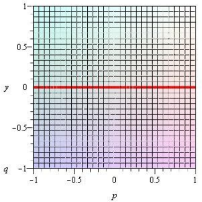
- A smooth mapping $f : M \to N$ (as the IKM from the workspace to the joint space).
- Restrict to dimension 2: M a neigborhood of $(0,0)$ in $\mathbb{R}^2$, with coordinates (x,y), N a neighborhood of $(0,0)$ in $\mathbb{R}^2$ with coordinates (p,q), $f(x,y) = (p(x,y), q(x,y))$.
- $f(0,0) = (0,0)$ and $(0,0)$ is singular point of f:
$$\mathrm{Jac}(f)(0,0) = \begin{vmatrix} \partial p/\partial x(0,0) & \partial p/\partial y(0,0) \\ \partial q/\partial x(0,0) & \partial q/\partial y(0,0) \end{vmatrix} = 0$$
- **Fact**: there are two types of stable singularities (persistent under small perturbation of f): *folds* (codimension 1) and *cusps* (codimension 2.

Folds

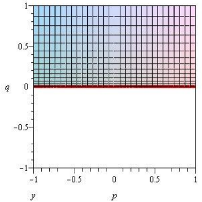
- Normal form (up to smooth coordinates change):
  $$\begin{cases} p = x \\ q = y^2 \end{cases}$$
- Jacobian: $2y$.
- Image singularity curve: $q = 0$
- Codimension 1: curve in dimension 2, surface in dimension 3
- Double solution to $f(x, y) = (0, 0)$. Two solutions for $q > 0$, zero for $q < 0$

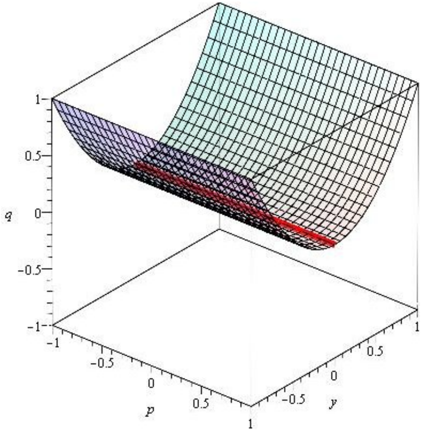
Views of a fold

- In the (x, y) plane



- In the (p, q) plane



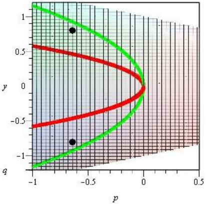


Cusps

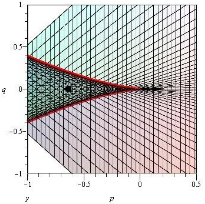
- Normal form: $\begin{cases} p = x \\ q = y^3 + xy \end{cases}$
- Jacobian: $3y^2 + x$. Smooth singularity curve (parabola).
- Image singularity curve: $4p^3 + 27q^2 = 0$.
- The cusp is at origin: codimension 2. Point in dimension 2, curve (cusp edge) in dimension 3.
- Triple solution to $f(x, y) = (0, 0)$. One solution for $4p^3 + 27q^2 > 0$, three for $4p^3 + 27q^2 < 0$; one can pass smoothly from the lowest solution to the highest by circling around the cusp in the (p, q)-plane.
- Cusp is stationary point on the image singularity curve.

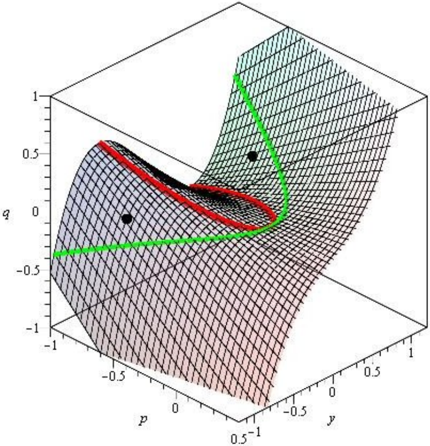
Views of a cusp

- In the (x, y) plane

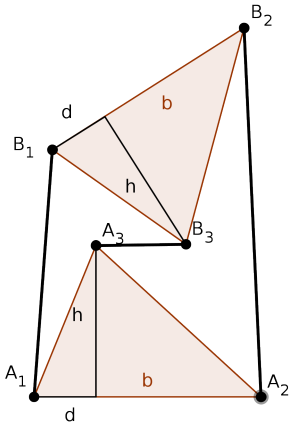


- In the (p, q) plane



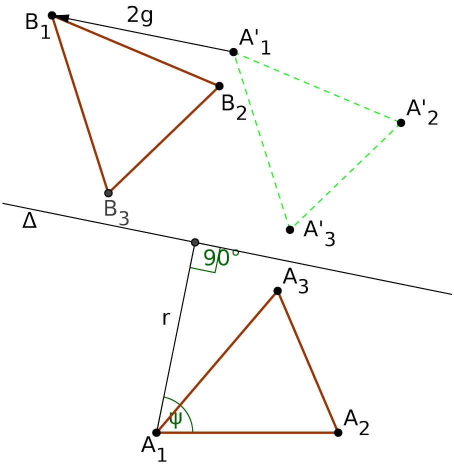


Symmetric 3-RPR



- *Geometry*: the base triangle and moving triangle are congruent modulo an indirect isometry (*glide reflexion*).
- *Analytic nature*: the 6th-degree DKP is split into a 3rd-degree problem followed by a 2nd-degree problem.

Alternative coordinates in the workspace



Coordinates (ψ, r, g) given by the glide reflexion sending $A_1A_2A_3$ to $B_1B_2B_3$:

- (ψ, r) for the reflexion axis,
- $g \in \mathbb{R}$ for the glide.

$(\psi + \pi, r, g)$ is identified with $(\psi, -r, -g)$.

Change of coordinates:

$$\begin{aligned} \varphi &= 2\psi + \pi \pmod{2\pi} \\ x &= 2\,(r\cos(\psi) - g\sin(\psi)) \\ y &= 2\,(r\sin(\psi) + g\cos(\psi)) \end{aligned}$$

Coordinates match the splitting of the DKP

- The IKM now writes

$$\begin{array}{rcl} \rho_1^2 & = & 4\left(r^2+g^2\right) \ , \\ \rho_2^2 & = & 4\left((b\,\cos(\psi)-r)^2+g^2\right) \ , \\ \rho_3^2 & = & 4\left((d\,\cos(\psi)+h\sin(\psi)-r)^2+g^2\right) \ . \end{array}$$

- Setting $\delta_2=(\rho_2^2-\rho_1^2)/4,\ \ \delta_3=(\rho_3^2-\rho_1^2)/4$, we get

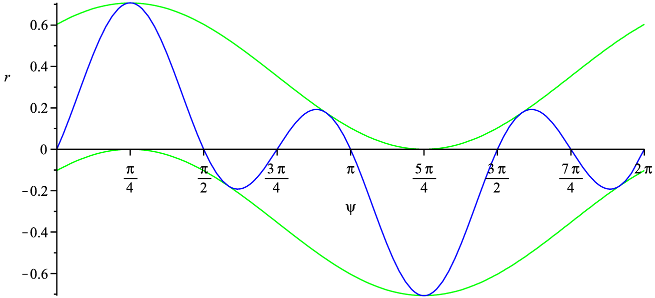
$$\left\{ \begin{array}{rcl} \delta_2 & = & -2\,b\cos(\psi)\,r+b^2\,(\cos(\psi))^2 \ , \\ \delta_3 & = & (-2\,d\cos(\psi)-2\,h\sin(\psi))\,r+(d\cos(\psi)+h\sin(\psi))^2 \ . \end{array} \right. \tag{1}$$

- System (1) enables solving for the *reflexion* (ψ, r) in terms of (δ_2,δ_3). By elimination of r, 3rd-degree equation in $t=\tan\psi$.
- 2nd-degree step: solving for the *glide* g $\rho_1^2=4\left(r^2+g^2\right)$.

Singularity curve of the 3rd-degree step

The jacobian of $(\psi, r) \to (\delta_2, \delta_3)$ vanishes on the singularity curve

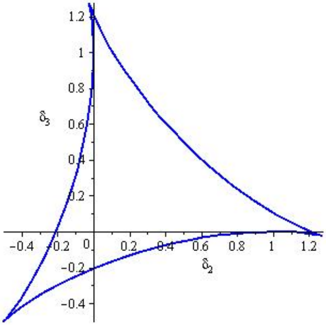
$$r = \cos\psi \left(\frac{h^2 + bd - d^2}{h} \cos\psi \sin\psi + (2d - b)\cos^2\psi + (b - d) \right)$$



(Figure for $b = h = 1$, $d = 0$, right isosceles triangle)

Image singularity curve

- Image singularity curve of the 3rd-degree step (solving for the reflexion): a deltoid with three cusps in the (δ_2, δ_3) plane.
- 3 solutions in (ψ, r) inside, 1 outside, triple solution at cusps, double along the arcs.
- Each of the 3 solutions inside can be smoothly continued to the unique solution outside through exactly one arc of the deltoid. Non-singular changing of solution possible by circling around cusps.



Singularity surface for the IKM

The singularity surface splits in two:

- a cylinder with generatrix parallel to the g-axis for the 3rd-degree step (black lines give cusp edges in the workspace);
- the plane $g = 0$ for the 2nd-degree step.

Two aspects, symmetric w.r.t. $g = 0$

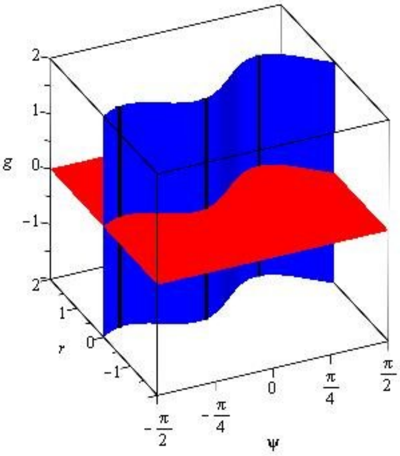


Image singularity surface in the joint space

Cuts at $\rho_1^2 = 1$ and 4 in the $(\rho_1^2, \rho_2^2, \rho_3^2)$-space. The component for the 3rd-degree step is a part of a cylinder with base the deltoid and generatrix parallel to $(1, 1, 1)$ and has 3 cusp edges.