Global stabilization of n-dimensional population models by a positive control

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Abstract

We consider the n-dimensional Lotka-Volterra differential system, describing the interactions between n species and the individual growth rates of each species. We suppose that there is a single positive control on these growth (or death) rates. We obtain in some cases a global stabilizing feedback around a reference equilibrium, that keeps the control positive.

1 Introduction

To describe the interaction between n species of ecological systems, the modellers often use the well-known Lotka-Volterra (LV) model:

$$\dot{x}_i = x_i\ (b_i + \sum_{j=1}^{n} A_{ij} x_j)\ \ (i = 1, \ldots, n) \qquad (1)$$

The $n \times n$ real matrix A describes the quadratic interactions, and is usually supposed to be bijective (the positive equilibrium will be unique if it exists). The vector b describes the individual growth rate of each species. A priori, the b_i can be positive (growth rate) or negative (death rate). The variables x_i, standing for populations, are supposed real non-negative. There are many results on these systems (see [4, 3]).

Let us suppose now that this system is subject to a control on the growth rates (the vector b). We can imagine, for example, that the growth rate of each species depends on one single factor that could be the lighting, the temperature, or the pollution of the ambient space .... The new system is now :

$$\dot{x}_i = x_i\ (ub_i + \sum_{j=1}^{n} A_{ij} x_j)\ \ (i = 1, \ldots, n) \qquad (2)$$

In this system the scalar control u is supposed to be positive ; it means that the external factor cannot change, for example, a death rate into a growth rate, which is quite realistic. The mathematical problem that we want to address here is the following:
" Given a reference equilibrium x^*, u^*, which turns out to be unstable, is it possible to stabilize the system around this equilibrium ?"
Of course, we have the constraint that u is positive ($(u - u^*)$ is bounded from above); we will also search for a smooth feedback, which is more realistic if we consider the interpretation of u as an external factor. Moreover, we demand that the equilibrium is stable in the whole space of positive variables. In general, the control or stabilization of systems by positive or bounded controls is a difficult problem ([1]), and our answers are very partial. In some cases, we will be able to obtain an explicit smooth positive feedback $u = \phi(x)$ which stabilizes globally the system.

2 Stabilization

The idea to stabilize the system is simple : we consider u as a new variable, and we choose its dynamic in order to verify the constraint of positivity and to stabilize the new system with the help of something like a Lyapunov function. In fact, we will use a criterion for global stability of LV systems developed in [2]. The equation for u is chosen of the following form :

$$\dot{u} = u(\sum_{i=1}^{n} c_i x_i + du)$$

where the coefficients c_i, d, and the initial condition $u(0)$ have to be chosen. The equilibrium gives us that

$$\sum_{i=1}^{n} c_i x_i^* + du^* = 0$$

Let us remark that u will stay positive if the initial control is positive, because the hyperplane $u = 0$ is

invariant by this equation.
The new state vector is now $y=(x,u)$ and is a solution of the system :

$$\dot{y}_i = y_i\,(\sum_{j=1}^{n+1} A'_{ij} y_j) \;\; (i=1,\dots,n+1) \qquad (3)$$

where $A' = \begin{pmatrix} A & b \\ {}^t c & d \end{pmatrix}$. We are interested in this system for $y \geq 0$ (because x_i and u have to be positive); the faces $y_i=0$ of the orthant are invariant, and so is the open positive orthant. Let us remark that this system is quadratic homogeneous and analogous to a LV system but without the linear terms. We scale the equilibrium (x^*, u^*) to $(1,1,\dots,1)$ and keep the same notation. Then we can apply a criterion of global stability for the above system ([2]), which uses properties of cooperative systems ([5]):

Proposition 1 *If the system (3) is such that* $\sum_{j=1}^{n+1} \min_{i\neq j} A'_{ij} > 0$ *, then the positive equilibrium is globally asymptotically stable.*

We obtain:

Theorem 1 *If the system (2) is such that*

$$\sum_{j=1}^{n} \min_{i \neq j} A_{ij} + \min_i b_i > 0$$

then there exists u positive stabilizing globally (2).

Indeed, we can choose $c_j \geq \min_{i\neq j} A_{ij}$, and the resulting system will verify the above proposition.
In fact, we can even obtain an explicit formulation of the feedback, because the system (3) has a first integral ${}^t q \ln y = C$, where q belongs to the kernel of ${}^t A'$. Finally $C=0$ because when $x=(1,\dots,1)$, then $u=1$. We obtain the following explicit form:

Corollary 1 *A smooth positive stabilizing feedback for the original system (2) is:*

$$u = u^* \prod_{i=1}^{n} (\frac{x_i}{x_i^*})^{-q_i/q_{n+1}}$$

where q belongs to the kernel of ${}^t A'$.

3 Example

Let us consider the following three-dimensional system:

$$\begin{cases} \dot{x} = x(\frac{4}{3}x - z) - \frac{1}{3}u \\ \dot{y} = y(3x - \frac{5}{2}y - z) + \frac{1}{2}u \\ \dot{z} = z(3x - \frac{3}{2}y - \frac{5}{2}z) + u \end{cases}$$

For the reference control $u=1$, the system has the equilibrium $x=y=z=1$, which is unstable: indeed, the jacobian matrix at the equilibrium is nothing but A, which has a positive eigenvalue. We add the following equation for u :

$$\dot{u} = u(\alpha x + \beta y + \chi z + \delta u)$$

If we write the whole matrix (3), the criterion will be verified if we can find the coefficients such that:

$$\min(\alpha, 3) + \min(\beta, -3/2) + \min(\chi, -1) - 1/3 > 0$$

We can choose $\alpha = 3, \beta = \chi = 0$. The coefficient δ is determined so that $u=1$ is an equilibrium. We obtain that :

$$\dot{u} = u(3x - 3u)$$

stabilizes globally the system for positive variables. The kernel of ${}^t A'$ is $(-171/40, -27/20, 9/4, 1)$, and the explicit form for the positive feedback is therefore :

$$u = x^{171/40} y^{27/20} z^{-9/4}.$$

4 Conclusion

To stabilize systems like (2) by a positive control, we can introduce the control as a new variable. The results will depend on our ability to predict the global stability of systems very similar to Lotka-Volterra systems. It would be interesting to examine if the known criteria of stability for LV systems (mainly based on Lyapunov functions) given in the literature (see [4]) can be useful for designing feedbacks.

References

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