A criterion of global convergence to equilibrium for differential systems with an application to Lotka-Volterra systems

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1 Introduction

The purpose of the present paper is to study the behavior, and especially the asymptotic behavior, of particular systems of non-linear ordinary differential equations that often arise in biological modelling. These systems are characterized by the fact that they can be written (in n-space):

$$\dot{x} = f(x) - g(x) \tag{1}$$

with the conditions, valid on an open set $W \subset \mathbf{R}^n$:

$$\frac{\partial f_i}{\partial x_j}(x) \geq 0 \text{ for } i \neq j \text{ and } \frac{\partial g_i}{\partial x_j}(x) \geq 0 \; (i, j = 1, \ldots, n) \tag{2}$$

The vector-function f is said to be quasi-monotone increasing and g monotone increasing (see [16]). If $g \equiv 0$, the system (1) is also called cooperative [4]. This last mathematical formulation has been used by Volterra and Lotka (see [15]) in connection with biological problems, and has been followed by many studies (see [7,8,9]). More recently, the study of cooperative systems has been accomplished ([4,5,6], see [12] for a review).

To our knowledge, there are few results on the general system (1) subject to condition (2). Roughly speaking, this system is the difference of two cooperative systems, in fact of a cooperative system and a system the jacobian matrix of which is nonnegative (it is stronger than cooperative). As we shall see in section 5, many differential systems can be written in this manner.

In this paper, we show that it is sometimes possible to describe in a simple way some of the global features of (1).We shall give sufficient conditions which ensure the existence of an equilibrium in a given region in $\mathbf{R}^n$, and a global convergence in this region towards the equilibrium. The key idea in our study of system (1) in $\mathbf{R}^n$ is to immerse it into a cooperative system in $\mathbf{R}^{2n}$, where it is possible to obtain an upper and a lower bound on the solution of (1). Of course the condition obtained is conservative and puts strong constraints on the system (1). We explain this in section 4.

In section 2, we write down the associate system in $\mathbf{R}^{2n}$ and study some of its properties. The main results on the existence of an equilibrium and the dynamic behavior of (1) are in section 3 and 4. In section 5, we apply these results to Lotka-Volterra systems.

Notations: For x in $\mathbf{R}^n$, we write $x > 0$ if $x_i > 0$ $(i = 1, \dots, n)$ and $x \geq 0$ if $x_i \geq 0$ $(i = 1, \dots, n)$. If $x \leq y$, let $[x, y] = \{z \; ; \; x \leq z \leq y\}$.The closed positive orthant is $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n; x \geq 0\}$. Let us denote by u^T the transpose of u, by e^x the vector $(e^{x_1}, \dots, e^{x_n})$, and similarly for $\ln x$.

If $V \subset \mathbf{R}^n$ is open, $h : V \to \mathbf{R}^n$ is C^1, and $x \in V$, we denote by $Dh(x)$ the jacobian $n \times n$ matrix $\partial h_i / \partial x_j(x)$.

If $V \subset \mathbf{R}^n$ is open, $h : V \to \mathbf{R}^n$ is C^1, and $x_0 \in V$, for the differential system $\dot{x} = h(x)$ ($\dot{x}$ is the derivative with respect to time t), we denote by $x(t, x_0)$ or sometimes by $x(t)$ the (maximally defined) solution in V with initial value x_0 for $t = 0$.

2 The associate system in $\mathbf{R}^{2n}$

We consider the system (1) for $x \in W$ with conditions (2) holding in W, and we suppose that W is an open subset of $\mathbf{R}^n$, and $f, g : W \to \mathbf{R}^n$ are C^1. We have therefore existence and unicity of a maximal solution of (1), and continuity with respect to the initial value.

To the system (1), we associate a new system in $\mathbf{R}^{2n}$:

$$\begin{cases} \dot{y} = f(y) - g(z) \\ \dot{z} = f(z) - g(y) \end{cases} \tag{3}$$

where $y \in \mathbf{R}^n$ and $z \in \mathbf{R}^n$, the initial value being (y_0, z_0). The system is defined for $(y, z) \in W \times W$ (one can find a similar idea, in the discrete context, in [2,10], see the remark at the end of section 4).

Lemma 1 *If x^* is an equilibrium of system (1), then (x^*, x^*) is an equilibrium for system (3). Moreover, the set $\{(y,z); y=z\}$ is invariant under the flow defined by (3).*

For we have $f(x^*) - g(x^*) = 0$ and if $y(t) = z(t)$, then in (3), $\dot{y}(t) = \dot{z}(t)$.

Lemma 2 *If $y_0 = z_0 = x_0$, then the solution $(y(t,(x_0,x_0)), z(t,(x_0,x_0)))$ of (3) is equal to $(x(t,x_0), x(t,x_0))$, where $x(t,x_0)$ is the solution of (1).*

For $y(t) = z(t)$ because of the preceding lemma, and $y(t, x_o)$ satisfies

$$\dot{y} = f(y) - g(y)$$

with $y_0 = x_0$.

Lemma 3 *By the change of variables:*

$$\begin{aligned} Y &= y \\ Z &= -z \end{aligned}$$

the system (3) is converted into the cooperative system (4).

$$\begin{cases} \dot{Y} = f(Y) - g(-Z) \\ \dot{Z} = g(Y) - f(-Z) \end{cases} \tag{4}$$

For the jacobian matrix of (4) is:

$$\begin{pmatrix} Df(Y) & Dg(-Z) \\ Dg(Y) & Df(-Z) \end{pmatrix}$$

Because of condition (2), the system is cooperative.

We need a basic result on cooperative systems, attributed to Kamke (see [5]):

Theorem: Let V an open convex set of $\mathbf{R}^n$, $h : V \to \mathbf{R}^n$ a C^1 quasi-monotone increasing map. If two solutions of the differential system $\dot{x} = h(x)$ are such that $x_0 \leq y_0$, then, for all $t \geq 0$ such that the solution is defined, $x(t) \leq y(t)$. This result is also valid with strict inequalities.

The next theorem gives bounds on the solution of (1):

Theorem 2.1 *Suppose W is convex. Let $x(t,x_0)$ be a solution of (1) and $(y(t,(y_0,z_0)),z(t,(y_0,z_0)))$ a solution of (3), and let $z_0 \le x_0 \le y_0$, then, for all $t \ge 0$, $z(t) \le x(t) \le y(t)$. This result is also valid with strict inequalities.*

Indeed, the inequalities $z_0 \le x_0 \le y_0$ are equivalent to:

$$\begin{pmatrix} x_0 \\ -x_0 \end{pmatrix} \le \begin{pmatrix} y_0 \\ -z_0 \end{pmatrix}$$

Because the system (4) is cooperative and W is convex (in fact p-convexity suffices, see [5]), we can apply Kamke's theorem to the preceding inequality between two initial values of (4). Then we use lemma 2 to obtain:

$$\begin{pmatrix} x(t) \\ -x(t) \end{pmatrix} \le \begin{pmatrix} y(t) \\ -z(t) \end{pmatrix}$$

which is the desired result.

Corollary 1 *The region $\{(y,z) \; ; \; y \ge z\}$ is positively invariant by (3).*

Corollary 2 *If the system (3) converges to (x^*,x^*), then the system (1) converges to x^*.*

3 Existence of an equilibrium

In this section, we will use the above results to obtain the existence of at least one stationary point of (1); we need some more geometrical hypotheses to construct an invariant region.

Theorem 3.2 *Assume there exists $(y_1,z_1) \in W \times W$ such that:*

$$\begin{aligned} z_1 &\le y_1 \\ f(y_1) &\le g(z_1) \\ f(z_1) &\ge g(y_1) \end{aligned}$$

Then the interval $[z_1,y_1]$ in W contains at least one equilibrium point of (1).

Proof. These inequalities mean that the vector field of (4) at $(y_1,-z_1)$ is nonpositive; because of the cooperativeness of (4), it implies that the orthant defined by $y \le y_1$ and $z \ge z_1$ is positively invariant for the system (3). Moreover, because of corollary 1, the region $z \le y$ is positively invariant.

Therefore, the region Q defined by the intersection of these two regions is also positively invariant. But by lemma 2, the edge of the region Q such that $y = z$ is also invariant by (3), and, because it is compact convex, there exists an equilibrium (x^*, x^*) in it, by a fixed point theorem (see e.g. [11]). The point x^* is therefore an equilibrium for (1).

Corollary 3 *The interval* $[z_1, y_1]$ *in* W *is positively invariant.*

4 Convergence towards the equilibrium

If, moreover, we have unicity of this equilibrium (x^*, x^*) for the system (3), we can obtain global convergence in the interval:

Theorem 4.3 *Assume there exists* $(y_1, z_1) \in W \times W$ *such that:*

$$\begin{aligned} z_1 &\leq y_1 \\ f(y_1) &\leq g(z_1) \\ f(z_1) &\geq g(y_1) \end{aligned}$$

and that (3) has at most one equilibrium in the rectangular region $(z_1, z_1) \leq (y, z) \leq (y_1, y_1)$.

Then there exists a single equilibrium x^* *of (1) in* $[z_1, y_1]$ *and all the solutions* $x(t, x_0)$ *of (1) such that* $z_1 \leq x_0 \leq y_1$ *converge towards this point. The interval* $[z_1, y_1]$ *is an invariant contracting interval.*

Proof. The preceding theorem shows that there exists an equilibrium (x^*, x^*) in Q. It is therefore the only equilibrium in Q. The vector field of the cooperative system (4) at $(-z_1, y_1)$ is nonpositive; we can then apply a well known theorem (cf [5, Th. 2.5]) to obtain that $(-z_1(t), y_1(t))$ have nonincreasing coordinates converging (because the trajectory remains in a compact invariant region) towards the equilibrium $(-x^*, x^*)$. Then we apply theorem (2.1) to obtain the result.

Other links can be stressed between systems (1) and (3): let us simply mention that, if system (1) admits a linear first integral u such that $u^T.x = const$, then, by symmetry, system (3) admits the linear first integral $u^T.(y + z) = const$.

Let us remark that this theorem gives a strong result of global convergence, but also requires strong properties from the functions f and g

of system (1). As an illustration, we remark that, because g is monotone increasing, the three inequalities of theorem 4.3 give:

$$z_1 \leq y_1 \quad f(z_1) \geq f(y_1)$$

which means (because of the signs of $Df(x)$) that the diagonal terms of $Df(x)$ are influential enough compared to the off-diagonal positive terms in order to reverse the inequality. It implies that the diagonal terms of $Df(x)$ must be sufficiently negative at least somewhere for x between z_1 and y_1 to "dominate" the off-diagonal ones.

We also remark, as K.P. Hadeler pointed out, that these theorems are the continuous analogues of theorems for discrete systems appearing originally in ([2, p. 277]).

5 Applications

We first state a short lemma that shows that the decomposition (1) includes roughly all systems with bounded off-diagonal derivatives:

Lemma 4 *Let $h : W \to \mathbf{R}^n$ be C^1 and such that:*

$$\left| \frac{\partial h_i}{\partial x_j}(x) \right| \leq a_{ij} \text{ for } i \neq j \text{ and } x \in W$$

then the differential system $\dot{x} = h(x)$ can be written in the form (1).

For we can define:

$$g_i(x) = \sum_{j \neq i} a_{ij} x_j$$

and $f = h + g$. It is easy to check f and g verify condition (2). Of course it remains to find a point verifying the inequalities of theorem 3.2.

We now apply theorem 4.3 to Lotka-Volterra systems. These systems appear in biological modelling ([15]) and have the form:

$$\dot{x}_i = x_i(b_i + \sum_{j=1}^{n} c_{ij} x_j) \text{ for } i = 1, \ldots, n \tag{5}$$

Many results are known on these systems ([13,14]). We do not assume, a priori, anything on the signs of c_{ij} or b_i. A known sufficient condition of global convergence to equilibrium in the whole space $\mathbf{R}^n_+$, using Lyapounov functions, is ([14,3]) the existence of a positive definite diagonal matrix D such that $DC + C^T D$ is negative definite. This condition seems to be not easy to check, except for simple cases, like the one where C is off-diagonal positive ([1]).

We know that the interior of the closed positive orthant is invariant by (5) and take only initial values x_0 in it and so, performing the change of variables $\xi = \ln x$, we obtain:

$$\dot{\xi} = b + Ce^{\xi} \tag{6}$$

We write $C = F - G$ with F and G as above. We suppose (6) has at least one equilibrium $\xi^* = \ln x^*$ ($x^* > 0$). The system (4) is now:

$$\begin{pmatrix} \dot{Y} \\ \dot{Z} \end{pmatrix} = A \begin{pmatrix} U \\ V \end{pmatrix} \tag{7}$$

with

$$A = \begin{pmatrix} F & G \\ G & F \end{pmatrix} \quad \text{and} \quad \begin{cases} U = e^Y - e^{\xi^*} \\ V = e^{\xi^*} - e^{-Z} \end{cases} \tag{8}$$

One equilibrium of (7) is $U = V = 0$ and we suppose it is unique (we suppose A bijective). The criterion with strict inequalities of theorem 4.3 is equivalent to:

$$A \begin{pmatrix} U_1 \\ V_1 \end{pmatrix} < 0 \text{ and } \begin{pmatrix} U_1 \\ V_1 \end{pmatrix} > 0$$

This condition is equivalent to the fact $-A$ is an M-matrix ([1]). Let us now study the region of convergence. Suppose $-A$ is an M-matrix, so we can find U_1 and V_1. Then λU_1 and λV_1 is also a solution for all $\lambda > 0$. But we have:

$$V = e^{\xi^*} - e^{-Z}$$

and so we must choose λ such that:

$$\lambda V_1 - e^{\xi^*} < 0$$

We choose λ as large as possible to obtain the biggest region and so $U_m = \lambda U_1, V_m = \lambda V_1$ such that one component (at least) of Z is arbitrarily large positive. Then system (6) converges for initial value ξ_0 such that:

$$\ln(e^{\xi^*} - V_m) < \xi_0 < \ln(e^{\xi^*} + U_m)$$

and the original system (5) converges for:

$$x^* - V_m < x_0 < x^* + U_m$$

where $x^* - V_m$ admits at least one component arbitrarily small.

We have therefore a theorem of global convergence:

Theorem 5.4 *If the Lotka-Volterra system (5) admits one equilibrium, and*

$$-\begin{pmatrix} F & G \\ G & F \end{pmatrix}$$

is a bijective M-matrix, then (5) converges to the unique equilibrium for all initial values in $[x_{min}, x_{max}]$, where x_{min} and x_{max} are determined as above, and where one component at least of x_{min} can be chosen to be arbitrarily small positive. The interval $[x_{min}, x_{max}]$ is a contracting invariant interval.

It is easy to check $-A$ is an M-matrix by computing its principal minors, that must be positive ([1]).

Example : Let the three-dimensional Lotka-Volterra system:

$$\begin{cases} \dot{x} = x(-4x - y + z + 4) \\ \dot{y} = y(2x - 5y + z + 2) \\ \dot{z} = z(-x - y - 3z + 5) \end{cases}$$

with the equilibrium $x = y = z = 1$. We can write:

$$F = \begin{pmatrix} -4 & 0 & 1 \\ 2 & -5 & 1 \\ 0 & 0 & -3 \end{pmatrix} \quad G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

and deduce from the above theorem that the convergence toward equilibrium is global in the three-dimensional interval $]0, 2]$.

As a conclusion, we have obtain a sufficient criterion of global convergence inside a given rectangular contractive region of the space, for a large class of differential systems; this criterion mainly consists of an inequality between two points around the equilibrium. We think that it is very conservative, but, even in the case of Lotka-Volterra systems, we don't know whether it is more or less conservative than the one previously known ([3]) we cited before.

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