Differential systems with positive variables

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Abstract. The variables of biological, ecological, or chemical systems are often positive, because they measure concentrations, numbers,... We study polynomial n-dimensional differential systems defined in the positive orthant. We use the tools of the usual linear algebra to exploit the specific structure of such systems, and obtain some indications on their behavior. In some cases, we are able to exhibit functions that decrease along the trajectories and therefore to give sufficient conditions for a regular global behavior: that is, all the trajectories either converge towards the equilibria or are unbounded. Our main example will be the n - dimensional Lotka-Volterra system (arising in biological modeling of species interactions). We apply the above results to stabilize the Lotka-Volterra system by controlling either the total growth rates of some species, or, alternatively, the individual growth rates of some species.

1 Introduction

Consider the n-dimensional differential system

$$\dot{x} = f(x) \tag{1}$$

We are going here to study this system for *positive* x only, i.e. the system is defined in a subset of the open positive n-dimensional orthant. Such situations, for example, arise in chemical or biological modeling (where the variables are constrained to be positive because they represent numbers or concentrations). Interestingly, some tools have been developed, that are deeply founded on the sign properties of the variables: let us cite the work on chemical kinetics of Feinberg, Horn and Jackson ([3]), of Clarke ([2]) and others (see the references in [2]). A more biological domain of application is population dynamics, where one models the way species interact together by competition, cooperation ...([13]). Some other tools have also been developed, that use the signs of the Jacobian matrix inside a region of the space. For example, there exist strong results when this matrix is off-diagonal positive, i.e. the species are cooperative (mutualistic) ([10, 14]). The fact that the elements of the Jacobian matrix are positive is strongly bound to the positivity of the variables. The positivity of the variables is therefore important; in some cases, when "strong positivity" is demanded, it leads to convergence towards equilibrium ([8]).

A famous related example is the generalized Lotka-Volterra (LV) systems in dimension n, that can have a complex behavior for $n \geq 3$ ([11]) :

$$\dot{x}_i = x_i(b_i + \sum_{j=1}^{n} A_{ij} x_j) \quad (i = 1, \dots, n) \tag{2}$$

The $(n \times n)$ real matrix A describes the quadratic interactions, and is usually supposed to be bijective (the equilibrium will be unique if it exists). The variables x_i, standing for populations, are real non-negative. The positive orthant is invariant.

Our aim in this paper is to give some results (related to [3]) concerning positivity, already obtained by the author ([5, 7, 6]). The tools used here are mainly changes of variables, tools from the usual linear algebra, results from positivity and theory of positive matrices ([1]) and auxiliary functions that decrease along the trajectories and Lasalle's theorem ([12]. The typical result that we obtain under some hypotheses is that all the trajectories either converge towards some attractor set S containing the set of equilibria, or that they cannot remain in any compact set of the positive orthant; they have therefore limit points either at infinity or on the faces of the orthant, or they leave the orthant if it is not invariant. If, moreover, the set S is actually the set of equilibria, we deduce that the system cannot have a complex behavior (such as periodic solutions, recurrent trajectories, chaos, ...) in any compact set of the interior of the positive orthant.

Notations: For x in $\mathbf{R}^n$, we write $x > 0$ if $x_i > 0$ $(i = 1, \dots, n)$ and $x \geq 0$ if $x_i \geq 0$ $(i = 1, \dots, n)$. The closed positive orthant is $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n; x \geq 0\}$. We will frequently use the open positive orthant $\mathbf{P}^n = \{x \in \mathbf{R}^n; x > 0\}$. Let us denote by ${}^t u$ the transpose of u, by e^x the vector ${}^t(e^{x_1}, \dots, e^{x_n})$, and similarly for $\ln x$. $\mathbf{1}$ is the vector ${}^t(1 \dots 1)$ and diag (x) the diagonal matrix with diagonal $(x_1, x_2, \dots, x_n)$.

2 Polynomial positive systems

The basic system will be in a polynomial form $\dot{x} = P(x)$, where P is a vector polynomial in the variables $x_i, i = 1, \dots, n$. More precisely:

$$\dot{x_i} = \sum_{j=1}^{q} a_{ij} v_j(x_1, \dots, x_n) \quad (i = 1, \dots, n) \tag{3}$$

where the a_{ij} are real and the $v_j(x_1, \dots, x_n)$ monomials of the form $x_1^{\beta_1} \dots x_n^{\beta_n}$, where the β_i are real; q is the number of distinct monomials.

We study the system when x is positive (the system is defined in $\mathbf{P}^n$). We remark that the powers β_j in the monomials can be real or negative: the system is always C^∞ in $\mathbf{P}^n$.

It is clear that this kind of polynomial system is very often met in the models. Let us make here an important remark: nearly all the biological models are in a rational fraction form (after reduction to the same denominator): $\dot{x} = \frac{P(x)}{Q(x)}$ where P and Q are two polynomials (Q is a real polynomial). It happens very frequently that, for $x > 0$, $Q(x) > 0$ (it is sufficient, for example, that Q have positive coefficients). Then we can multiply the right member by the positive number $Q(x)$, and the new system has the same geometrical phase portrait ([11]). The new system is now in a polynomial form. It is easy to see we can write (3) into the form :

$$\dot{x} = \text{diag }(x) A e^{(B \ln x)} \tag{4}$$

where A is a $n \times p$ matrix and B a $p \times n$ matrix. We can do that by factoring x_i in the i^{th} equation. The elements of B are nothing but the powers β_j or $(\beta_j - 1)$ of x_i in the monomials; p is the number of distinct monomials in the system after factorization of x_i.

The change of variables (well defined because $x > 0$) $y = \ln x$ gives:

$$\dot{y} = A e^{By} \tag{5}$$

We now study the equilibria of this system; let $s = \text{rank } A$. The equilibria are such that: $\dot{y} = 0 \Leftrightarrow e^{By} \in \ker A$ and therefore belong to the intersection, in a p-space, of $\ker A$ (a vector space of dimension $(p - s)$) and of a manifold (generated by e^{By}) of dimension $r = \text{rank } B$ ($r \leq (\max(\text{n}, \text{p}))$) and located in the positive orthant $\mathbf{P}^p$.

If $\ker A \cap \mathbf{P}^p = \emptyset$, then there is no equilibrium. In particular, if A is injective, there is no equilibrium; this result does not depend on B.

In this last case, we have shown that:

Theorem 1 *If $\ker A \cap \mathbf{P}^p = \emptyset$, then all positive orbits of (5) have an unbounded closure and every solution eventually leaves every compact of $\mathbf{P}^n$ and never comes back.*

Let us recall the basis of the proof : if $\ker A \cap \mathbf{P}^p = \emptyset$, then im ${}^tA \cap \mathbf{R}^p_+ \neq \{0\}$, and we can choose $r = {}^tAq$ in this intersection. Let $V(y) = {}^tqy$, then $\dot{\overline{V(y)}} = {}^tqAe^{By} = {}^tre^{By}$, where r is a nonnegative and non-zero vector. So this last expression never vanishes, and Lasalle's theorem ([12]) gives us the first result.

It means that, given an initial condition (and if $\ker A \cap \mathbf{P}^p = \emptyset$), the orbit cannot remain in any compact; in particular, there is no complicated behavior (like periodic or recurrent solutions, chaos...) inside any compact; for the original system (4), it means that, given a positive initial condition, the orbit is unbounded or has points of the faces of the orthant as limit points, or leaves the orthant: that is, the positive orbit definitely leaves any compact set of the interior of the positive orthant.

These results heavily depend on the number n and p of variables and distinct monomials; suppose that $n \geq p$ and that $s = p$; then A is injective and the condition $\ker A \cap \mathbf{P}^p = \emptyset$ is trivially verified. We have therefore the simple following corollary:

Corollary 1 *If $n \geq p$, then for almost any choice of the matrix A (if A is injective), all the trajectories of (5) are unbounded. This result does not depend on the matrix B.*

That is, the unboundedness of the trajectories does not depend on the precise form of the non-linear monomials, nor on the value of the elements of A, but only on the injectivity of A.

3 Lotka-Volterra systems

As we have seen before, the most interesting case is $n \leq p$. We make this hypothesis now, and we define $z \in \mathbf{R}^p$: $z = By$ to obtain

$$\dot{z} = BAe^z \tag{6}$$

This is a change of variables for $z \in \operatorname{im} B$. Let us remark that this system is quadratic homogeneous and analogous to a LV system but without the linear terms. In fact a LV system can be written into this form by taking one more state variable x_{n+1} with a zero dynamics.

We have built tools to study such systems; we just give the results and refer to the papers ([7, 6]).

If, given A and B, we can find an off-diagonal nonnegative and singular matrix M and two vectors $k > 0$ and w, such that ${}^tM\mathbf{1} = 0$, $Mk = 0$ and $\ln k = Bw$, and a symmetric matrix P, with $PA = {}^tBM$, we will summarize these hypotheses by saying the system (A, B) admits a "PM-decomposition". This property is a linear programming problem, and can be compute with finite algorithms.

Theorem 2 *If A and B admit a PM-decomposition, the trajectories of (5) are unbounded or converge towards the maximal invariant set included in the set of y such that:*

$$e^{By} = \sum_{j=1}^{l} \lambda_j k_j$$

Here, the graph of M is made of l disconnected classes, associated with matrices M_j $(j = 1, \ldots, l)$, k_j is a positive vector in $\ker M_j$ having the property that the components not corresponding to the vertices of the graph of M_j are zero, and the λ_j are real nonnegative.

See [7, 6, 5] for the proof.

Corollary 2 *If A and B admit a PM-decomposition with $\ker P \cap A\mathbf{P}^p = \{0\}$ (in particular if P is bijective), then the trajectories of (5) are unbounded or converge towards the (non empty) set of equilibria.*

Indeed, because of the preceding theorem, the bounded trajectories converge towards the points y such that $Me^{By} = 0 \Rightarrow PAe^{By} = 0$, and with the hypothesis of the corollary, it implies that $Ae^{By} = 0$, so y is an equilibrium.

If the hypothesis of the corollary are fulfilled, the differential system has a rather simple behavior: a trajectory either goes towards the set of equilibria or is unbounded (it can happen that there is no equilibrium on the linear first integrals of the solution, in such a case the trajectory is unbounded). For the original system (3), it means that a trajectory either converges towards the set of equilibria, or has limit points at infinity or on the faces of the orthant, or leaves the orthant. Intuitively, the trajectory converges to equilibrium or "leaves" the interior of the orthant. Of course, such a trajectory can reenter the orthant later, and infinitely often (this behavior makes no sense when the variables have a biological meaning). For more details, see ([4]).

These theorems applied to true Lotka-Volterra systems (2) give the following particular case. We can write a new n-dimensional quadratic homogeneous system:

$$\dot{x}_i = x_i(\sum_{j=1}^{n} B_{ij}x_j) \quad (i = 1, \ldots, n)$$

with the initial condition $x(0)$ satisfying $x_n(0) = 1$, and

$$B = \begin{pmatrix} A & b \\ 0 & 0 \end{pmatrix} \tag{7}$$

We obtain the new system :

$$\dot{y} = Be^y \tag{8}$$

and the theorem follows, giving the same results on the behavior as above:

Theorem 3 *Suppose that, for a given B, there exist a square off-diagonal nonnegative singular matrix M, such that ${}^t M\mathbf{1} = 0$, and a symmetric square matrix P such that $PB = M$. Then, if*

$$V(y) = \frac{1}{2}\,{}^t(y - \ln k)P(y - \ln k)$$

with $k \in \ker B \cap \mathbf{P}^n$, the function $V(y)$ decreases along the trajectories of (8) ($\overline{\dot{V}(y(t))} \leq 0$).

Moreover, if the graph of M is made of l disconnected classes, associated with matrices M_j $(j = 1, \ldots, l)$, then this derivative vanishes if and only if

$$e^y = \sum_{j=1}^{l} \lambda_j k_j$$

where k_j *is a positive vector in* $\ker M_j$ *having the property that the components not corresponding to the vertices of the graph of* M_j *are zero, and the* λ_j *are real nonnegative.*

4 Stabilization of LV systems by a positive control

Let us suppose now that the LV system (2) is subject to a control on the growth rates (the vector b). We can imagine, for example, that the growth rate of each species depends on one single factor that could be the lighting, the temperature, or the pollution of the ambient space .... The new system is now :

$$\dot{x}_i = x_i \ (ub_i + \sum_{j=1}^{n} A_{ij} x_j) \ \ (i = 1, \ldots, n) \tag{9}$$

In this system the scalar control u is supposed to be positive ; it means that the external factor cannot change, for example, a death rate into a growth rate, which is quite realistic. The mathematical problem that we want to address here is the following:
" Given a reference equilibrium x^*, u^*, which turns out to be unstable, is it possible to stabilize the system around this equilibrium ?"

The idea to stabilize the system is simple : we consider u as a new variable, and we choose its dynamic in order to verify the constraint of positivity and to stabilize the new system with the help of a Lyapunov function. The equation for u is chosen of the following form :

$$\dot{u} = u(\sum_{i=1}^{n} c_i x_i + du)$$

where the coefficients c_i, d, and the initial condition $u(0)$ have to be chosen. The equilibrium gives us that

$$\sum_{i=1}^{n} c_i x_i^* + du^* = 0$$

Let us remark that u will stay positive if the initial control is positive, because the hyperplane $u = 0$ is invariant by this equation.
The new state vector is now $y = (x, u)$ and is a solution of the system : (after change of variables $z = \ln y$)

$$\dot{z} = A' e^z; \ \ A' = \begin{pmatrix} A & b \\ {}^t c & d \end{pmatrix} \tag{10}$$

We can also scale the equilibrium $z^* = (\ln x^*, \ln u^*)$ to 0 by a translation. We keep the same notation for A' which verifies now $A'\mathbf{1} = 0$.

We want to choose the parameters c, d verifying ${}^t c\mathbf{1} + d = 0$ such that the system (10) is globally stable. This system is very similar to the previous one and we could use the same theorem (3). We use here a different Lyapunov function, based on cooperativity.

It is not difficult to obtain the result ([9]):

Proposition 1 *If the system (10) is such that*

$$\sum_{j=1}^{n+1} \min_{i \neq j} A'_{ij} > 0$$

then the positive equilibrium is globally asymptotically stable.

We apply this result to our control system to obtain a sufficient condition of global stabilization :

Theorem 4 *If the system (10) is such that*

$$\sum_{j=1}^{n} \min_{i \neq j} A_{ij} + \min_{i} b_i > 0$$

then there exists u positive stabilizing globally (9).

Indeed, we can choose $c_j \geq \min_{i \neq j} A_{ij}$, and the resulting system will verify the above proposition.

In fact, we can even obtain an explicit formulation of the feedback, because the system (10) has a first integral ${}^t qz = C$, where q belongs to the kernel of ${}^t A'$. Finally $C = 0$ because when $x = \mathbf{1}$, then $u = 1$. We obtain the following explicit form:

Corollary 3 *A smooth positive stabilizing feedback for the original system (9) is:*

$$u = u^* \prod_{i=1}^{n} (\frac{x_i}{x_i^*})^{-q_i/q_{n+1}}$$

where q belongs to the kernel of ${}^t A'$.

5 Example

Let us consider the following three-dimensional LV system :

$$\begin{cases} \dot{x} = x(\frac{4}{3}x - z - \frac{1}{3}u) \\ \dot{y} = y(3x - \frac{5}{2}y - z + \frac{1}{2}u) \\ \dot{z} = z(3x - \frac{3}{2}y - \frac{5}{2}z + u) \end{cases}$$

For the reference control $u = 1$, the system has the equilibrium $x = y = z = 1$, which is unstable: indeed, the Jacobian matrix at the equilibrium is nothing but A, which has a positive eigenvalue. We add the following equation for u :

$$\dot{u} = u(\alpha x + \beta y + \chi z + \delta u)$$

If we write the whole matrix (10), the criterion will be verified if we can find the coefficients such that:

$$\min(\alpha, 3) + \min(\beta, -3/2) + \min(\chi, -1) - 1/3 > 0$$

We can choose $\alpha = 3, \beta = \chi = 0$. The coefficient δ is determined so that $u = 1$ is an equilibrium. We obtain that $\dot{u} = u(3x - 3u)$ stabilizes globally the system for positive variables. The kernel of ${}^tA'$ is $(-171/40, -27/20, 9/4, 1)$, and the explicit form for the positive feedback is therefore :

$$u = x^{171/40} y^{27/20} z^{-9/4}.$$

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