TRANSFORMATION OF POLYNOMIAL DIFFERENTIAL SYSTEMS IN THE POSITIVE ORTHANT

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ABSTRACT

We study polynomial n-dimensional differential systems when the variable keeps the same sign; that is, the system is defined inside one of the orthants. We show it is always possible to transform the system into a quadratic Lotka-Volterra type system with first integrals. We give several tools to study these last systems; we deduce indications on the global behaviour of the original system. We give an application to global stabilization in the positive orthant.

Key Words: Polynomial differential systems, Lotka-Volterra systems, positivity, global behaviour.

1. INTRODUCTION

Consider a polynomial n-dimensional differential system. We can write it:

$$\dot{x}_i = \sum_{j=1}^{q} a_{ij} v_j(x_1, \ldots, x_n) \quad (i = 1, \ldots, n)$$

where the a_{ij} are real and the $v_j(x_1, \ldots, x_n)$ mononomials of the form $x_1^{\beta_1} \ldots x_n^{\beta_n}$; q is the number of different monomials in the whole system. If this system is defined for $x \in \mathbf{R}^n$, it is well known it can have a very complicated behaviour. In some particular cases, it is possible to study such systems, but global results are rather rare(cf. [5,6]).

We are going here to study this system for positive x only, i.e. the system is defined in an open subset of the open positive n-dimensional orthant. Of course, we could study the system in any of the open orthants: the important fact is that we are interested by the behaviour of the system as long as x keeps the same sign. Such situations, for example, arise in chemical or biological modelling (where the variables are positive)([2]).

We will show it is possible to transform the original polynomial system (defined as long as x keeps the same sign given the initial condition x_0) into a system of (generally) bigger dimension, but of stronger structure: more precisely, the trajectories of the original system will be transformed into trajectories of the new system constrained to stay on (generally) certain first integrals. The new system will be of dimension p (p is related to q), and is quadratic with a special form. In fact, it is a Lotka-Volterra type system ([11,10]), that keeps constant the signs of the variables, because each face of the orthant is invariant. Many results are known on these systems ([10,1,9,3]).

Because of the stronger structure of the new systems, we can then exhibit several tools to study their behaviour; it is possible then to describe many features of the behaviour of the original system. One typical result is that a trajectory of the original system either converges toward the set of equilibria (if it exists) or leave any compact of the orthant; that means that the system has no complicated behaviour (like limit cycles, chaos,...) inside any compact of the positive orthant. We also give an application to the global stabilization of such differential systems in the whole orthant.

Notations: For x in $\mathbf{R}^n$, we write $x > 0$ if $x_i > 0$ $(i = 1, \ldots, n)$ and $x \geq 0$ if $x_i \geq 0$ $(i = 1, \ldots, n)$. The closed positive orthant is $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n; x \geq 0\}$. We will frequently use the open positive orthant $\mathbf{P}^n = \{x \in \mathbf{R}^n; x > 0\}$. Let us denote by ${}^t u$ the transpose of u, by e^x the vector ${}^t(e^{x_1}, \ldots, e^{x_n})$, and similarly for $\ln x$. The Kronecker product between two vectors $x \otimes y$ is the vector with components $(x_i y_i)$. If x is a vector, we denote by x^{-1} the vector of components $(1/x_i)$. $\mathbf{1}$ is the vector ${}^t(1 \ldots 1)$ and $\mathrm{diag}(x)$ the diagonal matrix with diagonal x.

If $V \subset \mathbf{R}^n$ is open, $h : V \to \mathbf{R}^n$ is C^1, and $x_0 \in V$, for the differential system $\dot{x} = h(x)$ ($\dot{x}$ is the derivative with respect to time t), we denote by $x(t, x_0)$ or sometimes by $x(t)$ the (maximally defined) solution in V with initial value x_0 for $t = 0$.

2. TRANSFORMATION OF THE SYSTEM

We first expose the basic idea of the transformation. As x always keeps the same sign (does not vanish), we can write the original system:

$$\dot{x}_i = x_i \sum_{j=1}^{p} a_{ij} w_j(x_1, \ldots, x_n) \quad (i = 1, \ldots, n) \tag{1}$$

Here p is the number of different "monomials" $w_j(x_1, \ldots, x_n)$, in which we can have variables with negative power. Now we take the p functions $w_j(x)$ as new variables u_j. So we define

$$u_j = w_j(x) = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$$

where we have omitted the j-index for simplicity.

The time derivative is

$$\dot{u}_j = \alpha_1 x_1^{\alpha_1 - 1} \ldots x_n^{\alpha_n} \dot{x}_1 + \ldots + \alpha_n x_1^{\alpha_1} \ldots x_n^{\alpha_n - 1} \dot{x}_n$$

and therefore, by using (1):

$$\dot{u}_j = w_j(x)(\alpha_1 \sum_{k=1}^{p} a_{1k} w_1(x) + \ldots + \alpha_n \sum_{k=1}^{p} a_{nk} w_n(x))$$

and we obtain the system:

$$\dot{u} = u \otimes Cu \tag{2}$$

of dimension p, where C is a square p-matrix; this system is quadratic homogeneous. The u_j are, in general, constrained to stay on first integrals because of the relations $u_j = w_j(x)$, that create relations between the u_j if $p > n$.

Now we want to study more precisely this transformation; we will suppose (without any restriction of the problem) that x is positive; that is, the system (1) is defined in $\mathbf{P}^n$. First we remark that the powers α_j in the monomials can be real negative: the system is always C^∞.

It is easy to see we can write (1) into the form :

$$\dot{x} = x \otimes Ae^{(B \ln x)} \tag{3}$$

where A is a $n \times p$ matrix and B a $p \times n$ matrix. The elements of B are nothing but the powers α_j of x in the monomials. We transform first the equation by the change of variables :

$$y = \ln x$$

so that:

$$\dot{y} = Ae^{By} \tag{4}$$

Now we define

$$z = By \tag{5}$$

so

$$\dot{z} = BAe^{z} \tag{6}$$

In the sequel, we will often work on this system, with the relations (5). We can also come back to positive variables by writing:

$$u = e^{z}$$

The new system is :

$$\dot{u} = u \otimes BAu \tag{7}$$

with relations

$$u = e^{B \ln x} \tag{8}$$

the same as the system (2) with $C = BA$.

The system (7) can be seen as a p-dimensional Lotka-Volterra system without linear terms ([10]). Each face of the positive orthant, and the positive orthant himself, is invariant.

We can also refine the description for a particular case: suppose there is a constant monomial (equal to the constant 1) among the w_j, that is, say, $w_p(x) = 1$. Then $\dot{u}_p = 0$ and the system can be written :

$$\dot{\hat{u}} = \hat{u} \otimes (B_p A_p \hat{u} + B_p a_p) \tag{9}$$

where $\hat{u}$ is u without u_p, B_p is the matrix B without the last null line, A_p is made from the $(p-1)$ first columns of A, and a_p is the last column of A. We therefore obtain a real $(p-1)$-dimensional Lotka-Volterra system.

We can now apply known tools to study the new system. In the following, we will often obtain the result that, for every initial condition of system (4) or (6), either the forward orbit has an unbounded closure, either the solution go toward the set of equilibria. We will call this kind of behaviour "regular behaviour". For system (1), it implies that, for every initial condition, either a set of points belonging to the faces of the orthant, or infinity, are limit points of the forward orbit, either the solution go toward the set of equilibria in the positive orthant, either the solution leave the orthant by a face. If, moreover, the set of equilibria is empty, we will say that the behaviour is "regularly unbounded".

3. POSITIVITY

We first use tools from ([2]) (see also ([4])). We remark that, in the system (6), the term e^{z} is always positive; theorem 10.3 from ([2]) implies that:

Proposition 1 *If* $\ker C \cap \mathbf{P}^p = \emptyset$*, then there is no equilibrium for system (6); the behaviour is regularly unbounded; a fortiori, the behaviour is regularly unbounded for system (1).*

For the sake of completness, we give briefly the proof: first it is clear that if x^* is a positive equilibrium of (1), then $z^* = B \ln x^*$ is an equilibrium of (6), and therefore e^{x^*} is in the kernel of $C = BA$. If $\ker C \cap \mathbf{P}^p = \emptyset$, then $\mathrm{im}\ {}^tC \cap \mathbf{R}_+^p \neq \{0\}$, and if we take $r = {}^tCq$ in this set, and write $V(z) = {}^tqz$, then, along the trajectories, $\dot{\overline{V(z)}} = {}^tqCe^z = {}^tre^z$ where r in nonnegative and non equal to zero. So the last expression never cancels, and Lasalle's theorem tells us the trajectories have all an unbounded closure.

Example:
The two-dimensional system

$$\begin{cases} \dot{x} = x^2y - xy \\ \dot{y} = 2y^2 - 3xy^2 \end{cases}$$

defined for $x > 0, y > 0$ is such that:

$$A = \begin{pmatrix} 1 & -1 \\ -3 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and $C = BA$ is bijective, so there is no equilibrium and all the trajectories go to infinity or toward the faces of the two-dimensional positive orthant.

4. AUXILIARY FUNCTIONS

From above, we deduce that the only non-trivial case is when $\ker C$ intersects $\mathbf{P}^p$, and we have an equilibrium (at least) z^* for the system (6), with $Ce^{z^*} = 0$. We will suppose moreover that B *is injective*, that is, that the equilibria of (6) (with relations (5)) are the same as those of (4). In particular, it implies that $p \geq n$, which is the most interesting case ([2,4]).

We want to try $V(z) = {}^tde^z$, where d is a given vector, as an auxiliary function that decreases along trajectories (a weak version of Lyapunov functions). Then $\dot{\overline{V(z)}} = {}^te^z\mathrm{diag}(d)Ce^z$. Applying Lasalle's theorem ([8]), we have :

Theorem 1 *If there exists a p-vector d such that the quadratic form* $\mathrm{diag}(d)C + {}^tC\mathrm{diag}(d)$ *is semi-definite positive, then the trajectories of system (6) either are unbounded or go toward the points z such that* e^z *is in the kernel* K *of this form. Consequently, the trajectories of (6) constrained by (5) either go toward these points in* $\mathrm{im}\ B$ *or are unbounded.*

Let us remark that this form is never definite, because of the equilibrium.

We shall make the theorem more precise when $\ker C = \ker A$ is one-dimensional. We want that the kernel of the form be as small as possible, so we try to choose d such that the kernels of C and ${}^tC\mathrm{diag}(d)$ are the same, that is to say:

$${}^tC\mathrm{diag}(d)e^{z^*} = 0$$

We know indeed that it exists k such that ${}^tCk = 0$, so we choose

$$d = k \otimes (e^{z^*})^{-1}$$

Suppose the quadratic form is positive and of kernel $\ker C$; the trajectories $z(t)$ either are unbounded or go toward the set$\{z; Ce^z = 0\}$; if q is a positive vector of $\ker C$, we therefore have

$$e^z = \lambda q \Leftrightarrow z = (\ln \lambda)\mathbf{1} + \ln q$$

$(\lambda > 0)$ and the set of equilibria is a straight (affine) line L of vector $\mathbf{1}$ and containing the point $\ln q$. But ${}^tk\dot{z} = 0$ and we have one first-integral for z. So the trajectory must go toward the intersection of L and of the (affine) hyperplan orthogonal to $\ker\ {}^tC$; this intersection is reduced to one point if and only if ${}^tk\mathbf{1} \neq 0$.

Theorem 2 *Suppose* $\ker C$ *is one-dimensional and the quadratic form* $\mathrm{diag}(d)C + {}^tC\mathrm{diag}(d)$, *with* $d = k \otimes (e^{z^*})^{-1}$, *is positive with kernel* $\ker C$. *Then, if* ${}^tk\mathbf{1} \neq 0$, *the trajectories of system (6) either are unbounded or go toward the unique equilibrium on the first integral* ${}^tkz = const$. *Consequently, the system (1) has a regular behaviour.*

If k is negative, then d is negative, and $V(z)$ is actually a Lyapunov function, and the equilibrium is globally stable in the orthant. If k is positive, it is globally totally unstable (stable in reverse time).

Example:
Let the three-dimensional system:

$$\begin{cases} \dot{x} = x^2 - \frac{1}{3}xyz - \frac{2}{3}x^3z \\ \dot{y} = 2xy + \frac{19}{3}y^2z - \frac{25}{3}x^2yz \\ \dot{z} = -4xz - \frac{10}{3}yz^2 + \frac{22}{3}x^2z^2 \end{cases}$$

defined on $\mathbf{P}^3$. Then

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & -1/3 & -2/3 \\ -2 & 3 & -1 \\ -2 & -4 & 6 \end{pmatrix}$$

We have $C\mathbf{1} = 0$ and $\ker\ {}^tC$ contains the vector $k = {}^t(1\ 1/3\ 1/6)$, so we take $d = k$ and obtain the quadratic form

$$\begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix}$$

of kernel $\mathbf{1}$ and positive. Moreover, ${}^tk\mathbf{1} \neq 0$. We can therefore deduce that all the trajectories of the system have a regular behaviour. Moreover, because d is positive, the equilibrium is totally globally unstable in $\mathbf{P}^3$.

For the particular case of system (9), we can use a well-known theorem (cf ([11,1,7])).

Theorem 3 *Suppose B_pA_p is bijective and that there exits a positive equilibrium; then, if there exists a diagonal matrix D such that $DB_pA_p + {}^tB_p^tA_pD$ is definite negative, then the trajectories $z(t)$ and $y(t)$ either go toward the equilibrium or are unbounded.*

One often constrains D to be positive to obtain a Lyapunov function, and therefore global stability. But an arbitrary D gives also indications on the global behaviour.

Example: Let the two-dimensional system:

$$\begin{cases} \dot{x} = 7x^2y^3 - 8x^2y^2 + x \\ \dot{y} = -3xy^4 + 3xy^3 \end{cases}$$

then

$$B = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \; B_pA_p = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

and B_pA_p is negative definite, therefore (because D is positive), the equilibrium $x = 1, y = 1$ is globally stable in the positive orthant.

5. APPLICATION TO STABILIZATION

It is easy to apply the preceding results to the problem of stabilization in the positive orthant of a polynomial system like (3). For the sake of simplicity, let us consider only the controlled system:

$$\dot{y} = Ae^{By} + b + Cu \tag{10}$$

where b is a constant vector, u a k-vector, and C a $(n \times k)$ matrix. We suppose that A is square regular and that 0 is the equilibrium of the system. We can apply the results of the above section by taking a control of the form:

$$u = K(e^{By} - \mathbf{1})$$

where K is a $(k \times n)$ matrix. The theorem 3 gives us:

Corollary 1 *If there exists a positive diagonal matrix D and a matrix K such that the quadratic form associated with $DBA + DBCK$ is negative definite, then the control stabilizes the system in the whole positive orthant.*

Example: Consider the system defined in the positive quadrant:

$$\begin{cases} \dot{x} = -5x^2y^2 + x^2y^3 + 4x \\ \dot{y} = 2xy^3 + xy^4 - 3y + u_1 \end{cases}$$

The equilibrium $(1, 1)$ is unstable; we can stabilize it in the whole positive orthant by taking

$$u_1 = -2y(xy^3 - 1)$$

6. CONCLUSION

We have shown that every polynomial n-dimensional system can be transformed into a quadratic Lotka-Volterra type system. Then it is possible (with good hypotheses) to study the global behaviour of the original system by using the strong structure of the transformed system.

We have the feeling that, in general, it is simpler to study a differential polynomial system in an orthant than in the whole space; and that it is possible to find more specific tools to study such systems.

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