Interval observers for uncertain biological systems

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Abstract

We present a technique for the dynamic estimation of bounds on unmeasured variables (or parameters) of an uncertain dynamical system. Our approach is purely deterministic and lies on interval observers: from (possibly time varying) intervals on the uncertainty and measurements, we compute guaranteed intervals for the unmeasured variables. This method is applicable for a class of nonlinear systems met in several biological modellings: two examples are shown, one model of a three stages structured population, and one of biological water treatment.

Key-words: Nonlinear estimation, uncertain models, biological systems, interval observers.

1 Introduction

The problem we investigate in this paper deals with the estimation of unmeasured variables (or parameters) for a class of uncertain biological dynamical systems. "Uncertain" means here that some part of the model is not precisely known, or there may be some noise. Given some outputs (the measurements), one tries to estimate (or "identify") the unknown variables or parameters. This problem is classical, but we want to insist in this paper on the dynamical estimation approach.

Traditional approaches used in biological sciences are often based on regression methods (for instance least-squares methods, by minimization of the deviation between the data and the model, or more elaborated criterions [21]). These approaches are powerful and easy to implement, but, in the case of uncertain models, might present drawbacks:

- In the presence of uncertainties (or noises) on the model, the asymptotic convergence of the average estimates is guaranteed under some statistical hypotheses (mean values and covariances of the noises need usually to be known). An important issue for the biologist is then to decide if these hypotheses are fulfilled on the data. Very often, these hypotheses are difficult or impossible to check.
- Because of uncertainties, a static estimation of the initial values of the state variables is in general not adapted to the estimation of current values of these variables; there is a need for an adaptive dynamic method.
- Let us suppose that we want to estimate a parameter; if this parameter is in fact a forcing term slowly varying with time, it is interesting to reconstruct these variations with the help of a dynamic method.
- If there is a need for an on-line estimation (cf. the example of section 6), static minimization methods are clearly not adapted.

Other approaches are available from the Control Theory, such as the observers one ([12]). Let us take an example that will motivate us all along the paper.

A typical example: Consider a population (measured by its biomass) structured into three stages: larvae, juveniles and adults (see [14]) whose respective stocks $(x_1, x_2, x_3) \in X = \mathbb{R}^3_+$ follow the dynamics:

$$\begin{cases} \dot{x}_1 = -\alpha_1 x_1 - m_1 x_1 + r(t, x_3) \\ \dot{x}_2 = \alpha_1 x_1 - \alpha_2 x_2 - m_2 x_2 \\ \dot{x}_3 = \alpha_2 x_2 - m_3 x_3 - c(t) x_3 \\ y \;\; = x_3 \end{cases}$$

The positive coefficients α_i and m_i represent respectively the growth and mortality rates. We assume that the births in class x_1 are generated only by the adults class x_3 with a reproduction law of Beverton-Holt type (cf [14]):

$$r(t, x_3) = \frac{a(t) x_3}{b + x_3} \qquad a(t) > 0, \; b > 0.$$

The term $c(t)$ in the third equation represents an harvesting effort on the adult population. A typical instance of such a system is used for the modelling of population of fishes harvested by fishermen (cf [3]) but the same model is also met within the metabolic field (cf [14][p. 143]). Of course, we could have considered n stages with the same structure.

Assume now that the only measurement available on line (or "observation") concerns the adults stock (note that this hypothesis is quite reasonable from a biological point of view: adults are generally of bigger size, easier to be detected and counted):

$$y(t) = x_3(t), \quad \forall t \geq 0.$$

For example, to be more realistic, we could take the case of aquaculture, or of a terrestrial population that can be directly counted. For fishes, it could also be obtain by scientific measurements.

If one is interested in predicting future stocks x_3 (for instance to be able to detect a possible future extinction), the following problem is of relevant importance:

The basic problem: How to estimate at current time t the unmeasured stocks $x_1(t)$ and $x_2(t)$ from the set of past observations $\{y(\tau) = x_3(\tau)\}_{\tau \leq t}$?

When the parameters of the model $(\alpha_i, m_i, a(.), b, c(.))$ are known, there exists a classical method (observer theory, cf. [12] for a good introduction) that can be applied (cf section 2) to obtain such estimates with an arbitrary precision and arbitrarily fast. But, in more realistic situations, several uncertainties may appear:

- the recruitment function $r(t, x_3)$ is not well known. For instance, the parameter $a(t)$ is uncertain (it is often the case in fisheries that the stock-recruitment function is hard to obtain [20])
- the harvesting effort $c(t)$ is unknown, because of lack of official data; or there is an uncontrolled harvesting or poaching.

and one may wonder about the nature of this uncertainty: *Can it be considered as a random variable for which statistics can provide enough knowledge on the characteristics of its density function ?* If the answer is positive, then the state of the system is also a random variable, for which we seek an estimation of its expectation at the current time. The model can then be interpreted in terms of stochastic processes and the problem becomes relevant from the filtering theory. Unfortunately, apart the case of additive Gaussian noise processes in the equations of the dynamics, the filtering theory appears to be difficult (cf. [19]). Moreover, the assumptions of randomness are often difficult to justify (take for example the fishing effort above).

Nevertheless, in many biological modellings, even when no precise statistics are available, the uncertainty remains bounded with a priori known bounds. Then, an alternative from the stochastic approach is to consider the uncertainty as a *deterministic* but unknown function of the time (among the set of all time-measurable functions respecting the bounds and compatible with the observations). With these bounds on the parameters, one may hope to estimate bounds on the unobserved variables: this is the so-called "robust" approach. Because in our case these bounds are obtained under the form of n-dimensional intervals, we construct " interval observers".

The robust problem: Given bounds on the uncertainty, how to estimate at current time t bounds on the unmeasured stocks $x_1(t)$ and $x_2(t)$ from the past observations $\{y(\tau) = x_3(\tau)\}_{\tau \leq t}$?

This approach is strongly related to the theory of differential inequalities and cooperative systems, and has been used in the field of stability of uncertain systems ([15,13,17]).

The paper is organized as follows. In the next section, we quickly review the classical observer method for the basic problem (i.e. when the model is known), and the problem of robust estimation is investigated: we have adopted an expository style, believing that this theory may be not familiar to the readers. In the third section, we present in details our approach, based on deterministic interval observers. Finally, we illustrate our method on two applications. One concerns the canonical example while the second one comes from a concrete waste water treatment problem.

2 The observation and observers problem

We recall quickly some basic facts. A very readable introduction is [12].

Consider a dynamical system:

$$(\mathcal{S}_0) : \begin{cases} \dot{x}(t) = f(t, x(t)), \quad x(0) = x_0 \\ y(t) = h(t, x(t)) \end{cases}$$

where the state $x \in X \subset \mathbb{R}^n$ and $y \in \mathbb{R}^p$ is the "output" or observation of the state.

2.1 Definitions

We recall usual definitions from the system theory (cf [18]).

Definition 1 (Observability): *$(\mathcal{S}_0)$ is observable if and only if the mapping $\Phi : x_0 \in X \longmapsto \{y(t)\}_{t \geq 0}$ is injective.*

It means that there exists a unique initial condition x_0 giving the output $y(t)$.

Consider a family of systems, parameterized by $K \in \mathcal{K}$:

$$(\hat{\mathcal{S}}_K) : \begin{cases} \dot{z}(t) = g_K(t, z(t), y(t)), & z(0) = z_0 \\ \hat{x}(t) = l_K(t, z(t), y(t)) \end{cases}$$

where the internal state $z \in Z \subset \mathbb{R}^m$. For a given K, we consider the coupled system $(\mathcal{S}_0, \hat{\mathcal{S}}_K)$:

$$(\mathcal{S}_0, \hat{\mathcal{S}}_K) : \begin{cases} \dot{x}(t) = f(t, x(t)), & x(0) = x_0 \\ \dot{z}(t) = g_K(t, z(t), h(t, x(t))), & z(0) = z_0 \\ \hat{x}(t) = l_K(t, z(t), h(t, x(t))) \end{cases}$$

Definition 2 (Observer): *The system $(\hat{\mathcal{S}}_K)$ is an observer if $||\hat{x}(t) - x(t)||$ tends to zero as t goes to infinity.*

Definition 3 (Exponential observer): *$(\hat{\mathcal{S}}_K)$ is an exponential observer if for any positive number λ, there exists a parameter $K \in \mathcal{K}$ such that the coupled system $(\mathcal{S}, \hat{\mathcal{S}}_K)$ has the following property:*

$$\forall (x_0, z_0) \in X \times Z, \;\; \forall t > 0, \;\; ||\hat{x}(t) - x(t)|| \leq e^{-\lambda t} ||\hat{x}(0) - x(0)||.$$

That means that it is possible to adjust the gain K to have an arbitrary rate of exponential convergence of the estimate towards the actual value of the variables.

2.2 The linear case

For linear autonomous systems (with a scalar output):

$$(\mathcal{S}_L) : \begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

with $A \in \mathcal{M}^{n \times n}(\mathbb{R})$ $(n \geq 2)$, $C \in \mathcal{M}^{1 \times n}(\mathbb{R})$, a well known algebraic criterion of observability (see [10] is:

$$(\mathcal{S}_L) \text{ observable } \Leftrightarrow \text{rank} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n.$$

Then, there exists an exponential observer in Luenberger form:

$$\frac{d\hat{x}(t)}{dt} = A\hat{x}(t) + K(C\hat{x}(t) - y(t))$$

where K is a vector called the gain vector, such that all the eigenvalues of the matrix $A + KC$ belong to the left complex plane; the error converges therefore towards zero; the rate of convergence of the error can then be arbitrarily adjusted with the gain K (cf. [10] for the choice of the gains).

Remark: There exists a weaker concept of observability, called "detectability" or "asymptotic observability" (cf [18]) that does not require the injectivity of the mapping Φ but states that the difference between two trajectories with the same output history converges asymptotically towards zero. In this case, the speed of convergence of an observer cannot be tuned as desired: the convergence is dictated by the time scale of the dynamics.

2.3 The affine output injection feedback case

In contrast to nonlinear systems, for which there does not exist a general design theory for observers (cf [8]), the theory of linear observer is fully developed. Nevertheless, there exist classes in "almost" linear form, like the "affine output injection feedback" one we consider in this paper and for which observer can be easily designed:

$$(\mathcal{S}) : \begin{cases} \dot{x}(t) = Ax(t) + \phi(t, y(t)) \\ y(t) = Cx(t) \end{cases} \tag{2}$$

with $A \in \mathcal{M}^{n\times n}(\mathbb{R})$ $(n \geq 2)$, $C \in \mathcal{M}^{1\times n}(\mathbb{R})$. The nonlinearity depends only on the output, thanks to the application $\phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^n$ assumed to be measurable w.r.t. time and Lipschitz continuous w.r.t. y (these conditions are sufficient to provide the existence of a unique solution of the differential equation, see [4]). Note that our canonical example is of this form. There exists also geometric conditions on the vector fields f and g under which a nonlinear system $(\mathcal{S})$ can be written in such a form (see [9]).

If the pair (A, C) is observable and ϕ known, one can design the same observer as above:

$$\frac{d\hat{x}(t)}{dt} = A\hat{x}(t) + \phi(t, y(t)) + K(C\hat{x}(t) - y(t))$$

with an exponential error dynamics (see Krener and Respondek [11]). Let us illustrate this with our three-stages population model. If we suppose that we know the model, an observer can be written as follows (the output being still x_3):

$$\begin{cases} \dfrac{d\hat{x}_1}{dt} = -\alpha_1\hat{x}_1 - m_1\hat{x}_1 + r(t, y(t)) + k_1(\hat{x}_3 - y(t)) \\ \dfrac{d\hat{x}_2}{dt} = \alpha_1\hat{x}_1 - \alpha_2\hat{x}_2 - m_2\hat{x}_2 + k_2(\hat{x}_3 - y(t)) \\ \dfrac{d\hat{x}_3}{dt} = \alpha_2\hat{x}_2 - m_3\hat{x}_3 - c(t)y(t) + k_3(\hat{x}_3 - y(t)) \end{cases}$$

Then the errors vector $e(t) = \hat{x}(t) - x(t)$ is solution of the autonomous linear system:

$$\frac{de}{dt} = \begin{pmatrix} -(\alpha_1 + m_1) & 0 & k_1 \\ \alpha_1 & -(\alpha_2 + m_2) & k_2 \\ 0 & \alpha_2 & -m_3 + k_3 \end{pmatrix} e.$$

It is easy to check that the eigenvalues of this last matrix can be arbitrarily chosen thanks to the freedom on the parameters k_1, k_2, k_3 ([12]).

2.4 Observation for uncertain models

It is often the case in models that some part is not well known. For instance in biological modelling, the analytical form of the functions involved may be difficult to obtain, or some parameters can fluctuate with time in an unknown manner. Here we assume that the uncertainty concerns only the nonlinear term ϕ in equation (2).

A classical approach consists in choosing a reference function $\tilde{\phi}$ and to consider $w(t) = \tilde{\phi}(t, y(t)) - \phi(t, y(t))$ as a disturbance. The system $(\mathcal{S}_w)$ can then be written:

$$(\mathcal{S}_w) : \begin{cases} \dot{x}(t) = Ax(t) + \tilde{\phi}(t, y(t)) + w(t) \\ y(t) = h(x(t)) \end{cases}$$

Three families of well-known methods can be investigated:

- *Observers with unknown inputs:* It consists in studying conditions under which there exist observers completely decoupled from w (what is usually called in the literature *observers with unknown inputs*). For the linear case, necessary and sufficient conditions for the existence of such observers (with the same dimension as the system state) are known (see for instance [5]). Extensions for affine systems with nonlinear terms in output injection is straightforward. Unfortunately, these conditions are quite restrictive (and never fulfilled for the scalar case i.e. when $\dim y = 1$ cf. [16]; that is our case in the example).

- *Linear filtering:* If w is a white Gaussian noise then the theory of optimal filtering applies (see [19]) and gives the optimal estimates with respect to the noise (*i.e.* of minimal asymptotic variance of the error). The main problem comes from the assumption of white noise to be justified: if the process w has an unknown mean (or even does not have at all any mean), the estimation is biased.
- *L_2 disturbance attenuation:* If we abandon statistical assumptions on w for a deterministic framework, the theory of (linear) H^∞ filtering proposes to design an observer $\hat{x}$ attenuating as much as possible the effect of the disturbance w on the error $e = \hat{x} - x$. It consists in solving the following optimization problem (see Başar, Bernhard [1]): given γ a positive number and R a definite symmetric matrix, find an observer $\hat{x}$ such that:

  $$\sup_{w()} \int_0^\infty |\hat{x}(t) - x(t)|_R^2 - \gamma^2 |w(t)|^2 \, dt \ \leq 0.$$

  Then, the solution (when it exists) leads to a bounded error, whose dynamics is of the following form:

  $$\dot{e} = \tilde{A}_\gamma e + \tilde{B}_\gamma w \quad \text{with } \tilde{A}_\gamma \text{ Hurwitz.}$$

  Furthermore, if w belongs to L_2 (which implies that $\lim_{t\to\infty} w(t) = 0$), the error exponentially converges towards 0. But when w does not belong to L_2 (which is often the case with biological models), this approach is not well suited: how to choose γ ?

In fact, it is easy to exhibit elementary examples where, because of the uncertainty, the system is not observable (it is possible to have the same output history with different histories of unobserved variables), even if the exact model is observable. Therefore, in such cases, the asymptotic bias of estimation is unavoidable; how can we cope with that?

In biological modelling, even if the uncertain w does not fulfill any of the assumptions required by one of these methods, it is usually at least bounded. Our objective in this work is to present explicit methods to build guaranteed (dynamic) bounds on the unmeasured variables from bounds on the uncertainty and measurements y. We shall illustrate them on the robust problem described in the introduction.

Remark: For technical reasons, we do not consider in this paper errors on the measurement, contrary to the general filtering theory. This should be the subject of a forthcoming work.

3 The interval observers

As explained above, we adopt a deterministic approach of the uncertainty (or "worst-case" consideration). Moreover, we want to obtain an estimation even during the transients of the system, i.e. bounds that are valid since the initial time. Our spirit is the following:

Given bounds on the uncertainties in the model, we are looking for dynamic bounds on the estimation of the variables.

Because bounds are usually given by intervals, we shall build what we call "interval observers":

Definition 4 *A pair of systems $(\mathcal{S}^-, \mathcal{S}^+)$:*

$$(\mathcal{S}^-): \begin{cases} \dot{z}^-(t) = g^-(t, z^-(t), y(t)) \\ x^-(t) = l^-(t, z^-(t), z^+(t), y(t)) \end{cases} \qquad (\mathcal{S}^+): \begin{cases} \dot{z}^+(t) = g^+(t, z^+(t), y(t)) \\ x^+(t) = l^+(t, z^-(t), z^+(t), y(t)) \end{cases}$$

is an interval estimator for system $\mathcal{S}_0$ if for any (relatively) compact set $X_0 \subset X$, there exist $(z_-(0), z_+(0))$ such that the coupled system $(\mathcal{S}_0, \mathcal{S}^-, \mathcal{S}^+)$ verifies:

$$x^-(t) \leq x(t) \leq x^+(t), \quad \forall t \geq 0$$

for any initial condition $x(0) \in X_0$.

In other words, we are able to compute *two* estimates, an upper one and a lower one, that bounds the unmeasured variables. Of course, we are searching for the best final estimate (the interval $[x^-(t), x^+(t)]$ must be small or ideally tends to $\{x(t)\}$ as the time t is large) and we also wish, if possible, to adjust in some way the rate of convergence: in such cases we will say that we have computed an interval observer. But these goals are more difficult to achieve, and we insist here on the spirit of obtaining guaranteed estimation bounds at any time when bounds on the uncertainties are known.
Remark: The operator $\leq$ applied to vectors should be understood as a collection of inequalities applied between their components.

Now we are giving a detailed construction of these observers in the case of almost-linear system $(\mathcal{S})$ given by formula (2). We suppose that the function ϕ is uncertain with known bounds ϕ^-, ϕ^+.

Assumption: For system (2), there exist known applications $\phi^-, \phi^+ : \mathbb{R}^+ \times$

$\mathbb{R} \rightarrow \mathbb{R}^n$, measurable w.r.t. t and Lipschitz continuous w.r.t. y such that:

$$\phi^-(t,y) \leq \phi(t,y) \leq \phi^+(t,y), \quad \forall (t,y) \in \mathbb{R}^+ \times \mathbb{R} \tag{3}$$

For the "upper" estimate x^+, we write the system:

$$\dot{x}^+(t) = Ax^+(t) + \phi^+(t, y(t)) + K(Cx^+(t) - y(t)) \tag{4}$$

in the same spirit as the Luenberger one, and for the "lower" estimate:

$$\dot{x}^-(t) = Ax^-(t) + \phi^-(t, y(t)) + K(Cx^-(t) - y(t)). \tag{5}$$

Let us then consider the upper error $e^+(t) = x^+(t) - x(t)$, it verifies the linear equation:

$$\dot{e}^+ = (A + KC)e^+ + b^+(t)$$

with

$$b^+(t) = \phi^+(t, y(t)) - \phi(t, y(t)).$$

Because of known bounds on ϕ, $b^+(t)$ is always positive, and we can apply a result of the theory of cooperative systems, or more simply of positive linear systems [12]:

Lemma 1 *If the matrix $(A + KC)$ is cooperative (i.e. has nonnegative off-diagonal elements) then $e^+(0) \geq 0$ implies $e^+(t) \geq 0$ for all positive time t.*

For the proof, it is enough to check that the vector field is repulsive at the boundaries. We have similar equations and lemma for the lower error $e^-(t) = x(t) - x^-(t)$ and the total error $e(t) = e^-(t) + e^+(t)$.

We deduce the following theorems on interval observer:

Theorem 1 *Suppose that it is possible to choose the gains K such that $(A + KC)$ is cooperative, and suppose that we have initial estimated bounds on the initial values:*

$$x^-(0) \leq x(0) \leq x^+(0)$$

then the pair (4,5) is an interval estimator for system 2.

Theorem 2 *Suppose that the hypotheses of the above theorem are fulfilled and, moreover, that $(A + KC)$ is stable and that the total uncertainty $b(t) = \phi^+(t, y) - \phi^-(t, y)$ is bounded by a fixed positive vector B, then the total non-negative error $e(t)$ is asymptotically lower (term by term) than the nonnegative vector*

$$e_{max} = -(A + KC)^{-1}B$$

In particular, if some components of e_{max} are equal to zero, the corresponding components of $e(t)$ converge exponentially towards zero.

The proof of theorem 1 is only a consequence of the lemma 1. The proof of theorem 2 comes from the differential vector inequality:

$$(A + KC)e + b(t) \leq (A + KC)e + B$$

which implies (for the same initial conditions, because of the positivity of the linear system)

$$e(t) \leq e_m(t), \quad \forall t \geq 0$$

$e_m(t)$ being the solution of the autonomous linear equation $\dot{e}_m = (A+KC)e_m + B$. If $(A + KC)$ is stable, the output interval will be bounded if the input $b(t)$ is, and $e_m(t)$ will converge towards e_{max}.

4 An application to the population model

Firstly, we remark that it is easy to show that the variables of the model are positive and bounded when there exists a non-trivial equilibrium; we will suppose its existence (see [20]).

For the canonical example, we have (with the notations $\beta_1 = \alpha_1 + m_1$, $\beta_2 = \alpha_2 + m_2$ and $\beta_3 = m_3$):

$$A = \begin{pmatrix} -\beta_1 & 0 & 0 \\ \alpha_1 & -\beta_2 & 0 \\ 0 & \alpha_2 & -\beta_3 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

If we choose for the vector K of gains: $k_1 = k_2 = 0$ and $k_3 < 0$ then

$$A + KC = \begin{pmatrix} -\beta_1 & 0 & 0 \\ \alpha_1 & -\beta_2 & 0 \\ 0 & \alpha_2 & -\beta_3 + k_3 \end{pmatrix}$$

We can apply the results of the above section, because the matrix $A + KC$ is cooperative and stable and we have assumed that the uncertainties are bounded:

$$a^- < a(t) < a^+ \Longrightarrow \frac{a^- x_3}{b + x_3} < r(t, x_3) < \frac{a^+ x_3}{b + x_3}$$

and

$$c^- < c(t) < c^+ \Longrightarrow -c^+ x_3 < -c(t) x_3 < -c^- x_3$$

The upper observer is then:

$$\begin{cases} \dot{x}_1^+ &= -\beta_1 x_1^+ + \dfrac{a^+ x_3}{b + x_3} \\ \dot{x}_2^+ &= \alpha_1 x_1^+ - \beta_2 x_2^+ \\ \dot{x}_3^+ &= \alpha_2 x_2^+ - \beta_3 x_3^+ - c^- y + k_3(x_3^+ - y) \end{cases}$$

and the lower one:

$$\begin{cases} \dot{x}_1^- &= -\beta_1 x_1^- + \dfrac{a^- x_3}{b + x_3} \\ \dot{x}_2^- &= \alpha_1 x_1^- - \beta_2 x_2^- \\ \dot{x}_3^- &= \alpha_2 x_2^- - \beta_3 x_3^- - c^+ y + k_3(x_3^- - y) \end{cases}$$

By theorem 2, we obtain that the total error is always smaller than the vector $(A + KC)^{-1}B$ with (x_3^{max} is an upper bound on x_3):

$$B = \begin{pmatrix} (a^+ - a^-) \\ 0 \\ (c^+ - c^-) x_3^{max} \end{pmatrix}$$

This yields two consequences:

- if we choose k_3 negative enough, then the error on x_3 goes toward zero (one sees that by computing $(A + KC)^{-1}B$)
- if k_3 is negative enough and if uncertainty on the recruitment $(a^+ - a^-)$ is small, the error on the estimations of the variables will also be small. In other words, if one knows the recruitment, one can estimate exactly the young classes by measuring the oldest, even if there is an *unknown* harvesting effort.

Here we did not consider the problem of obtaining the best final asymptotic interval, or the fastest rate of convergence. In fact, it is possible to obtain more elaborated results (see [16]) and to show, for example in the case where there is no harvesting ($c(t) = 0$), that it is possible to estimate the variables with arbitrary convergence rate and accuracy. To illustrate, we show a simulation given in [16]. Fig. 2 show the unknown input (with random variations between two constant bounds), and Fig. 3 the three estimated variables (a-c and d-f) for two different choices of the gains K. The interval bounds are given in dashed lines.

5 An application to a wastewater treatment model

Now we consider a more complicated case, related to biological treatment of wastewater by cells feeding on a substrate in a bioreactor. The basic plant components of the activated sludge process include two tanks: the aerator and the settler. The aeration tank is supposed to be continuously stirred, involving the growth of suspended micro-organisms at the expense of organic material dissolved in wastewater. This biodegradation results in the production of more cell mass, carbon dioxide and water in the presence of the oxygen dissolved in the water. We assume that the total solid part is separated from the liquid by sedimentation in the settler. A portion of the sludge is recycled to the aerator, while the rest is regularly wasted to other sludge treatment units to maintain the biomass level in the system (FIG. 1).

We consider the cell growth of one bacterial population x on a single limiting organic substrate s as the basic biological reaction in the aeration tank. We suppose that due to the gravity all solid components will settle and concentrate in the bottom of the settling tank, whereas the sedimentation of soluble organic matter is insignificant. The goal is to obtain an output concentration of the substrate low enough, for a given fluctuating input s_{in} . A simplified model of activated sludge process is described by the equations (cf. [7]):

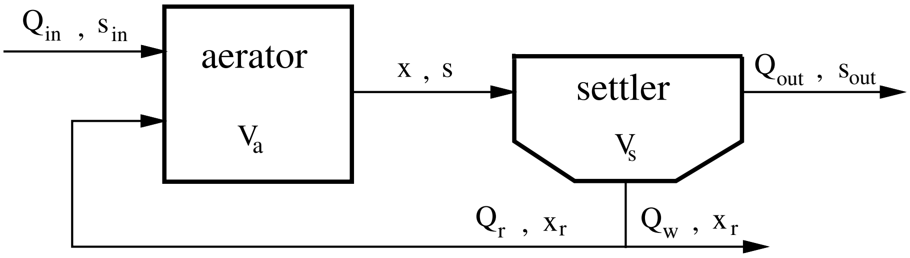

Fig. 1. Schematic diagram of the activated sludge process

$$\begin{cases} \dot{x} = \mu(\cdot)x - (1+r)D(t)x + rD(t)x_r \\ \\ \dot{s} = -\dfrac{\mu(\cdot)x}{Y} - (1+r)D(t)s + D(t)s_{in}(t) \\ \\ \dot{x_r} = v(1+r)D(t)x - v(w+r)D(t)x_r \end{cases} \tag{6}$$

with:

$$D(t) = \frac{Q_{in}}{V_a} \quad ; \quad r = \frac{Q_r}{Q_{in}} \quad ; \quad w = \frac{Q_w}{Q_{in}} \quad ; \quad v = \frac{V_a}{V_s}$$

where x, s and x_r are the state variables representing respectively the biomass, the substrate and the recycled biomass concentrations. Q_{in}, Q_{out}, Q_r and Q_w are successively the influent, effluent, recycle and waste flow rates. V_a and V_s represent the aerator and settler volumes. s_{in} corresponds to the substrate concentration in the feed stream. Y is the yield coefficient of biomass on substrate. $\mu(\cdot)$ is the specific growth rate.

The hypotheses are the following:

- We measure only the substrate s. We want to estimate x and x_r.
- Usually, the key function $\mu(\cdot)$ is a very complex time-varying function of state process and other several factors (pH, temperature, ...) and its analytical modeling is often uncertain. We suppose here that it is completely unknown.
- We assume that only bounds on the inflow concentration are known:

$$s_{in}^-(t) \leq s_{in}(t) \leq s_{in}^+(t) \quad \forall\, t \geq 0.$$

- We suppose also that a priori bounds (even loose) on initial values x_0 and x_{r0} are known.

We build an observer giving bounds on x and x_r, with the same technique as above, in addition to a total decoupling of the unknown function $\mu(\cdot)$. We make a change of variables to eliminate μ:

$$Z = X + \begin{bmatrix} Y \\ 0 \end{bmatrix} s; \quad Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad ; \quad X = \begin{bmatrix} x \\ x_r \end{bmatrix} \tag{7}$$

and obtain the reduced system:

$$\dot{Z} = D(t)\Big(A\ Z + B(s)\Big) \quad ; \quad Z_0 = \begin{bmatrix} x_0 + Ys_0 \\ x_{r0} \end{bmatrix}$$

$$A = \begin{bmatrix} -(1+r) & r \\ v(1+r) & -v(w+r) \end{bmatrix} \quad ; \quad B(s) = \begin{bmatrix} Ys_{in}(t) \\ -Yv(1+r)s \end{bmatrix}$$

we build the two estimators:

$$
\begin{cases}
\dot{\hat{Z}}^{+} = D(t)\Big(A\ \hat{Z}^{+} + B^{+}(s)\Big) \quad ; \quad \hat{Z}^{+}(0) = X_0^{+} + \begin{bmatrix} Y \\ 0 \end{bmatrix} s_0 \\
\\
\dot{\hat{Z}}^{-} = D(t)\Big(A\ \hat{Z}^{-} + B^{-}(s)\Big) \quad ; \quad \hat{Z}^{-}(0) = X_0^{-} + \begin{bmatrix} Y \\ 0 \end{bmatrix} s_0 \\
\\
\hat{X}^{+} = \hat{Z}^{+} - \begin{bmatrix} Y \\ 0 \end{bmatrix} s \\
\\
\hat{X}^{-} = \hat{Z}^{-} - \begin{bmatrix} Y \\ 0 \end{bmatrix} s
\end{cases}
\tag{8}
$$

with $B^{+}(s) = \begin{bmatrix} s_{in}^{+}(t) \\ -v(1+r)s \end{bmatrix} Y \quad ; \quad B^{-}(s) = \begin{bmatrix} s_{in}^{-}(t) \\ -v(1+r)s \end{bmatrix} Y$

Dropping technical details (cf. [7,6]), we can prove that it is an interval observer and obtain the following simulations: the variations of the measured variable s are represented in figure (FIG. 4.b). The time histories of the estimates of the bounds on x and x_r under the constraint that s_{in} is uncertain, are given in picture (FIG. 4, c and d). These two last figures show that, even with loose initial estimations on each bound of unmeasured variables, we obtain estimates of the uncertainty intervals with a reasonable accuracy.

6 Conclusion

We have presented a way of coping with uncertainties in the model when estimating the variables, given the outputs. Our basic idea is simple: *given deterministic bounds on the uncertainties, we obtain deterministic guaranteed dynamic intervals containing the variables to estimate.* Let us make some remarks:

- It is necessary to provide at initial time an estimate for the unknown variables. This can seem to be strong, but this estimate can be very loose (it is often the case in biology that it is possible to say that the variables are surely within some bounds).

- It remains to obtain practical conditions to guarantee a final asymptotic interval width in an optimal way, and to choose the rate of convergence towards this asymptotic interval. These problems can be solved in some cases (see [16]), but a lot remains to be done.
- We emphasize on that fact that uncertainties can depend on time, and so their bounds also. This can be used, for example, for inputs depending on an environment that fluctuates with time (light, temperature, ...)
- The method can also be applied when the uncertainties are of a more general form than the one given here, in particular when there is also uncertainty on the measurements y.
- The same method can also be used to estimate fixed parameters or slowly time varying parameters. The problem is that the system being often truly nonlinear with respect to some parameters, it may be hard to design an observer.

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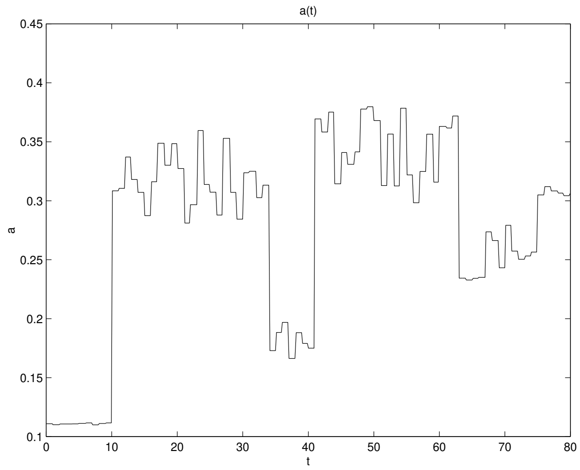
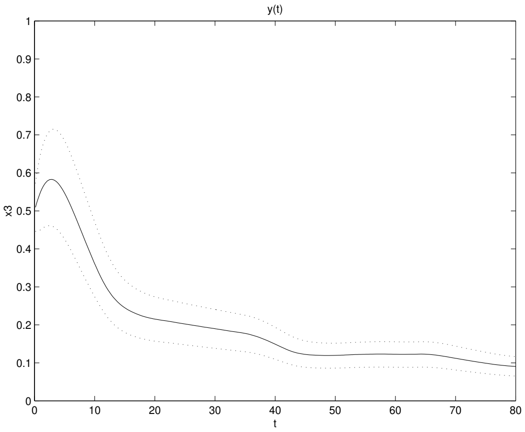
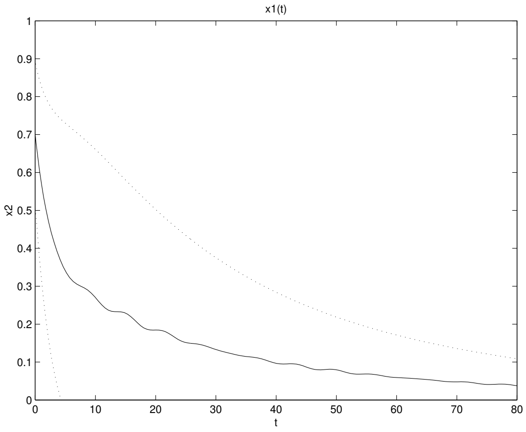


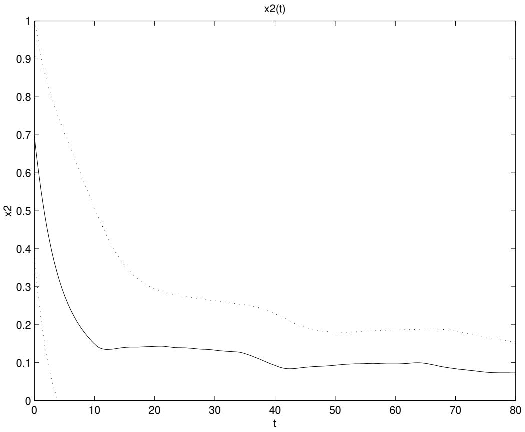
Fig. 2. Unknown input $a(t)$: the bounds are constant and equal to 0.1(lower) and 0.4(upper)



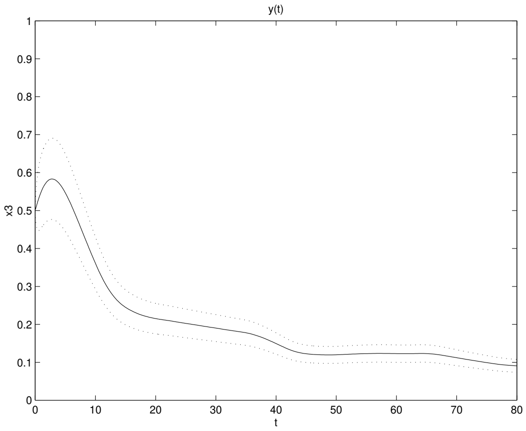
(a) Measured variable x_3



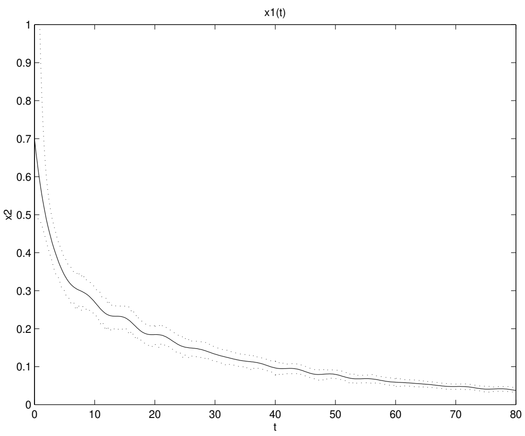
(b) Interval estimates of x_1



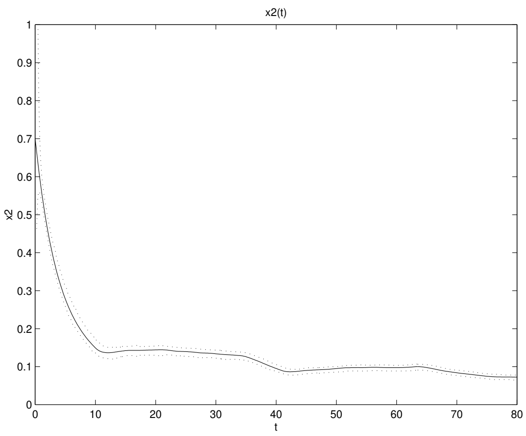
(c) Interval estimates of x_2



(d) Measured variable x_3, higher gains

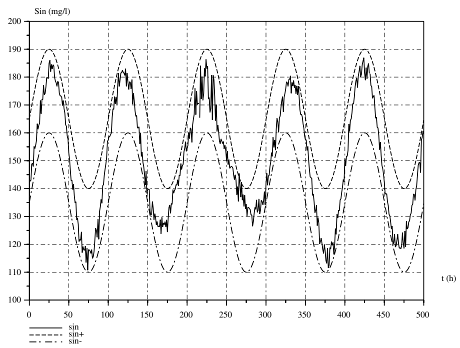


(e) Interval estimates of x_1, higher gains

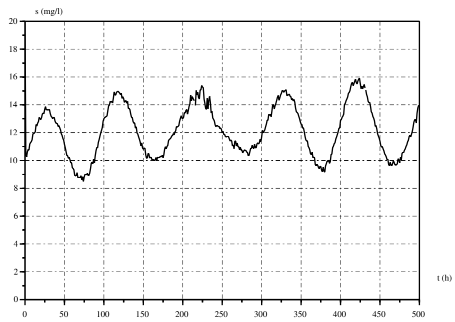


(f) Interval estimates of x_2, higher gains

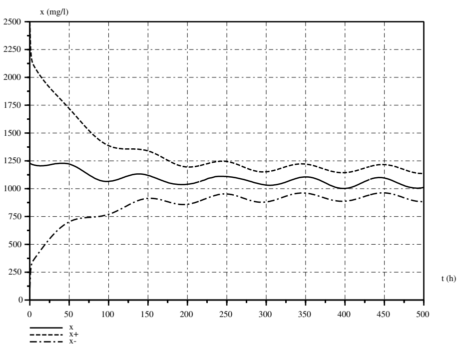
Fig. 3. Interval estimation (first example)



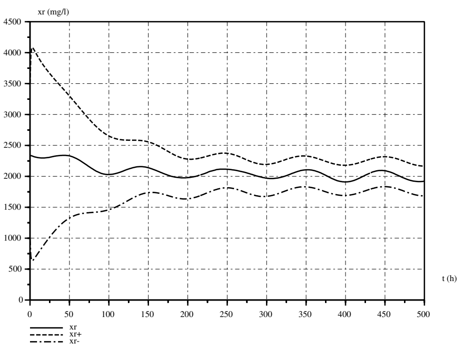
(a) Interval bounds for the input



(b) Measured variable s



(c) Bound estimates of x



(d) Bound estimates of x_r

Fig. 4. Interval estimation (second example)