Parallelotopic and practical observers for nonlinear uncertain systems

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Abstract : For a class of dynamical systems, with uncertain nonlinear terms considered as “unknown inputs”, we give sufficient conditions for observability. We show also that there does not exist any exact observer independent of the unknown inputs. Under the additional assumption that the uncertainty is bounded, we build *practical observers* whose error converges exponentially towards an arbitrarily small neighborhood of the origin. Under the hypothesis that bounds are available for the uncertain terms, we build *parallelotopic observers* providing time-varying bounds for the state variables, even when the system is not observable for unknown inputs These results are illustrated on a biological model of a structured population.

Key-words : Robust estimation, nonlinear observer, unknown inputs, interval observers, parallelotopes.

1 Introduction

This paper deals with the estimation (or observation) of the state of uncertain systems. Consider a nonlinear model of a system described by the continuous-time state equation :

$$(\mathcal{S}) : \left\{ \begin{array}{rcl} \dot{x} & = & f(t,x), \quad x(0)=x_0 \\ y & = & Cx \end{array} \right. \tag{1}$$

where the state x belongs to $\Omega \subset \mathbb{R}^n$ ($n \geq 2$), where the measured output y is scalar and where $C \in \mathcal{M}^{n\times 1}(\mathbb{R})$. Assume that the map $(t,x) \mapsto f(t,x)$ is measurable w.r.t to t and Lipschitz w.r.t. x. When f is perfectly known and the model is observable, one might be able to design an observer of the following form :

$$\dot{\hat{x}} = f(t,\hat{x}) + K(t,\hat{x})(C\hat{x} - y)$$

with an asymptotic or even exponential convergence towards zero of the error $x - \hat{x}$ (under some additional hypotheses, see the “high-gain” observers in [5, 10]). The rate of convergence of the error towards zero can then be adjusted by tuning the gain vector K.

In this paper, the function f will be assumed to be, in some sense, "uncertain", which is often much more realistic, especially when modeling biological systems [23]. Here, we shall assume that the output vector y is measured in continuous time, without noise. Our aim is then to estimate the state variables, taking into account this uncertainty of the model. Consider, for instance, the dynamics of a biological population structured in three age classes (larvae, young adults and adults), whose stocks (or densities) are $(x_1, x_2, x_3) \in \mathbb{R}_+^3$. Assume that these state variables satisfy

$$\left\{\begin{array}{lcl} \dot{x}_1 & = & -\alpha_1 x_1 - m_1 x_1 + r(t, x_2, x_3) \\ \dot{x}_2 & = & \alpha_1 x_1 - \alpha_2 x_2 - m_2 x_2 \\ \dot{x}_3 & = & \alpha_2 x_2 - m_3 x_3 \end{array}\right.$$

where the positive coefficients α_i and m_i respectively represent the growth and mortality rates. The reproduction law $r(t, x_2, x_3)$ is usually not well known but can often be assumed to fluctuate over time between two known functions $\underline{r}$, $\overline{r}$:

$$\underline{r}(t, x_2, x_3) \leq r(t, x_2, x_3) \leq \overline{r}(t, x_2, x_3)$$

The only measured output is the stock of the adults population : $y(t) = x_3(t)$. What can then be said about the estimation of x_1 and x_2 ? We shall return to this example in Section 6.

More generally, we shall assume that the nonlinear map f is not well known, but is functionally bounded, *i.e.* there exist known maps $\underline{f}$, $\overline{f}$ such that :

$$\underline{f}(t, x) \leq f(t, x) \leq \overline{f}(t, x), \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \tag{2}$$

A first possible approach to designing robust observers with respect to the uncertainty on f is to choose an arbitrary map $\widehat{f}$ (bounded below by $\underline{f}$ and above by $\overline{f}$) and to consider the vector

$$w(t) = \widehat{f}(t, x(t)) - f(t, x(t)) \tag{3}$$

as a disturbance. It then becomes possible to test whether there exist observers that do not depend on w (what is usually called in the literature *observers with unknown inputs*). For the linear case, necessary and sufficient conditions for the existence of such observers (with the same dimension as the state of the system) are known, see for instance Darouach and al. [7]. But, for the nonlinear case, even when the model $(\mathcal{S})$ is differentially observable with unknown f (see the definition 1 below), we shall show that there does not exist any *exact* observer (see Section 2). Designing discontinuous or *sliding-mode* observers is another available technique but the convergence of the error towards zero requires restrictive matching conditions of the uncertainty (see, for instance, [8]).

The most classical approach may be to use a Kalman filter [1], but such a filter will be optimal only under quite restrictive assumptions, the most important of which being that the nominal system has to be linear, and the disturbance $w(t)$ must be generated by a white noise. In its deterministic (or robust) version, the disturbance has to fulfill an integral quadratic constraint [25].

Yet another approach could be to design an observer $\widehat{x}$ attenuating the effect of the disturbance w on the error as much as possible. For this purpose, H^∞ filtering leads to studying theory the following optimization problem (see Başar, Bernhard [2]) : given a positive number γ and a symmetric positive-definite matrix R, find $\widehat{x}$ such that

$$\sup_{w()} \int_0^\infty ||\widehat{x}(t) - x(t)||_R^2 - \gamma^2 ||w(t)||^2 \, dt \ \leq 0$$

When the solution exists, the resulting estimation error satisfies

$$\dot{e} = F_\gamma e + G_\gamma w$$

with F_γ Hurwitz. When w belongs to L^2, the error exponentially converges towards 0. This hypothesis, however, implies that $\lim_{t\to\infty} w(t) = 0$, *i.e.* that the disturbance vanishes with time, which is often fairly unrealistic. Such an hypothesis will not be necessary here.

We shall show in Section 3 that there exist sufficient conditions on the set $\mathcal{F}$ of unknown f for the system $(\mathcal{S})$ to be differentially observable with unknown f. We shall assume in what follows that the model $(\mathcal{S})$ is differentially observable with unknown f. The problem is then to build an observer, which cannot be of the usual form, as will be made clearer in Section 3. We propose to consider a weaker kind of observers , which we shall call *practical observers*. For these observers, the error is no longer required to converge asymptotically towards 0 but it must converge instead to an arbitrarily small neighborhood of the origin. We shall show in Section 4 that this is feasible when the uncertainty on f is bounded (*i.e.* when $\overline{f} - \underline{f}$ is uniformly bounded).

An alternative approach, which can be used even when the model is not observable for unknown inputs, is based on a *dynamical interval analysis*, in the spirit of the interval computation of solutions of o.d.e. [6, 27] but with control theoretical tools. Given guaranteed intervals on uncertainties on the components of f, we compute guaranteed intervals for the variables in some coordinates; we thus build what we have call *interval observers* [12]. In the original coordinates, we then obtain a guaranteed time varying parallelotope for the unmeasured variables. When the model is differentially observable, the asymptotic diameter of this parallelotope can be made arbitrarily small.

Several state bounding techniques for uncertain dynamical systems already exist in the literature. Most of them deal with linear discrete-time systems and attempt to compute recursively sets guaranteed to contain all possible state vectors, subject to bounded uncertainties (see for instance [28, 4]). More recently, in a non-linear discrete-time context, new techniques have been proposed to compute guaranteed or exact sets, consistent with past observation and known bounds on the uncertainties [15, 18]. Their algorithms are based on interval analysis [22, 16] but suffer from their complexity for models with large dimension. For linear continuous time systems, ellipsoidal techniques provide an effective way to compute guaranteed sets [20]). For nonlinear continuous time systems, a few attempts for the characterization of the set of possible states conditioned to past observations have been proposed, using level sets of functions [14, 3]. These results are expressed in terms of non-stationary p.d.e., which do not seem currently practical.

Our aim here is, at the expense of stronger hypotheses, to obtain mathematical conditions for some kind of observability, to build explicit finite-dimensional observers, and to characterize their asymptotic behavior. The present work is a generalization of a simpler case [26], where the uncertain functions depended only on the measured output y.

What could be the advantages of such an approach, compared with more classical ones, where one seeks bounds on the norm of the error? Firstly, we are able to characterize in a more subtle way the influence of uncertainty on the behavior of the model (cf. the decomposition of f in Section 5). Moreover, when bounds on the uncertain parts of the inputs are provided independently on each coordinate (*i.e.* by intervals), we would like to take advantage of this information. So we naturally seek guaranteed bounds on each coordinate of the unmeasured variables (*i.e.* in terms of intervals or parallelotopes).

The paper is organized as follows: after some definitions (Section 2), we propose testable conditions for the observability with unknown inputs (Section 3), and compute practical observers if the model is observable (Section 4). An alternative route, which can be followed even if the model is not observable is to compute "parallelotopic observers" as suggested in Section 5, where the asymptotic behavior of these observers is also studied. Finally, these results are illustrated on an example inspired by biology in Section 6.

2 Definitions and assumptions

We assume, without loss of generality (see the remarks below), that the nonlinear model (1) can be written as

$$(\mathcal{S}) : \left\{ \begin{array}{rcl} \dot{x} & = & Ax + \psi(t,x), \quad x(0) = x_0 \\ y & = & Cx \end{array} \right. \qquad x \in \mathbb{R}^n \tag{4}$$

where the map ψ is measurable w.r.t. t and Lipschitz w.r.t. x. $A \in \mathcal{M}^{n\times n}(\mathbb{R})$ and $C \in \mathcal{M}^{1\times n}(\mathbb{R})$ are such that

Assumption A1 : The pair (A,C) is observable.

Remark 1 *If (1) is not defined on all $\mathbb{R}^n$ or if ψ is not Lipschitz w.r.t. x on all $\mathbb{R}^n$ but only on a domain $\Omega \subset \mathbb{R}^n$, then one can consider a Lipschitz extension of ψ on all $\mathbb{R}^n$ (see for instance the MacShane formula [21]).*

Remark 2 *When Assumption A1 is not fulfilled, provided that $C \neq 0$, it is always possible to choose a matrix $\widetilde{A}$ such that the pair $(\widetilde{A},C)$ is observable and define $\widetilde{\psi}(t,x) = \psi(t,x) + (A-\widetilde{A})x$. Then, (1) can be rewritten as*

$$(\mathcal{S}) : \left\{ \begin{array}{rcl} \dot{x} & = & \widetilde{A}x + \widetilde{\psi}(t,x), \\ y & = & Cx \end{array} \right.$$

Although the solutions of the system (4) are well defined for functions ψ that are only measurable w.r.t. t and Lipschitz w.r.t. x, we shall consider, when studying differential observability with inputs, a dense subset of smooth "inputs" ψ (see Gauthier and Kupka

[11]), for which the map $t \to y(t)$ can be differentiated $n-1$ times :

Assumption A2 : The known set Ψ to which the function ψ belongs is such that $\Psi \cap \mathcal{C}^{n-2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is dense in Ψ [1].

We now define a notion of observability (cf. [24]) that will be sufficient for our needs.

Definition 1 *Fix a set Ψ that fulfills A2. For any $\psi \in \Psi \cap \mathcal{C}^{n-2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, consider the map :*

$$x \longmapsto \Gamma_t(\psi, x) = \begin{pmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ \vdots \\ y^{(n-1)}(t) \end{pmatrix},$$

1. the model $(\mathcal{S})$ is said to be differentially observable (of rank n) *(for known $\psi \in \Psi$) if and only if the map $x \mapsto \Gamma_t(\psi, x)$ is injective for any time t and any $\psi \in \Psi \cap \mathcal{C}^{n-2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$.*

2. the model $(\mathcal{S})$ is said to be differentially observable (of rank n) for unknown $\psi \in \Psi$ *if and only if*

$$\forall \psi^a, \psi^b \in \Psi \cap \mathcal{C}^{n-2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n), \left\{ \Gamma_t(\psi^a, x^a) = \Gamma_t(\psi^b, x^b) \right\} \implies \left\{ x^a = x^b \right\}$$

for any time t.

3 Observability with unknown inputs

Let $\mathcal{O}$ be the observability matrix of the pair (A, C), S the anti-shift operator and $\kappa(P)$ the condition number of a nonsingular matrix P :

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}, \qquad \kappa(P) = ||P^{-1}|| \, ||P|| \, .$$

It is easy to write down sufficient conditions for differential observability for unknown ψ.

Proposition 1 *If*

1. *$(\mathcal{S})$ is differentially observable (for known $\psi \in \Psi$),*
2. *For any ψ^a, ψ^b in $\Psi \cap \mathcal{C}^{n-2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, $CA^i(\psi^a(t,x) - \psi^b(t,x)) = 0$, $\forall i \in 0, \cdots, n-2$,*

then $(\mathcal{S})$ is differentially observable for unknown $\psi \in \Psi$.

[1] $\mathcal{C}^{n-2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is the set of maps $\mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}^n$ continuously differentiable $n-2$ times.

Proof For an arbitrary known function $\psi \in \Psi \cap \mathcal{C}^{n-2}(\mathrm{I\!R}_+ \times \mathrm{I\!R}^n, \mathrm{I\!R}^n)$, we have :

$$\Gamma_t(\psi, x) = \mathcal{O}x \ + \ \sum_{k=0}^{n-2} S^{k+1}\mathcal{O}\, \frac{d^k\psi}{dt^k}(t,x).$$

So, for $\psi^a, \psi^b \in \Psi \cap \mathcal{C}^{n-2}(\mathrm{I\!R}_+ \times \mathrm{I\!R}^n, \mathrm{I\!R}^n)$, we have :

$$\begin{aligned}
\Gamma_t(\psi^a, x^a) = \Gamma_t(\psi^b, x^b) \ &\Rightarrow \ \mathcal{O}(x^a - x^b) + \sum_{k=0}^{n-2} S^{k+1}\mathcal{O}\left(\frac{d^k\psi^a}{dt^k}(t,x^a) - \frac{d^k\psi^b}{dt^k}(t,x^b)\right) = 0 \\
&\Rightarrow \ \mathcal{O}(x^a - x^b) + \sum_{k=0}^{n-2} S^{k+1}\mathcal{O}\left(\frac{d^k\psi^a}{dt^k}(t,x^a) - \frac{d^k\psi^a}{dt^k}(t,x^b)\right) \\
&\qquad + \sum_{k=0}^{n-2} S^{k+1}\mathcal{O}\left(\frac{d^k\psi^a}{dt^k}(t,x^b) - \frac{d^k\psi^b}{dt^k}(t,x^b)\right) = 0
\end{aligned}$$

From Condition 2, we have :

$$\sum_{k=0}^{n-2} S^{k+1}\mathcal{O}\left(\frac{d^k\psi^a}{dt^k}(t,x^b) - \frac{d^k\psi^b}{dt^k}(t,x^b)\right) = 0.$$

Then,

$$\begin{aligned}
\Gamma_t(\psi^a, x^a) = \Gamma_t(\psi^b, x^b) \ &\Rightarrow \ \mathcal{O}(x^a - x^b) + \sum_{k=0}^{n-2} S^{k+1}\mathcal{O}\left(\frac{d^k\psi^a}{dt^k}(t,x^a) - \frac{d^k\psi^a}{dt^k}(t,x^b)\right) = 0 \\
&\Rightarrow \ \Gamma_t(\psi^a, x^a) = \Gamma_t(\psi^a, x^b).
\end{aligned}$$

According to Condition 1, $(\mathcal{S})$ is differentially observable for any known ψ^a, so $x^a = x^b$, thus the differential observability of $(\mathcal{S})$ for unknown ψ. ∎

Remark 3 *As soon as there exist two maps $\psi^a, \psi^b \in \Psi \cap \mathcal{C}^{n-2}(\mathrm{I\!R}_+ \times \mathrm{I\!R}^n, \mathrm{I\!R}^n)$ and $(t,x) \in \mathrm{I\!R}_+ \times \mathrm{I\!R}^n$ such that $\psi^a(t,x) - \psi^b(t,x) \notin ker(S\mathcal{O})$, then $(\mathcal{S})$ is not differentially observable for unknown ψ.*

When A1 is satisfied, $\mathcal{O}$ has full rank and the dimension of $\ker(S\mathcal{O})$ is one. This leads us to consider particular models for which the set Ψ has the structure specified by the following hypothesis.

Hypothesis H1 : There exist a known map $\varphi : \mathrm{I\!R}_+ \times \mathrm{I\!R} \to \mathrm{I\!R}^n$ and a fixed vector d_1 in $\ker(S\mathcal{O}) \setminus \{0\}$ such that :

$$\forall \psi \in \Psi, \quad \psi(t,x) = \varphi(t, Cx) + \psi_1(t,x) d_1, \quad \forall (t,x) \in \mathrm{I\!R}_+ \times \mathrm{I\!R}^n$$

where the unknown functions ψ_1 are scalar.

Remark 4 *H1 means that the models considered are basically in output feedback form : $\dot{x} = Ax + \varphi(t,y)$ (for which the observability of the pair (A,C) is a sufficient condition to build an observer with linear error dynamics [13]) with an additional uncertain term in $ker(S\mathcal{O})$.*

Corollary 2 *Under H1, $(\mathcal{S})$ is differentially observable for unknown $\psi \in \Psi$.*

Proof For any known $\psi \in \Psi \cap \mathcal{C}^{n-2}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, we have

$$\Gamma_t(\psi, x) = \mathcal{O}x \; + \; \sum_{k=0}^{n-2} S^{k+1}\mathcal{O}\frac{d^k\varphi}{dt^k}(t,y)$$

x satisfies

$$x = \mathcal{O}^{-1}\left(\Gamma_t(\psi, x) - \; \sum_{k=0}^{n-2} S^{k+1}\mathcal{O}\frac{d^k\varphi}{dt^k}(t,y)\right)$$

the right-hand side of which depends only on the measured output $y(t)$ and its derivatives, thus the map $x \to \Gamma_t(\psi, x)$ is injective. Condition 1 of Proposition 1 is then fulfilled. Condition 2 is also fulfilled under H1. So, by Proposition 1, $(\mathcal{S})$ is differentially observable for unknown $\psi \in \Psi$. ∎

Under H1, it seems natural to look for *exact* observers *i.e.* finite-dimensional dynamical systems that can be written as

$$(\widehat{\mathcal{S}}) : \; \left\{ \begin{array}{rcl} \dot{z} & = & \widehat{f}(t,z,y), \quad z(0) = z_0 \\ \widehat{x} & = & g(z,y) \end{array} \right. \qquad z \in \mathbb{R}^p \quad (p \geq n) \tag{5}$$

with $\widehat{f}$ measurable w.r.t. t and $\widehat{f}, g$ Lipschitz w.r.t. (z, y), such that for any pair $(x_0, z_0) \in \mathbb{R}^n \times \mathbb{R}^p$, the solution of the coupled system $(\mathcal{S}, \widehat{\mathcal{S}})$ satisfies $\lim_{t\to\infty} ||\widehat{x}(t) - x(t)|| = 0$ for any unknown $\psi \in \Psi$. The next proposition indicates that this search is bound to fail.

Proposition 3 *Under H1, there does not exist any exact observer for unknown ψ.*

Proof An exact observer must leave the manifold $x = g(z, Cx)$ invariant for any ψ, which implies, at points (z, y) where g is differentiable (by Rademacher theorem [9], the Lipschitz map g is differentiable almost everywhere) :

$$Ax + \varphi(t,y) + \psi_1 d_1 = g_z\widehat{f}(t,z,Cx) + g_yC(Ax + \varphi(t,y) + \psi_1 d_1)$$

or

$$(I - g_yC)(Ax + \varphi(t,y)) - g_z\widehat{f}(t,z,Cx) = \psi_1(g_yC - I)d_1$$

for any unknown scalar ψ_1. So $d_1 \in \ker(g_yC - I)$ *i.e.* $d_1 = g_yCd_1$. But, under H1, $Cd_1 = 0$, which implies $d_1 = 0$, thus a contradiction. ∎

Remark 5 *For differentially observable systems with known inputs, it is possible to build exact observers under some additional regularity properties [10]. Proposition 3 show that this is no longer possible in the presence of unknown inputs.*

Remark 6 *For the scalar-output case considered in this paper, it comes as no surprise that there does not exist any observer with unknown inputs, as the conditions of [7] are never fulfilled under H1.*

4 Practical observers

In this section, we assume that H1 is fulfilled, so the system is differentially observable with unknown inputs, according to Corollary 2. Since an exact observer does not exist, as indicated by Proposition 3, we shall search for what we shall call *practical observers*, converging towards a small neighborhood of the actual state vector.

We consider the estimates $\widehat{x}$ for the state of $(\mathcal{S})$ as outputs of a family $(\widehat{\mathcal{S}})$ of finite-dimensional dynamical systems of the form (5) parameterized by a vector of gains K. We begin by a definition that imposes a practical convergence of the estimation error.

Definition 2 **A weak practical observer** *is a family* $(\widehat{\mathcal{S}})$ *such that for any* $\epsilon > 0$, *a vector* K *of gains can be found ensuring*

$$\forall (x_0, \widehat{x}_0),\ \exists T > 0,\ ||\widehat{x}(t) - x(t)|| \leq \epsilon, \quad \forall t > T$$

The next definition imposes that the rate of convergence towards the ball of radius ϵ can be adjusted arbitrarily (with the gain K).

Definition 3 **A strong practical observer** *is a family* $(\widehat{\mathcal{S}})$ *such that for any* $\epsilon > 0$, *a vector* K *of gains can be found ensuring*

$$\exists \lambda(\epsilon) > 0,\ \forall (x_0, \widehat{x}_0),\ ||\widehat{x}(t) - x(t)|| \leq \epsilon + e^{-\lambda(\epsilon)t}(||\widehat{x}(0) - x(0)|| - \epsilon)$$

with $\lambda(\epsilon) \to +\infty$ *when* $\epsilon \to 0$.

We shall now consider observers $(\widehat{\mathcal{S}})$ of the following form :

$$\dot{\widehat{x}} = A\widehat{x} + \varphi(t, y) + \widehat{\psi}_1(t, \widehat{x})d_1 + K(C\widehat{x} - y) \tag{6}$$

where the vector K and the function $\widehat{\psi}_1$ have to be chosen.

When (A, C) is observable, it is well known [17] that A is similar to a companion matrix : take

$$H = \mathcal{O}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathrm{P} = \begin{pmatrix} H & AH & \dots A^{n-1}H \end{pmatrix},$$

then $CH = CAH = \ldots = CA^{n-2}H = 0$ and $CA^{n-1}H = 1$. So

$$\mathcal{O}P = \begin{pmatrix} \mathrm{O} & & & 1 \\ & & 1 & \\ & \cdots & & \\ 1 & & & * \end{pmatrix}$$

P is non singular and we have :

$$\tilde{A} = P^{-1}AP = \begin{pmatrix} 0 & \dots & 0 & -a_n \\ 1 & & \mathrm{O} & \vdots \\ & \ddots & & \vdots \\ \mathrm{O} & & 1 & -a_1 \end{pmatrix} \qquad \tilde{C} = CP = \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}$$

where $X^n + a_1 X^{n-1} + \ldots + a_n = 0$ is the characteristic polynomial of A.

We shall use the following notation for diagonal and Vandermonde matrices :

$$\Delta_{\{\lambda_j\}} = \begin{pmatrix} \lambda_1 & & & O \\ & \lambda_2 & & \\ & & \ddots & \\ O & & & \lambda_n \end{pmatrix} \qquad V_{\{\lambda_j\}} = \begin{pmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \ldots & \lambda_2^{n-1} \\ \vdots & & & \\ 1 & \lambda_n & \ldots & \lambda_n^{n-1} \end{pmatrix}$$

Hypothesis H2 : Under H1, there exists a positive number L such that ψ_1 is Lipschitz w.r.t. x with constant L, for any $\psi \in \Psi$. Moreover, there exists functions $\underline{\psi}_1$ and $\overline{\psi}_1$ measurable w.r.t. t and Lipschitz w.r.t. x such that :

$$\forall \psi \in \Psi, \quad \underline{\psi}_1(t,x) \leq \psi_1(t,x) \leq \overline{\psi}_1(t,x), \quad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$$

and a positive number M such that $\overline{\psi}_1 - \underline{\psi}_1$ is uniformly bounded by M.

Remark 7 *If the functions $\psi \in \Psi$ are not Lipschitz with constant L globally on $\mathbb{R}^n$, but there exists a compact set $\mathcal{K}$ of $\mathbb{R}^n$ which is positively invariant by $(\mathcal{S})$ for any $\psi \in \Psi$, then one can consider, instead of Ψ, the set $\widetilde{\Psi}$ of unknown functions $\widetilde{\psi}$ defined as follows :*

$$\widetilde{\psi}(t,x) = \left| \begin{array}{ll} \psi(t,x) & \text{if } ||x|| \leq r \\ \psi\left(t, r\dfrac{x}{||x||}\right) & \text{if } ||x|| > r \end{array} \right. \qquad \psi \in \Psi$$

where r is a positive number such that $\mathcal{K} \subset \mathbb{B}(0,r)$. Then, H2 is fulfilled.

Proposition 4 *Under H1 and H2, for any function $\widehat{\psi}_1$ measurable w.r.t. t and Lipschitz w.r.t. x such that $\underline{\psi}_1(t,x) \leq \widehat{\psi}_1(t,x) \leq \overline{\psi}_1(t,x)$, $\forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, K can be chosen such that (6) is a strong practical observer.*

Proof The matrix $P^{-1}(A+KC)P$ is in companion form for any K. If K is chosen such that $A+KC$ has n distinct real negative eigenvalues $\lambda_1, \ldots \lambda_n$, then $P^{-1}(A+KC)P$ is diagonalizable by the Vandermonde matrix $V_{\{\lambda_j\}}$:

$$V_{\{\lambda_j\}} P^{-1}(A+KC)PV_{\{\lambda_j\}}^{-1} = \Delta_{\{\lambda_j\}}.$$

Make the change of variable :

$$z = V_{\{\lambda_j\}} P^{-1} x, \quad \widehat{z} = V_{\{\lambda_j\}} P^{-1} \widehat{x},$$

the error $e = z - \widehat{z}$ then satisfies

$$\dot{e} \; = \; \Delta_{\{\lambda_j\}} e \; + \; (\psi_1(t,x) - \widehat{\psi}_1(t,\widehat{x})) V_{\{\lambda_j\}} P^{-1} d_1.$$

H1 implies that

$$\mathcal{O} d_1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \delta \end{pmatrix} \;\Leftrightarrow\; d_1 = \delta H \;\Leftrightarrow\; P^{-1} d_1 = \delta \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \;\Rightarrow\; V_{\{\lambda_j\}} P^{-1} d_1 = \delta \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$$

If $\overline{\lambda} = \min_j |\lambda_j|$, then

$$\begin{aligned}
\frac{d}{dt}||e|| &\leq -\overline{\lambda}||e|| + |\psi_1(t,x) - \widehat{\psi}_1(t,\widehat{x})|\,||V_{\{\lambda_j\}}P^{-1}d_1|| \\
&= -\overline{\lambda}||e|| + |\psi_1(t,x) - \widehat{\psi}_1(t,\widehat{x})|\sqrt{n}|\delta| \\
&= -\overline{\lambda}||e|| + (|\psi_1(t,x) - \psi_1(t,\widehat{x})| + |\psi_1(t,\widehat{x}) - \widehat{\psi}_1(t,\widehat{x})|)\sqrt{n}|\delta| \\
&\leq (-\overline{\lambda} + L||PV^{-1}_{\{\lambda_j\}}||\sqrt{n}|\delta|)||e|| + M\sqrt{n}|\delta| \quad \text{(by H2)}
\end{aligned}$$

Since the coefficients of $V^{-1}_{\{\lambda_j\}}$ are rational functions of λ_j of degree at most zero, we can choose eigenvalues of the form $\lambda_j(\theta) = \theta\lambda'_j$ (where the set $\{\lambda'_j\}$ is fixed and θ is a positive number) to obtain :

$$\lim_{\theta\to\infty} -\overline{\lambda}(\theta) + L||PV^{-1}_{\{\lambda_j(\theta)\}}||\sqrt{n}|\delta| = -\infty$$

So, for θ large enough, $e(t)$ converges exponentially towards the ball :

$$\mathbb{B}\left(0, \frac{M\sqrt{n}|\delta|}{\left|-\overline{\lambda}(\theta) + L||PV^{-1}_{\{\lambda_j(\theta)\}}||\sqrt{n}|\delta|\right|}\right).$$

Define

$$\rho(\theta) = ||PV^{-1}_{\{\lambda_j(\theta)\}}||\frac{M\sqrt{n}|\delta|}{\left|-\overline{\lambda}(\theta) + L||PV^{-1}_{\{\lambda_j(\theta)\}}||\sqrt{n}|\delta|\right|}, \text{ then } \rho(\theta) \to 0 \text{ when } \theta \to \infty$$

The error in the original coordinates $(x - \widehat{x})$ then converges exponentially towards the ball $\mathbb{B}(0, \rho(\theta))$, which can be made arbitrary small by taking large values of θ. The rate of convergence can thus be made arbitrarily fast. ∎

5 Guaranteed parallelotopic observers

We relax now the hypothesis of differential observability and focus on obtaining guaranteed estimates of $x(t)$, under the assumption that bounds are available for all uncertain quantities.

Hypothesis H2b : There exists a positive number L such that any $\psi \in \Psi$ is Lipschitz w.r.t. x with constant L. Moreover, there exist known maps $\underline{\psi}$, $\overline{\psi}$ in Ψ and a known number $M < +\infty$ such that :

$$\left\{\begin{array}{l} \underline{\psi}(t,x) \leq \psi(t,x) \leq \overline{\psi}(t,x), \\ ||\overline{\psi}(t,x) - \underline{\psi}(t,x)|| \leq M, \end{array}\right. \qquad \forall \psi \in \Psi,\ \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$$

Note : An inequality between two vectors of $\mathbb{R}^n$ should be understood component wise, as a collection of n inequalities between their coordinates.

Remark 8 *If the functions ψ in Ψ are not globally Lipschitz but there exists a compact set invariant by the system $(\mathcal{S})$ for any $\psi \in \Phi$, then one might consider an alternative set $\widetilde{\Psi}$ of unknown $\widetilde{\psi}$, as suggested in the remark 7.*

In what follows, we shall use the interval notation [22] :

$$\psi(t,x) \in [\psi](t,x) = [\underline{\psi}(t,x), \overline{\psi}(t,x)].$$

Let $\mathcal{I}^n$ be the set of intervals of $\mathbb{R}^n$: $\mathcal{I}^n = \{[\underline{\xi}, \overline{\xi}],\ (\underline{\xi}, \overline{\xi}) \in \mathbb{R}^{2n} \,|\, \underline{\xi} \leq \overline{\xi}\}$, We can also use the notation of interval computation on $\mathcal{I}^n$: when F is a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^m$, the notation $[q] = F([p])$ stands for $[q] = [\underline{q}, \overline{q}]$ where :

$$\underline{q}_i = \min_{\pi \in [p]} F_i(\pi), \quad \overline{q}_i = \max_{\pi \in [p]} F_i(\pi) \quad (i = 1 \cdots m).$$

When p is a function of time and $[p(t)] = [\underline{p}(t), \overline{p}(t)]$, by convention $[\dot{p}]$ denotes $[\dot{\underline{p}}, \dot{\overline{p}}]$.

Lemma 5 *If a function $\phi : \mathbb{R}^n \mapsto \mathbb{R}$ is Lipschitz with constant L, then the functions $\underline{\phi}$, $\overline{\phi} : \mathbb{R}^{2n} \mapsto \mathbb{R}$ such that :*

$$\phi([\underline{\xi}, \overline{\xi}]) = [\, \underline{\phi}(\underline{\xi}, \overline{\xi}) \,,\, \overline{\phi}(\underline{\xi}, \overline{\xi}) \,], \quad \forall [\underline{\xi}, \overline{\xi}] \in \mathcal{I}^n$$

are also Lipschitz with constant L.

Proof Fix $\underline{\xi}$ and consider $\overline{\xi}_1$, $\overline{\xi}_2$.
If $\overline{\phi}(\underline{\xi}, \overline{\xi}_1) = \overline{\phi}(\underline{\xi}, \overline{\xi}_2)$, the result is trivial.
Assume that $\overline{\phi}(\underline{\xi}, \overline{\xi}_2) > \overline{\phi}(\underline{\xi}, \overline{\xi}_1)$ (otherwise, the roles of $\overline{\xi}_1$ and $\overline{\xi}_2$ can be exchanged). Consider $\widehat{\xi}_2$ such that $\overline{\phi}(\underline{\xi}, \overline{\xi}_2) = \phi(\widehat{\xi}_2)$. Necessarily, $\widehat{\xi}_2 \in [\underline{\xi}, \overline{\xi}_2] \setminus [\underline{\xi}, \overline{\xi}_1]$. Take now $\xi_1 \in [\underline{\xi}, \overline{\xi}_1]$ such that $||\xi_1 - \widehat{\xi}_2|| \leq ||\overline{\xi}_1 - \overline{\xi}_2||$ (for instance $\xi_1 = \mathrm{Proj}_{[\underline{\xi}, \overline{\xi}_1]}(\widehat{\xi}_2)$). Then, we have $0 \leq \overline{\phi}(\underline{\xi}, \overline{\xi}_2) - \overline{\phi}(\underline{\xi}, \overline{\xi}_1) \leq \phi(\widehat{\xi}_2) - \phi(\xi_1) \leq L||\xi_1 - \widehat{\xi}_2|| \leq L||\overline{\xi}_1 - \overline{\xi}_2||$. So, $\overline{\phi}$ is Lipschitz with constant L, w.r.t. $\overline{\xi}$. Similarly, $\overline{\phi}$ is Lipschitz with constant L, w.r.t. $\underline{\xi}$. Substituting $-\phi$ for ϕ, we obtain that $\underline{\phi}$ is also Lipschitz with constant L. ∎

Consider again the matrix P defined in the Section 4 and define the map

$$\gamma :\ (t,x) \mapsto \gamma(t,x) = P^{-1}\psi(t,x).$$

Under H2b, the maps γ obtained for all $\psi \in \Psi$ are Lipschitz with constant $||P^{-1}||L$ and bounded by two maps $\underline{\gamma}, \overline{\gamma}$, measurable w.r.t. t and Lipschitz w.r.t. x :

$$[\underline{\gamma}(t,x), \overline{\gamma}(t,x)] = [\gamma](t,x) = P^{-1}\left([\psi](t,x)\right) \tag{7}$$

such that $\overline{\gamma} - \underline{\gamma}$ is uniformly bounded by $||P^{-1}||M$.

Remark 9 *The coordinates of γ corresponds to the decomposition of ψ on the columns of the matrix P :*

$$\psi(t,x) = \sum_{i=1}^{n} \gamma_i(t,x) A^{i-1} H$$

which could be written:

$$\psi(t,x) = \sum_{i=1}^{n} \gamma_i(t,x) d_i$$

where the vectors d_i belong to $ker(S^i\mathcal{O}) \setminus \{0\}$. In this expression, we recognize the vector d_1 of Section 4. If H1 is fulfilled, then $\gamma_i = 0$, $i = 2 \ldots n$.

For $(y,[z]) \in \mathbb{R} \times \mathcal{I}^n$, define the parallelotope in $\mathbb{R}^n$:

$$\mathsf{Q}(y,[z]) = \left\{ PV_{\{\lambda_j\}}^{-1} z \mid z \in [z] \right\} \cap \{ x \mid Cx = y \}$$

and the set $\mathcal{X} \subset \mathbb{R} \times \mathcal{I}^n$ of pairs $(y,[z])$ that are compatible, *i.e.* such that $\mathsf{Q}(y,[z]) \neq \emptyset$. Define also the maps γ^-, γ^+ :

$$\left[\gamma^-(t,y,[z]), \gamma^+(t,y,[z])\right] = [\gamma](t, \mathsf{Q}(y,[z]))$$

Proposition 6 *Under H2b, given a vector K such that $A+KC$ has n distinct eigenvalues λ_j in $\mathbb{R} \setminus \{0\}$ and a compact set X_0 of $\mathbb{R}^n$, then there exists $[z_0] \in \mathcal{I}^n$ such that $X_0 \subset PV_{\{\lambda_j\}}^{-1}[z_0]$ and, for any unknown initial condition $x(0) \in X_0$ and unknown $\psi \in \Psi$, the trajectory $x(.)$ of the model $(\mathcal{S})$ belongs to the time-varying parallelotope :*

$$x(t) \in \mathsf{Q}(y(t),[z](t)), \qquad \forall t \geq 0 \tag{8}$$

where $[z](.)$ is solution of the interval observer :

$$(\mathsf{O}) : \left\{ \begin{array}{lcl} [\dot{z}] & = & \Delta_{\{\lambda_j\}}[z] \; - \; V_{\{\lambda_j\}}P^{-1}Ky \; + \; V_{\{\lambda_j\}}[\gamma](t, \mathsf{Q}(y,[z])) \\ [z](0) & = & [z_0] \end{array} \right. \tag{9}$$

Proof Take $[x_0] \in \mathcal{I}^n$ such that $X_0 \subset [x_0]$ and consider :

$$[z_0] = V_{\{\lambda_j\}}P^{-1}[x_0]. \tag{10}$$

$$\begin{array}{rcl} z & = & V_{\{\lambda_j\}}P^{-1}x, \\ [\underline{z}, \overline{z}] & = & [z] \end{array} \tag{11}$$

By H2b, γ is Lipschitz w.r.t. x with constant $||P^{-1}||L$, and by Lemma 5, γ^+ and γ^- are also Lipschitz w.r.t. $(\underline{z}, \overline{z})$ with constant $||P^{-1}||L$. So, the right-hand side of the state equation in (9) is Lipschitz w.r.t. $(\underline{z}, \overline{z})$ and the solutions of (O) are well defined as long as $(y, [\underline{z}, \overline{z}]) \in \mathcal{X}$.

We have :

$$\left\{ \begin{array}{lcl} \dfrac{d}{dt}(\overline{z} - z) & = & \Delta_{\{\lambda_j\}}(\overline{z} - z) + V_{\{\lambda_j\}}\left(\widetilde{\gamma}^+(t,y,[z]) - \gamma(t,x)\right) \\ \dfrac{d}{dt}(z - \underline{z}) & = & \Delta_{\{\lambda_j\}}(z - \underline{z}) + V_{\{\lambda_j\}}\left(\gamma(t,x) - \widetilde{\gamma}^-(t,y,[z])\right) \end{array} \right. \quad \text{where } x = PV_{\{\lambda_j\}}^{-1}z$$

But,

$$x \in \mathsf{Q}(Cx, \underline{z}, \overline{z}) \Rightarrow \left\{ \begin{array}{lcl} V_{\{\lambda_j\}}\left(\widetilde{\gamma}^+(t,y,[z]) - \gamma(t,x)\right) & \geq & 0 \\ V_{\{\lambda_j\}}\left(\gamma(t,x) - \widetilde{\gamma}^-(t,y,[z])\right) & \geq & 0 \end{array} \right.$$

So, $\dot{\overline{z}} - \dot{\underline{z}} \geq \Delta_{\{\lambda_j\}}(\overline{z} - \underline{z})$, *i.e.* , the solution of (O) leaves $\mathcal{I}^n$ invariant. Furthermore, we can write :

$$z \in [z] \quad \Longrightarrow \quad x \in \mathsf{Q}(Cx,[z]) \quad \Longrightarrow \quad \left\{ \begin{array}{lcl} \dfrac{d}{dt}(\overline{z} - z) & \geq & \Delta_{\{\lambda_j\}}(\overline{z} - z) \\ \dfrac{d}{dt}(z - \underline{z}) & \geq & \Delta_{\{\lambda_j\}}(z - \underline{z}) \end{array} \right.$$

So, for any $x(0) \in X_0$ we have $z(0) \in [z_0] \Rightarrow z(t) \in [z](t)$, $\forall t > 0$ and we conclude that $x(t) \in \mathsf{Q}(Cx(t),[z](t))$ and that the solution of (O) is well defined : $(y(t), \underline{z}(t), \overline{z}(t)) \in \mathcal{X}$ for any time t. ∎

Remark 10 *Proposition 6 means that if initial bounds can be obtained for the state variables to be estimated, then it is possible to build an interval observer. Of course, these initial estimates can be very loose, which facilitates their obtention.*

Remark 11 *When $X_0 \subset [\underline{x}_0, \overline{x}_0]$, the interval bounds of $[\gamma]$ in (7) and $[z_0]$ in (10) can be computed as follows :*

$$\begin{array}{lcl}
\overline{\gamma}(t,x) & = & \frac{1}{2}\left[P^{-1}(\overline{\psi}(t,x)+\underline{\psi}(t,x))+|P^{-1}|(\overline{\psi}(t,x)-\underline{\psi}(t,x))\right] \\
\underline{\gamma}(t,x) & = & \frac{1}{2}\left[P^{-1}(\overline{\psi}(t,x)+\underline{\psi}(t,x))-|P^{-1}|(\overline{\psi}(t,x)-\underline{\psi}(t,x))\right] \\
\overline{z}_0 & = & \frac{1}{2}\left[V_{\{\lambda_j\}}P^{-1}(\overline{x}_0+\underline{x}_0)\ +\ |V_{\{\lambda_j\}}P^{-1}|(\overline{x}_0-\underline{x}_0)\right] \\
\underline{z}_0 & = & \frac{1}{2}\left[V_{\{\lambda_j\}}P^{-1}(\overline{x}_0+\underline{x}_0)\ -\ |V_{\{\lambda_j\}}P^{-1}|(\overline{x}_0-\underline{x}_0)\right]
\end{array}$$

where $|P^{-1}|$ stands for the matrix whose coefficients are the absolute values of the coefficients of P^{-1}.

When the eigenvalues λ_j are negative, the maps $\widetilde{\gamma}^{\pm}(t,y,[z])$ defined in (11) are :

$$(\widetilde{\gamma}_i^+(t,y,[z]),\widetilde{\gamma}_i^-(t,y,[z])) = \left|\begin{array}{ll}
\left(\max\limits_{x\in\mathsf{Q}(y,[z])}\overline{\gamma}_i(t,x),\ \min\limits_{x\in\mathsf{Q}(y,[z])}\underline{\gamma}_i(t,x)\right) & \text{if } i\equiv 1 \mod 2 \\
\left(\min\limits_{x\in\mathsf{Q}(y,[z])}\underline{\gamma}_i(t,x),\ \max\limits_{x\in\mathsf{Q}(y,[z])}\overline{\gamma}_i(t,x)\right) & \text{if } i\equiv 0 \mod 2
\end{array}\right.$$

Remark 12 *A known function $\varphi(t,y)$ could have been added to the function $\psi(t,x)$, as in H1.*

Of course, the proposed bounds may tend to infinity, hence the interest for sufficient conditions that guarantee an asymptotic upper bound on the diameter of the set $\mathsf{Q}(y(t),\underline{z}(t),\overline{z}(t))$. In order to establish such conditions, we shall use the decomposition of ψ mentioned above,

$$\psi(t,x) = \sum_{i=1}^{n}\gamma_i(t,x)d_i$$

where the vectors d_i belong to $\ker(S^i\mathcal{O})\setminus\{0\}$, and build the procedure step by step.

The first step is already known : assume that only γ_1 is non-zero, which corresponds to the case covered by H1 :

Corollary 7 *Under H1 and H2, K can be chosen such that the diameter of $\mathsf{Q}(y(t),[z](t))$ converges arbitrarily fast towards an arbitrarily small value.*

Proof If $l=\overline{z}-\underline{z}$, then

$$\dot{l}_j = \lambda_j l_j + \sum_{i=1}^{n}\lambda_j^{i-1}(\widetilde{\gamma}_i^+(t,y,[z])-\widetilde{\gamma}_i^-(t,y,[z])),\quad j=1\dots n. \tag{12}$$

As $\ker(S\mathcal{O})=\mathrm{span}\{H\}$, we have $\gamma_j\equiv 0$ and $\gamma_j^+\equiv\gamma_j^-\equiv 0$ for any $j\neq 1$. So, with λ_j negative and $\overline{\lambda}=\min\limits_j|\lambda_j|$,

$$\begin{array}{rcl}
\dfrac{d}{dt}||l|| & \le & -\overline{\lambda}||l|| + (\widetilde{\gamma}_1^+(t,y,[z])-\widetilde{\gamma}_1^-(t,y,[z]))\sqrt{n} \\
& \le & -\overline{\lambda}||l|| + ||P^{-1}||(M+L||PV_{\{\lambda_j\}}^{-1}||\ ||l||)\sqrt{n} \\
& \le & (-\overline{\lambda}+\kappa(P)L||V_{\{\lambda_j\}}^{-1}||\sqrt{n})||l|| + ||P^{-1}||M\sqrt{n}
\end{array}$$

As in Proposition 4, we can choose eigenvalues of the form $\theta\lambda_j$ and obtain exponential convergence of the set $\mathsf{Q}(y(t),[z](t))$ towards a ball of arbitrarily small radius. Convergence can be made arbitrarily fast by taking sufficiently large values of θ. ∎

Next, we assume that only γ_1 and γ_2 are non-zero.

Hypothesis H1b : $\forall\psi\in\Psi,\ \psi(t,x)\in\ker(S^2\mathcal{O}),\ \forall(t,x)\in\mathbb{R}_+\times\mathbb{R}^n$.

Proposition 8 *Under H1b and H2b, when $L<(\kappa(P)\sqrt{n})^{-1}$, for any $\epsilon>0$, K can be chosen such that the diameter of $\mathsf{Q}(y(t),[z](t))$ converges arbitrarily fast towards $[0, M/((\kappa(P)\sqrt{n})^{-1}-L)+\epsilon]$.*

Proof As $\ker(S^2\mathcal{O})=\text{span}\{H,AH\}$, we have $\gamma_j\equiv 0$ and $\gamma_j^+\equiv\gamma_j^-\equiv 0$ for $j>2$ in (12). So

$$\begin{aligned}
\frac{d}{dt}||l|| &\leq -\overline{\lambda}||l||+\left(\widetilde{\gamma}_1^+(t,y,[z])-\widetilde{\gamma}_1^-(t,y,[z])+\overline{\lambda}(\widetilde{\gamma}_2^-(t,y,[z])-\widetilde{\gamma}_2^+(t,y,[z]))\right)\sqrt{n}\\
&\leq -\overline{\lambda}||l||+||P^{-1}||(M+L||PV_{\{\lambda_j\}}^{-1}||\,||l||)(1+\overline{\lambda})\sqrt{n}\\
&\leq (-\overline{\lambda}+\kappa(P)L||V_{\{\lambda_j\}}^{-1}||(1+\overline{\lambda})\sqrt{n})||l||+||P^{-1}||M(1+\overline{\lambda})\sqrt{n}
\end{aligned}$$

From [5], if the n eigenvalues of $A+KC$ are chosen as $\lambda_j=\lambda_j(\sigma)=-\sigma^j$, then

$$\lim_{\sigma\to\infty}||V_{\{\lambda_j(\sigma)\}}^{-1}||=1.$$

So, when $\kappa(P)L\sqrt{n}<1$, there exists $\sigma=\overline{\lambda}$ large enough such that $\mu_\sigma=-\sigma+\kappa(P)L||V_{\{\lambda_j(\sigma)\}}^{-1}||(1+\sigma)\sqrt{n}<0$. Then $l(t)$ converges towards the ball :

$$\mathbb{B}\left(0,\frac{||P^{-1}||M(1+\sigma)\sqrt{n}}{|\mu_\sigma|}\right),$$

which implies the convergence of $\mathsf{Q}(y(t),[z](t))$ towards a ball of radius :

$$\rho_\sigma=\frac{\kappa(P)||V_{\{\lambda_j(\sigma)\}}^{-1}||M(1+\sigma)\sqrt{n}}{|\mu_\sigma|}.$$

We also have :

$$\lim_{\sigma\to+\infty}\mu_\sigma=-\infty\quad\text{and}\quad\lim_{\sigma\to+\infty}\rho_\sigma=\frac{M}{1/(\kappa(P)\sqrt{n})-L}.\ ∎$$

Remark 13 *The hypotheses H1b and H2b of Proposition 8 imply that $(\mathcal{S})$ is differentially observable for any* known $\psi\in\Psi$ *: in the coordinates $\xi=P^{-1}x$, $(\mathcal{S})$ can be rewritten as*

$$\left\{\begin{array}{rcl}\dot{\xi}&=&\widetilde{A}\xi+\gamma(t,P\xi)\\ y&=&\widetilde{C}\xi\end{array}\right.$$

Thus

$$\Gamma_t(\psi,x^a)=\Gamma_t(\psi,x^b)\ \Rightarrow\ (\xi^a-\xi^b)+\begin{pmatrix}\gamma_2(t,P\xi^a)-\gamma_2(t,P\xi^b)\\0\\\vdots\\0\end{pmatrix}=0$$

which implies that $\xi_j^a = \xi_j^b$ for $j = 2 \dots n$. There remains one equation in ξ_1. If we consider the scalar function $\xi_1 \to \xi_1 + \gamma_2(t, P\xi)$, where the other coordinates of ξ are fixed, it is easy to see that this function is increasing (because γ_2 is Lipschitz with constant $||P^{-1}||L$ with $L < (\kappa(P)\sqrt{n})^{-1}$) and therefore $\xi_1^a = \xi_1^b$. The two observers defined in Proposition 6 are then strong observers (when $M = 0$). Note that we do not require Γ_t to be a global diffeomorphism neither the system $(\mathcal{S})$ to leave a compact invariant, as required for the exact convergence of the "high-gain" observers (see [11]).

The next step is to assume that γ_3 is also non-zero. This, however, may yield unbounded dynamics when the eigenvalues become large. Further work needs therefore to be done for the determination of the optimal eigenvalues, with respect to a compromise between speed and accuracy.

6 Example

Consider again the introductive example of Section 1. This model may describe the evolution of the dynamics of a biological population structured in three classes [23] whose stocks are : $(x_1, x_2, x_3) \in \Omega = \mathbb{R}_+^3$:

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & -\alpha_1 x_1 - m_1 x_1 + r(t, x_2, x_3) \\ \dot{x}_2 & = & \alpha_1 x_1 - \alpha_2 x_2 - m_2 x_2 \\ \dot{x}_3 & = & \alpha_2 x_2 - m_3 x_3 \end{array} \right.$$

The positive coefficients α_i and m_i respectively represent the growth and mortality rates of the ith class. The classes could be age classes (corresponding for instance to larvae, young adults and old adults) or stages in the life cycle. The same type of model can be used for the description of metabolisms [23].

Assume that the births in class x_1 are produced only by the classes x_2 and x_3 (the young and old adults, say) with a reproduction law of the classical Beverton-Holt type [23] :

$$r(t, x_2, x_3) = a(t)\, \frac{x_2 + x_3}{b + x_2 + x_3} \qquad a(t) > 0,\ b > 0 \tag{13}$$

Note that the model leaves Ω invariant. Assume further that the stock of old adults can be measured online and without error :

$$y(t) = x_3(t), \quad \forall t \geq 0$$

One can always choose loose bounds on the initial values of the other states variables, so

$$x(0) \in [\underline{x}(0), \overline{x}(0)].$$

To simplify notation, we shall write $\beta_1 = \alpha_1 + m_1$, $\beta_2 = \alpha_2 + m_2$ and $\beta_3 = m_3$, so

$$A = \left(\begin{array}{ccc} -\beta_1 & 0 & 0 \\ \alpha_1 & -\beta_2 & 0 \\ 0 & \alpha_2 & -\beta_3 \end{array} \right), \quad C = \left(\begin{array}{ccc} 0 & 0 & 1 \end{array} \right)$$

To simplify computation, we shall assume that $\beta_1 = \beta_2 = \beta_3 = \beta$ and $\alpha_1 = \alpha_2 = \alpha$ with $\beta \geq \alpha > 0$, so

$$P = \left(\begin{array}{ccc} 1/\alpha^2 & -\beta/\alpha^2 & \beta^2/\alpha^2 \\ 0 & 1/\alpha & -2\beta/\alpha \\ 0 & 0 & 1 \end{array} \right)$$

6.1 Uncertainty on the reproduction law

In the reproduction law (13), the parameter b is assumed to be known but $a(t)$ is unknown and may fluctuate over time between two known functions :

$$a(t) \in [\underline{a}(t), \overline{a}(t)].$$

Then, the state equation of the model can be written as

$$(\mathcal{S}) : \left\{ \begin{array}{lcl} \dot{x} & = & Ax + \psi_1(t,x)d_1 \\ y & = & Cx \end{array} \right.$$

with

$$d_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

One can easily check that the pair (A, C) is observable and $d_1 \in \ker(S\mathcal{O})$, so H1 is fulfilled and $(\mathcal{S})$ is observable for unknown ψ_1. Furthermore,

$$\psi_1(t,x) \in \left[\underline{\psi}_1(t,x_2,x_3), \overline{\psi}_1(t,x_2,x_3)\right]$$

where

$$(\underline{\psi}_1(t,x_2,x_3), \overline{\psi}_1(t,x_2,x_3)) = \left(\underline{a}(t)\,\frac{x_2+x_3}{b+x_2+x_3}, \overline{a}(t)\,\frac{x_2+x_3}{b+x_2+x_3}\right)$$

with

$$\overline{\psi}_1(t,x_2,x_3) - \underline{\psi}_1(t,x_2,x_3) \leq \overline{a}(t) - \underline{a}(t) \leq M,$$

where M is a given constant, so H2 is fulfilled. Consider then the matrix

$$A + KC = -\beta I + \alpha I \begin{pmatrix} 0 & 0 & \alpha^{-1}k_1 \\ 1 & 0 & \alpha^{-1}k_2 \\ 0 & 1 & \alpha^{-1}k_3 \end{pmatrix}$$

We can fix the eigenvalues of $A + KC$ to be $\lambda_1, \lambda_2, \lambda_3$ with :

$$k_i = \alpha(-1)^{i+1}\sigma_i\left(\mu_1, \mu_2, \mu_3\right), \qquad i = 1 \ldots 3$$

where $\mu_i = (\lambda_i + \beta)/\alpha$ and $\sigma_i(\mu_1, \mu_2, \mu_3)$ are the symmetric functions of the roots :

$$\left\{ \begin{array}{lcl} \sigma_1(\mu_1,\mu_2,\mu_3) & = & \mu_1\mu_2\mu_3 \\ \sigma_2(\mu_1,\mu_2,\mu_3) & = & \mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 \\ \sigma_3(\mu_1,\mu_2,\mu_3) & = & \mu_1 + \mu_2 + \mu_3 \end{array} \right.$$

For any set $\{\lambda_j\}_{j=1\ldots3}$ of distinct negative real numbers, $A + KC$ is diagonalizable by the Vandermonde matrix $V_{\{\mu_j\}}$ and the interval observer can be written in the coordinates $z = V_{\{\mu_j\}}x$:

$$\left\{ \begin{array}{lcl} \dot{\overline{z}}_i & = & \lambda_i\overline{z}_i - (k_1 + k_2\mu_i + k_3\mu_i^2)y + \psi_1^+(t,y,[\underline{z},\overline{z}]) \\ \dot{\underline{z}}_i & = & \lambda_i\underline{z}_i - (k_1 + k_2\mu_i + k_3\mu_i^2)y + \psi_1^-(t,y,[\underline{z},\overline{z}]) \end{array} \right. \qquad i = 1 \ldots 3$$

with :

$$\left\{\begin{array}{rcl}\psi_1^+(t,y,[\underline{z},\overline{z}]) & = & \overline{\psi}_1\left(t,y,\displaystyle\sum_{i=1}^{3}\max\{w_{2i}\overline{z}_i, w_{2i}\underline{z}_i\}\right)\\ \psi_1^-(t,y,[\underline{z},\overline{z}]) & = & \underline{\psi}_1\left(t,y,\displaystyle\sum_{i=1}^{3}\min\{w_{2i}\overline{z}_i, w_{2i}\underline{z}_i\}\right)\end{array}\right.$$

where the w_{ij}s are the coefficients of the matrix $V_{\{\mu_j\}}^{-1}$:

$$\left\{\begin{array}{rcl} w_{21} & = & -\dfrac{\mu_2+\mu_3}{\mu_1^2+\mu_2\mu_3-\mu_1\mu_2-\mu_1\mu_3}\\ w_{22} & = & \dfrac{\mu_1+\mu_3}{-\mu_2^2+\mu_2\mu_3+\mu_1\mu_2-\mu_1\mu_3}\\ w_{23} & = & -\dfrac{\mu_1+\mu_2}{\mu_3^2-\mu_2\mu_3+\mu_1\mu_2-\mu_1\mu_3}\end{array}\right.$$

because $\underline{\psi}_1$, $\overline{\psi}_1$ are non decreasing functions w.r.t. x_2.

With the initial condition :

$$\left\{\begin{array}{rcl}\overline{z}_i(0) & = & \overline{x}_1(0)+\max\{\mu_i\overline{x}_2(0),\ \mu_i\underline{x}_2(0)\}+\mu_i^2 y(0)\\ \underline{z}_i(0) & = & \underline{x}_1(0)+\min\{\mu_i\overline{x}_2(0),\ \mu_i\underline{x}_2(0)\}+\mu_i^2 y(0)\end{array}\right. \qquad i=1\ldots3,$$

Proposition 6 guarantees that $x(t)$ belongs to the parallelotope :

$$\mathsf{Q}(y(t),[\underline{z}(t),\overline{z}(t)]) \;=\; \left\{\, x \mid V_{\{\mu_j\}}x\in[\underline{z}(t),\overline{z}(t)] \text{ and } x_3=y(t)\,\right\},\quad \forall t\geq 0.$$

and Corollary 7 ensures that the asymptotic size of this parallelotope can be reduced as desired by taking large negative values λ_j.

Remark : In this example, the conditions of [7] are not satisfied.

Numerical experimentation : We consider the modeling of the evolution of a population with uniform growth rate $\alpha_1=\alpha_2=0.3$ and a mortality that occurs only on old adults : $m_1=m_2=0$ and $m_3=0.3$. Thus, we have $\alpha=0.3$ and $\beta=0.3$. For the reproduction law, we have chosen the parameter $b=1$ and a stationary interval $[\,\underline{a},\overline{a}\,]=[\,0.1,0.4\,]$ for the uncertain function $a()$.

We have made numerical simulations for a function $t\to a(t)$ which graph is shown on figure 1 and a given initial condition $(x_1,x_2,x_3)=(0.7,0.7,0.5)$. This has generated the observation $t\to y(t)$ which graph is shown on figure 1. The interval observers have been run for different values of the λ_is, which have been chosen such that $\mu_i=-\theta^i$, which ensure reasonable values for the norm of the matrix $V_{\{\mu_j\}}^{-1}$.
Simulation 1 : For $\theta=8/3$, we have $Sp(A+KC)=\{-1.1,-2.4,-6\}$. The results are presented on figures 2 and 3.
Simulation 2 : For $\theta=17/3$, we have $Sp(A+KC)=\{-2,-10,-55\}$. The results are presented on figures 4 and 5.

These simulations illustrate the concept of practical observer : the asymptotic error is not equal to 0 but can be made arbitrarily small, taking large negative eigenvalues λ_i. Note that the effect of the unknown input on the width of the guaranteed parallelotope is larger in the direction of the vector $(1\,0\,0)'$ (the first column of P), which corresponds to the image of the vector $(1\,1\,1)'$ by the matrix $PV_{\{\mu_j\}}^{-1}$.

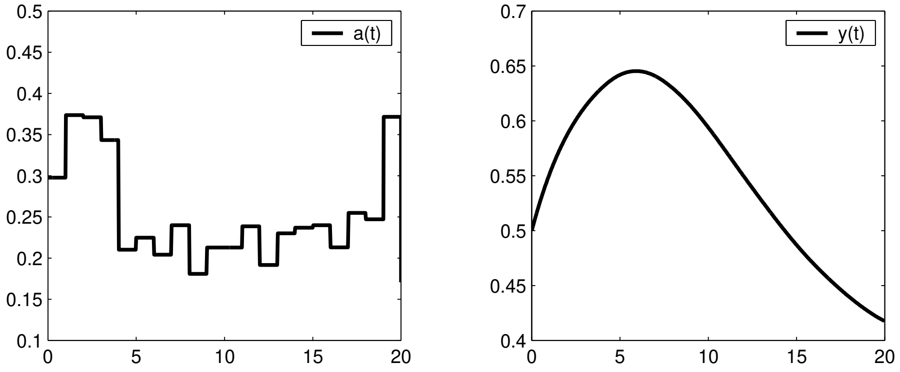

Figure 1: Unknown input $a(t)$ and known observation $y(t)$

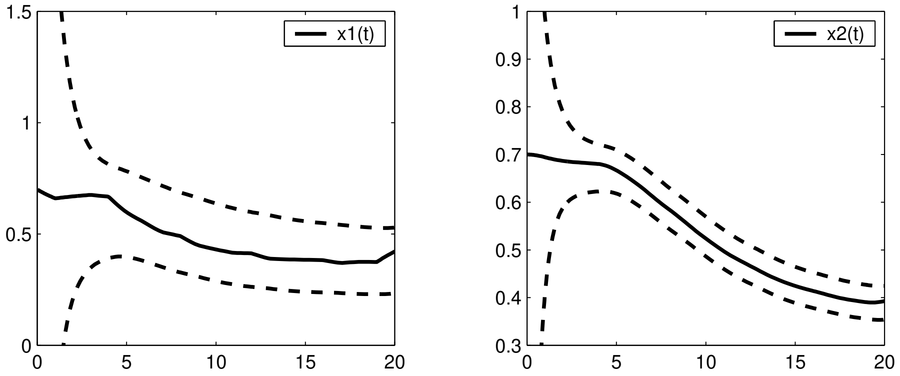

Figure 2: Unmeasured variables $x_1(t)$ and $x_2(t)$ with intervals guaranteed to contain them (for $\theta = 8/3$)

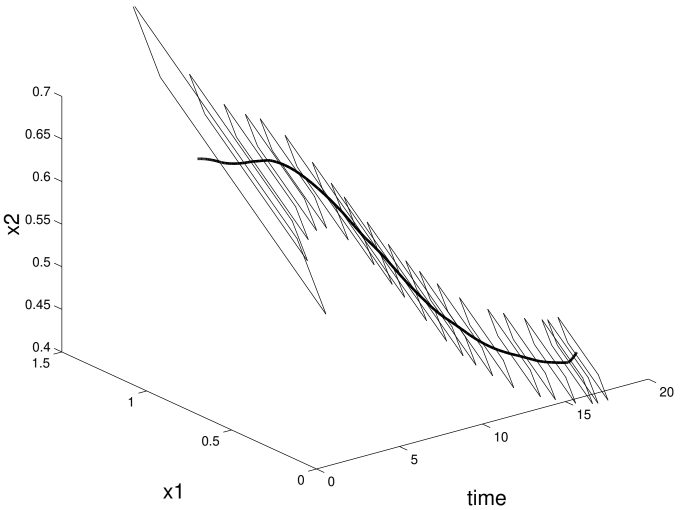

Figure 3: Projection of the actual state trajectory onto the (x_1, x_2) and time-varying parallelotope guaranteed to contain it, generated by the interval observer (for $\theta = 8/3$)

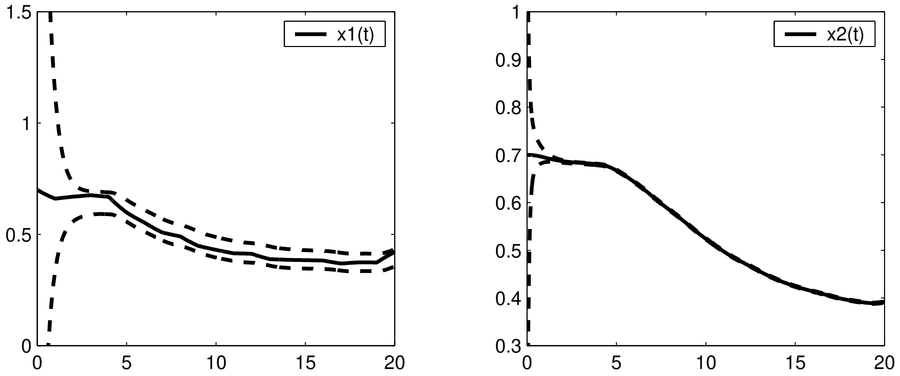

Figure 4: Unmeasured variables $x_1(t)$ and $x_2(t)$ with intervals guaranteed to contain them (for $\theta = 17/3$)

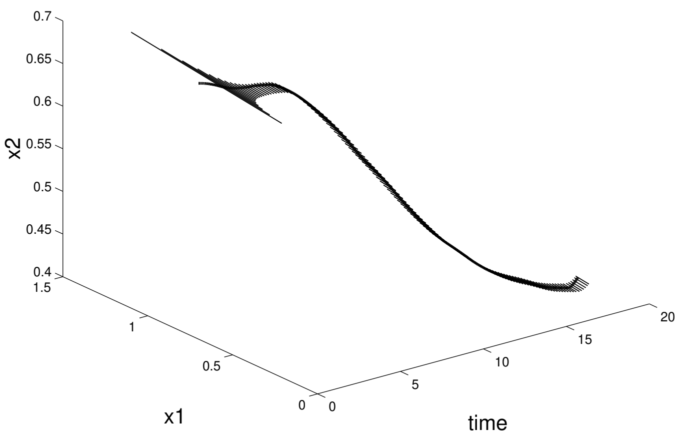

Figure 5: Projection of the actual state trajectory onto the (x_1, x_2) and time-varying parallelotope guaranteed to contain it, generated by the interval observer (for $\theta = 17/3$)

6.2 Uncertainty on an harvesting effort

In addition to the uncertainty on the reproduction law, assume now that an uncertain mortality or harvesting effort $e(t) \geq 0$ exists on the class x_2, so the state equation becomes

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & -\alpha_1 x_1 - m_1 x_1 + r(t, x_2, x_3) \\ \dot{x}_2 & = & \alpha_1 x_1 - \alpha_2 x_2 - m_2 x_2 - e(t) x_2 \\ \dot{x}_3 & = & \alpha_2 x_2 - m_3 x_3 \end{array}\right.$$

where $e(t)$ is bounded between two known functions :

$$e(t) \in [\underline{e}(t), \overline{e}(t)] \subset [0, E^{\max}], \quad \forall t \geq 0.$$

It is easy to check that all the variables are bounded (and non-negative). With the same notation as before, the state equation of the model can be written as :

$$(\mathcal{S}) : \left\{\begin{array}{rcl} \dot{x} & = & Ax + \psi_1(t,x) d_1 + \psi_2(t,x) d_2 \\ y & = & Cx \end{array}\right.$$

with

$$d_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

One can easily check that $d_2 \in \ker(S^2 \mathcal{O})$, so the system is not observable for the unknown input ψ_2. But H1b and H2b are fulfilled, as

$$\psi_2(t,x) \in [\underline{\psi}_2(t,x_2), \overline{\psi}_2(t,x_2)] = [\underline{e}(t)x_2, \overline{e}(t)x_2]$$

and

$$\overline{\psi}_2(t,x_2) - \underline{\psi}_2(t,x_2) \leq H^{\max}$$

where $H^{\max}$ is some constant depending on the bounds on the variables, for instance $H^{\max} = E^{\max} x_2^{\max}$ (where $x_2^{\max}$ is a global upper bound on x_2). Then, a guaranteed interval observer can be written in the coordinates z :

$$\left\{\begin{array}{rcl} \dot{\overline{z}}_i & = & \lambda_i \overline{z}_i - (k_1 + k_2\mu_i + k_3\mu_i^2) y + \overline{\psi}_1(t, y, \overline{x}_2) - \mu_i \overline{\psi}_2(t, \underline{x}_2) \\ \dot{\underline{z}}_i & = & \lambda_i \underline{z}_i - (k_1 + k_2\mu_i + k_3\mu_i^2) y + \underline{\psi}_1(t, y, \underline{x}_2) - \mu_i \underline{\psi}_2(t, \overline{x}_2) \end{array}\right. \qquad i = 1 \ldots 3$$

where

$$(\underline{x}_2, \overline{x}_2) = \left(\sum_{i=1}^{3} \min\{w_{2i}\overline{z}_i, w_{2i}\underline{z}_i\}, \sum_{i=1}^{3} \max\{w_{2i}\overline{z}_i, w_{2i}\underline{z}_i\} \right)$$

If the λ_is are large enough for all μ_is to be non-positive then Proposition 8 guarantees that the parallelotope $\mathsf{Q}(y(t), \underline{z}(t), \overline{z}(t))$ converges exponentially towards a bounded set when $E^{\max} < 1/(\sqrt{3}\kappa(P))$.

Numerical experimentation : We used the same parameters as before. In addition, we have chosen the values $E^{\max} = 0.25$ and $H^{\max} = 1$ for the harvesting term. For the simulations, we have generated a function $t \to e(t)$ respecting the bounds $[0, E^{\max}]$ (see the figure 6) and run the interval observers for different values of θ.

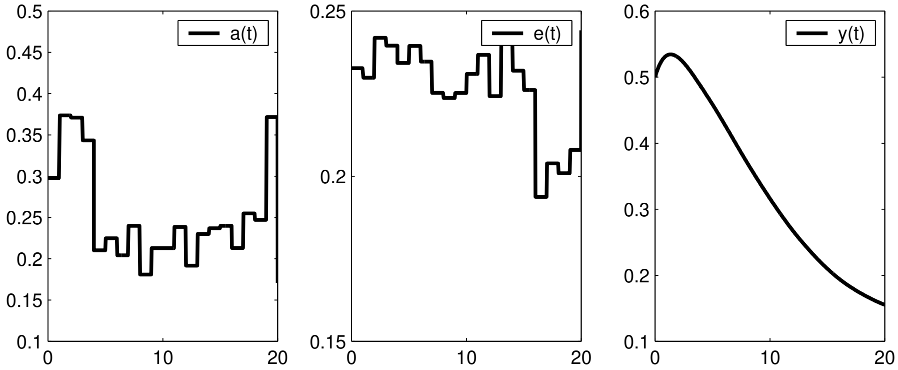

Figure 6: Unknown inputs $a(t)$ and $e(t)$, and known observation $y(t)$

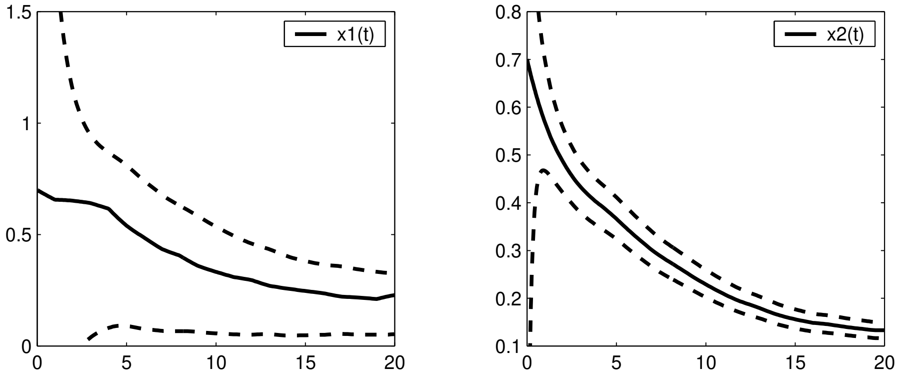

Figure 7: Unmeasured variables $x_1(t)$ and $x_2(t)$ with intervals guaranteed to contain them (for $\theta = 4$ and unknown harvesting effort)

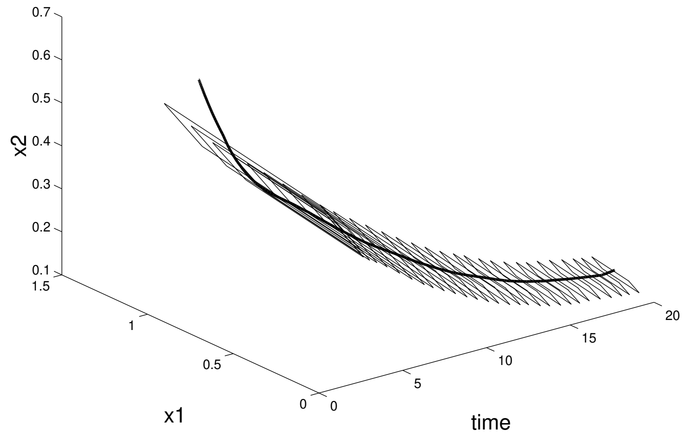

Figure 8: Projection of the actual state trajectory onto the (x_1, x_2) and time-varying parallelotope guaranteed to contain it, generated by the interval observer (for $\theta = 4$ and unknown harvesting effort)

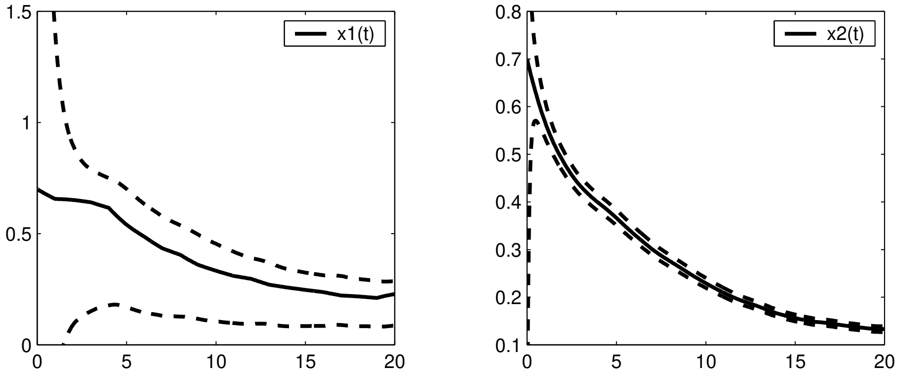

Figure 9: Unmeasured variables $x_1(t)$ and $x_2(t)$ with intervals guaranteed to contain them (for $\theta = 9$ and unknown harvesting effort)

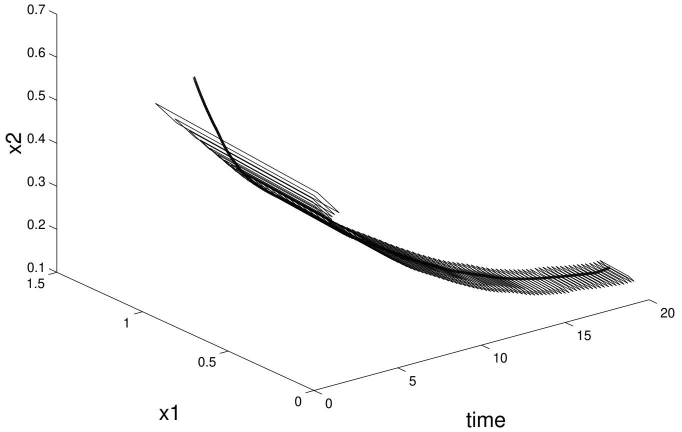

Figure 10: Projection of the actual state trajectory onto the (x_1, x_2) and time-varying parallelotope guaranteed to contain it, generated by the interval observer (for $\theta = 9$ and unknown harvesting effort)

Simulation 1 : For $\theta = 4$, we have $Sp(A+KC) = \{-1.5, -5.1, -19\}$. The results are presented on figures 7 and 8.
Simulation 2 : For $\theta = 9$, we have $Sp(A+KC) = \{-3, -25, -220\}$.The results are presented on figures 9 and 10.

Even when taking large negative eigenvalues λ_i, there is always a residual error on x_1, but outer bounding parallelotope stays bounded.

7 Conclusion

In this paper, we have studied the observability for unknown inputs for a class of nonlinear systems, for which we proposed practical observers. For more general nonlinear systems, which may not be observable for unknown inputs, we have proposed a construction of guaranteed parallelotopic observers. The philosophy of the approach is simple : given deterministic bounds on the uncertainties, we obtain deterministic guaranteed dynamic bounds on the variables to estimate. In this work, we have also investigated the possible asymptotic behaviors of the parallelotopes, and linked them to the decomposition of the uncertain terms of the dynamics on a particular basis. We emphasize also on the fact that this approach allows the bounds on the uncertainties to be functional and time-dependent.

Finally, we have considered in this paper that the measurements $y(t)$ were available continuously, free from any error. It is possible to relax this assumption, considering deterministic and bounded uncertainties on the measurements of the form $y(t) \in [\underline{y}(t), \overline{y}(t)]$ using similar interval techniques. This will be the purpose of a forthcoming work.

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