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Hybrid bounded error observers for uncertain bioreactor models

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Abstract In this paper, we build bounded error observers for a common class of partially known bioreactor models. The main idea is to construct hybrid bounded observers "between" high gain observer, which has an adjustable convergence rate but requires perfect knowledge of the model, and asymptotic observer which is very robust towards uncertainty but has a fixed convergence rate. An hybrid bounded error observer which reconstructs the two state variables is constructed considering two steps: first step is similar to a high gain observer meaning that fast convergence rate but error depending on the knowledge of the model are obtained; second step is a switch to an observer similar to the asymptotic one meaning that fixed convergence rate towards an error as small as desired is obtained. Thus, a better convergence rate of estimated variables than the classical asymptotic observer is obtained.

Keywords Uncertainty modeling · Nonlinear observers · Bioreactor models

Introduction

The bioreactor is a continuous device where microorganisms consume nutrient to grow. This nutrient is provided by a constant inflow q, and a blend of nutrient and of microorganisms is retrieved in the outflow q [1]. Generally, no reliable biological sensor for each variable of a biological system exist. In this context, building observers is very interesting in order to estimate concentration of the main chemical or biological species in the bioreactor.

Firstly, let us recall the classical observer definition. Consider the dynamical system:

$$\begin{aligned} \dot{x} &= F(x,u), \\ y &= h(x), \end{aligned} \tag{1}$$

with $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ $m \leq n$, $y \in \mathbb{R}$.

An observer for Eq. 1 is the following dynamical system

$$\dot{\hat{x}} = \hat{F}(\hat{x}, u, y),$$

whose task is state estimation. It is expected to provide an estimated state $\hat{x}$ of x. One usually requires at least that $\| \hat{x} - x \|$ goes to zero when t tends to ∞ ; in some cases, exponential convergence is also required [12].

Often it happens that some functions of the state variables are partially known in the original model [9]. Then, we define a bounded error observer giving $\hat{x}$ with $\| \hat{x} - x \|$ bounded by a "reasonable" constant; "reasonable" meaning that it is small enough to have a good approximation of the unmeasured states.

In all the paper, we always consider the following class of bioreactor models [1]:

$$\begin{cases} \dot{x} = \mu(s)x - dx \\ \dot{s} = -\alpha\mu(s)x + ds_{\text{in}} - ds, \end{cases} \tag{2}$$

where $d = q/V$ is the dilution rate with V the volume of the bioreactor and q the constant flow passing through the bioreactor, α is the growth yield, s_{in} is the input substrate concentration, $\mu(s)$ is the specific growth rate per unit of biomass. Let us notice that the inputs d and s_{in} are fixed (see Remark 3.5).

Different models exist in the literature; for example, the Monod specific function $\mu(s) = \mu_m s/(k+s)$ is often used (μ_m maximum growth rate and k half saturation constant).

Moreover, we assume that the output is:

$$y = s.$$

The goal of the paper is to adapt the observer design to the available knowledge of the growth rate $\mu(s)$. First, we recall classical observers built for bioreactor model 2.

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When the growth rate μ(s) is perfectly known, a high gain observer which has an adjustable convergence rate is recalled; if μ(s) is unknown, then an asymptotic observer which has a fixed convergence rate is considered.

Then an intermediate approach is proposed to deal with partial knowledge of μ(s). A bound on the error depending on the knowledge we have on the model is obtained: it can be adjusted in some way as in [4]. These hybrid observers evolve between two limit cases: high gain observer and asymptotic one in the same way than the asymptotic-Kalman observer proposed by [3].

Finally, we illustrate all the results by simulation studies.

Classical observers for the bioreactor model

The high gain observer

First, we recall briefly the notion of high gain observer for general system. Consider the differential system defined on a domain $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} \dot{x} = f(x) \\ y = h(x), \end{cases} \tag{3}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are smooth. Moreover, if we assume that Hypotheses 2.1 hold [5], we can design high gain observers.

Hypotheses 21

1. *The system 2 is observable*. rank $(dy, d\dot{y}, \ldots, dy^{(n-1)}) = n$.
2. *The map* Φ *defined such that*:

$\Phi : \mathbb{R}^n \to \mathbb{R}^n$

$$x \mapsto \begin{pmatrix} y = z_1 \\ \dot{y} = z_2 \\ \vdots \\ y^{(n-1)} = z_n \end{pmatrix}$$

is a diffeomorphism of $\Omega \subset \mathbb{R}^n$ *on* $\Phi(\Omega)$.

Under the hypotheses 2, the system 3 becomes:

$$\begin{cases} \dot{z} = \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_n \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ \vdots \\ \varphi(z) \end{pmatrix} = F(z) \\ y = z_1 \end{cases} \tag{4}$$

3. *φ can be extended to* $\mathbb{R}^n$ *in a map* $\mathscr{C}^\infty$ *global Lipschitz on* $\mathbb{R}^n$.

Notice that for biological systems, these hypotheses are often verified [2]. Then we obtain the high gain observer definition as follows.

Proposition 22 *For θ large enough, the following differential system 5 is an exponential observer for 3*

$$\dot{\hat{z}} = F(\hat{z}) + S^{-1}C^t(y - C\hat{z}), \tag{5}$$

with S the solution of the equation $\theta S + A^t S + SA = C^t C$, *where* $A \in \mathscr{M}_n(\mathbb{R})$ *with* $a_{i,i+1} = 1$ *and* $a_{i,j} = 0$ *for all* $i, j = 1, \ldots, n-1$ *and* $C = (1\ 0 \ldots\ 0)$

$S = (s_{i,j}) \in \mathscr{M}_n(\mathbb{R})$ can be analytically computed

$$s_{i,j} = \frac{(-1)^{i+j}}{\theta^{i+j-1}} \frac{(i+j-2)!}{(i-1)!(j-1)!}.$$

In particular, assuming that the specific growth rate μ(s) is given by the "Monod function", we get the differential standard equations for the model (2):

$$\begin{cases} \dot{s} = -\alpha \frac{\mu_m s x}{k+s} - ds + ds_{\text{in}} \\ \dot{x} = \frac{\mu_m s x}{k+s} - dx \\ y = s \end{cases} \tag{6}$$

We obtain the following high gain observer for the system 6 applying Proposition 22 [5]:

$$\begin{cases} \dot{\hat{s}} = \frac{-\alpha\mu_m \hat{x}\hat{s}}{k+s} + d(s_{\text{in}} - \hat{s}) - 2\theta(\hat{s} - y) \\ \dot{\hat{x}} = \frac{\mu_m \hat{s}\hat{x}}{s+k} - d\hat{x} + \left(2\theta \frac{k\hat{x}}{(k+\hat{s})\hat{s}} + \theta^2 \frac{\hat{s}+k}{\alpha\mu_m \hat{s}}\right)(\hat{s} - y). \end{cases}$$

Simulations

We make two simulations: one when the model is well known and one when the model is partially known (i.e., we take $\hat{\mu}(s)$ instead of μ(s) in the high gain observer (Fig. 1)).

We take for parameters values $s_{\text{in}} = 50$, $d = 0.1$, $\alpha = 1$, $\theta = 3$ and for initial conditions $\hat{s}(0) = 10, \hat{x}(0) = 20$. In the model, we choose $\mu(s) = s/(140 + s)$ and when the model is not well known $\hat{\mu}(s) = 0.8s/(140 + s)$.

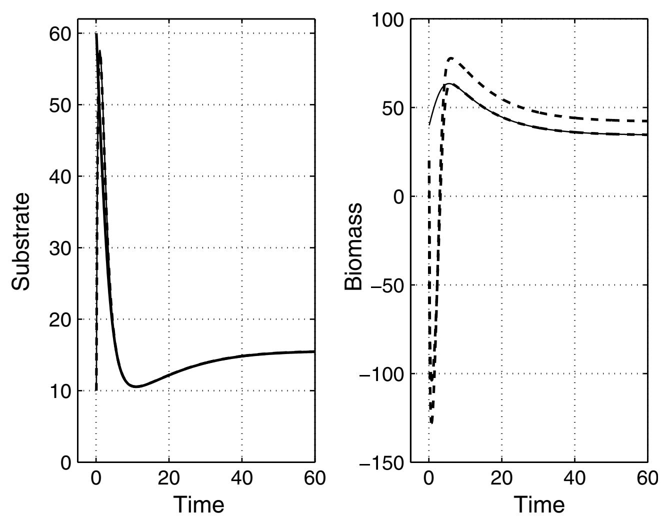

Fig. 1 In *dash line* the high gain observer when the growth rate is partially known (*bold*) and when the model is perfectly known, in *plain line* the model

In the two following simulations, θ is fixed and s is measured. In dash bold line, we can see the high gain observer when the growth rate is partially known, and in dash line when the model is perfectly known, in plain line the model.

A very strong peak (giving negative values for the observed biomass) appears at the beginning of the simulations. The large value of the gain θ and the large initial output error explain this phenomenon. Moreover, to obtain this exponential observer, the model must be perfectly known: we can see that the observer converges towards the model rapidly. If we do not know the model (μ(s) is replaced by $\hat{\mu}(s)$ in the observer equation), we can see that the error does not go to zero. Then a better bound for this error must be obtained.

The asymptotic observer

The main idea of asymptotic observer is to eliminate the unknown function and then obtain an error between estimated and modeled variables equal to zero.

Consider the dynamical system 2 and take $z = \alpha x + s$. Assume that s is exactly measured and that μ(s) is unknown [1].

The dynamics of z is given by the following equation:

$$\dot{z} = ds_{\text{in}} - dz. \tag{7}$$

An asymptotic observer for Eq. 7 is given by

$$\dot{\hat{z}} = ds_{\text{in}} - d\hat{z}.$$

If we consider the error $e = \hat{z} - z$, we can immediately conclude that $\dot{e} = -de$ that is to say the asymptotic observer converges towards z with a constant convergence rate e^{-dt}.

Moreover, x can be reconstructed considering $\hat{x} = (\hat{z} - s)/\alpha$.

The advantage of this kind of observer is its robustness opposed to high gain observer but its convergence rate is fixed by the model.

Bounded error observers

We define a bounded error observer as a dynamical system such that we do not require the error between the estimated and the modeled variables to converge to zero anymore but to be bounded by a "reasonable" constant; "reasonable" meaning that this constant is small enough with respect to measurement errors. Moreover, this bound is zero if the model is perfectly known.

Definition 31 *A bounded error observer of* Eq. 1 *will be a dynamical system*

$$\dot{\hat{x}} = \hat{F}(\hat{x}, u, y) \quad \text{with} \quad \limsup_{t\to\infty} \|\hat{x} - x\| \le m$$

m a positive real constant depending on the knowledge of F such that m = 0 if F is perfectly known.

For the class of bioreactor model we consider in this paper, we assume that μ (s) is partially known (*i.e.*, μ (0) = 0), $\hat{\mu}(s)$ is perfectly known (i.e., $\hat{\mu}(s)$ is a Monod function for example). Then the knowledge of μ (s) is defined such that:

$$|\hat{\mu}(s) - \mu(s)| \le a,$$

with a a positive real constant. In this case, m given in definition 3.1 depends on a.

One-dimensional bounded error observer

In this section, we assume that s is measured exactly. Then we only reconstruct the biomass variable x. This is what we define as a one-dimensional bounded error observer.

Consider the system 2 and make the change of variable $(s,x) \to (s,z)$ with

$$z = \alpha x + \theta s,$$

where θ is a fixed real constant. The dynamics of z is given by:

$$\dot{z} = (1-\theta)\mu(s)\alpha x - dz + \theta d s_{\text{in}} \tag{8}$$

Proposition 32 The system

$$\dot{\hat{z}} = (1-\theta)\hat{\mu}(s)\alpha\hat{x} - d\hat{z} + \theta d s_{\text{in}}$$

is a bounded error observer of 8 *where* $\hat{\mu}(s)$ *is chosen such as* $|\hat{\mu}(s) - \mu(s)| \le a$ *with* $a \in \mathbb{R}^{*+}$ and θ is a gain (θ > 1).

Proof See [7].

This bounded observer has a positive static error depending on θ. Indeed, this error is equal to zero if θ = 1 and is fixed if θ is large.

Then to improve this observer, a good idea seems to choose θ time dependent (large at the beginning of the integration and equal to 1 at the end). Thus, this bounded observer can be seen as a switch between a kind of high gain observer and a kind of asymptotic one.

A proof of convergence when θ is time dependent can be found in [10, 11].

Simulations

We take for parameters values: $s_{\text{in}} = 50$, $d = 0.1$, and the difference a between μ(s) and $\hat{\mu}(s)$ equal to 0.2. Moreover, we take θ = 1, θ = 2 and θ time dependent (sigmoidal or exponential) (see Figs. 2 and 3).

We can see in this simulation that a better convergence rate is obtained by taking a bounded observer with θ time dependent rather than with an asymptotic observer (*i.e.*, θ = 1). Moreover, the bounded observer with θ time dependent converges towards 0 whereas the bounded observer with θ = 3 converges towards a fixed bound.

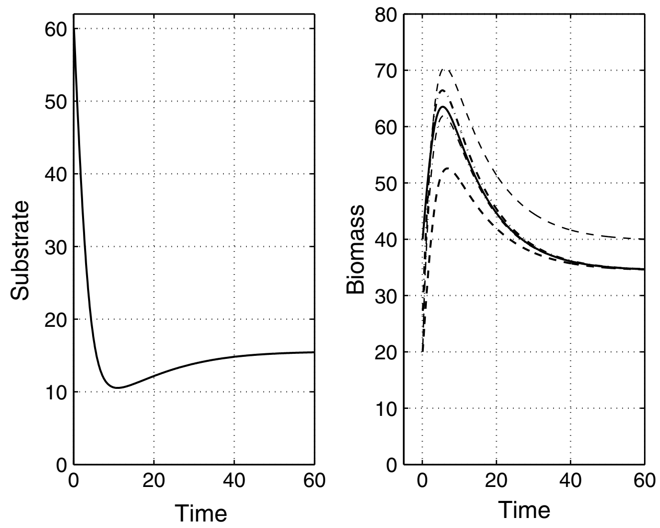

Fig. 2 In *dash line* the one-dimensional bounded observer for θ = 1 and θ = 3 , in *dash dot line* for θ time dependent, in *plain line* the model

Comments

The main problem of this bounded observer is the difficulty to adjust the gain to have a better convergence rate: the time of the switch between a large gain and a gain equal to 1 is hard to obtain. Indeed, as we don't reconstruct the measured variable we cannot use the output error (i.e the difference between the measured and the estimated state) as a control parameter.

To avoid this problem, we construct a two-dimensional hybrid bounded error observer (i.e., the measured variable is reconstructed) with the following view: when the error between the measured and the observed variable is large, a kind of high gain observer is constructed; when the error is small enough, a kind of asymptotic observer is built.

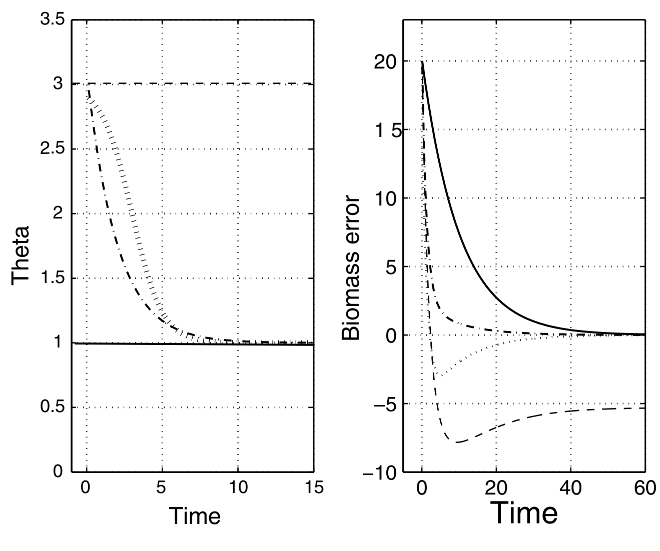

Fig. 3 θ values and biomass error for θ = 1 in *plain line*, for θ = 3 in *dash line*, for θ time dependent in *dotted line* and in *dash dotted line*

Two-dimensional hybrid bounded error observer

In this section, we assume that s is measured. Then the biomass variable x and the substrate variable s are reconstructed. This is what we define as a two-dimensional hybrid bounded error observer.

We consider the system 2. We make the change of variables $(s,\ x) \to\ (s,\ z)$ with $z = \alpha\ x + s$. The new dynamical system is obtained as follows:

$$\begin{cases} \dot{s} = -\hat{\mu}(s)(z-s) - ds + ds_{\text{in}} + (\hat{\mu}-\mu)(z-s) \\ \dot{z} = -dz + ds_{\text{in}} \\ y = s. \end{cases} \tag{9}$$

We assume that Hypotheses 2.1 and in [6] are verified.

Proposition 33 *The dynamical system*

$$\begin{cases} \dot{\hat{s}} = -\hat{\mu}(s)(\hat{z}-\hat{s}) - d\hat{s} + ds_{\text{in}} - k_1\theta(\hat{s}-s) \\ \dot{\hat{z}} = -d\hat{z} + ds_{\text{in}} - k_2\theta^2(\hat{s}-s) \end{cases} \tag{10}$$

with θ *a positive fixed gain* , k_1 and k_2 *fixed gains verifying* 11, k_2 *depends on the error*$\hat{s}-s$ *such that* $k_2 = 0$ *when*$\hat{s}-s$ (ϵ *a fixed small constant), is a bounded error observer for* Eq. 9 *where*$\hat{\mu}(s)$ *is chosen such that*: $|\hat{\mu}(s)-\mu(s)| \leq a$ *with* $a \in \mathbb{R}^{+*}$.

To prove the proposition, we need the following lemma.

Lemma 34 *It exists a constant* $\epsilon > 0$, *such that* $s(0) > \epsilon$ *implies* $s(t) > \epsilon$ *and* $\mu(s(t)) > \mu(\epsilon)$ *for all* t.

The proof is easy using standard techniques for invariant sets [8]. Thanks to this lemma, we can always choose ϵ such that $\hat{\mu}(s) > \hat{\mu}(\epsilon) = l$.

Another useful property of the system 2 is the boundedness of s and x.

Proof The ideas are the same as in [6] for the high gain observer. The main steps of the proof are:

First step (kind of high gain observer)

- Consider the error e between modeled and estimated variables.
- Consider the change of variable $e_1 = \Delta_\theta^{-1} e$.
- Prove that the positive definite function $V = \frac{1}{2} e_1^t S e_1$ is bounded.
- Then come back to the initial variable e and prove that e is bounded. More precisely, the limit of the bound when t tends to ∞ and θ is fixed (large) does not depend on θ.

Second step (kind of asymptotic observer)

- Switch to the asymptotic observer taking $k_2 = 0$.
- Inject $e_z(t)$ in $\dot{e}_s$.
- Prove that e_s is as small as we want with a fixed convergence rate.

Third step (conclusion)

- Conclude that e is as small as we want.

In the following, we detail these steps.

First step

– Consider e the error in s, and z such that

$$e = \begin{pmatrix} e_s = \hat{s} - s \\ e_z = \hat{z} - z \end{pmatrix}.$$

It verifies the following equation:

$$\dot{e} = \begin{pmatrix} -k_1\theta & -\hat{\mu}(s) \\ -k_2\theta^2 & 0 \end{pmatrix} e + \begin{pmatrix} \hat{\mu}(s) - d & 0 \\ 0 & -d \end{pmatrix} e + (\mu(s) - \hat{\mu}(s))(z - s)\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

– Taking $e_1 = \Delta_\theta^{-1} e = \begin{pmatrix} \frac{1}{\theta} e_s \\ \frac{1}{\theta^2} e_z \end{pmatrix}$ with $\Delta_\theta^{-1} = \begin{pmatrix} \theta & 0 \\ 0 & \theta^2 \end{pmatrix}$, we obtain the following equation for e_1:

$$\begin{aligned} \dot{e}_1 &= \Delta_\theta^{-1}\begin{pmatrix} -k_1\theta & -\hat{\mu}(s) \\ -k_2\theta^2 & 0 \end{pmatrix}\Delta_\theta e_1 \\ &+\Delta_\theta^{-1}\begin{pmatrix} \hat{\mu}(s) - d & 0 \\ 0 & -d \end{pmatrix}\Delta_\theta e_1 \\ &+\Delta_\theta^{-1}(\mu(s) - \hat{\mu}(s))(z - s)\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

That is to say:

$$\dot{e_1} = \theta\begin{pmatrix} -k_1 & -\hat{\mu}(s) \\ -k_2 & 0 \end{pmatrix} e_1 + \begin{pmatrix} \hat{\mu}(s) - d & 0 \\ 0 & -d \end{pmatrix} e_1 + (\mu(s) - \hat{\mu}(s))(z - s)\begin{pmatrix} \frac{1}{\theta} \\ 0 \end{pmatrix}.$$

Consider the matrix $A = \begin{pmatrix} 0 & -\hat{\mu}(s) \\ 0 & 0 \end{pmatrix}$, $C = (1\ 0)$. Then it exists a real constant $\lambda > 0$, a vector $K \in \mathbb{R}^2, K^t = (k_1\, k_2)$ and a symmetric, positive definite 2×2 matrix S only depending on the bounds of $\hat{\mu}(s)$ such that:

$$S(A - KC) + (A - KC)^t S \leq -\lambda Id. \tag{11}$$

A proof of this lemma can be found in [6]. We can notice that $A - KC$ is stable, meaning $k_1 > 0$ and $k_2 < 0$. With matrix notation, we obtain the equation for e_1: $\dot{e_1} = \theta(A - KC)e_1 + Be_1 + (\mu(s) - \hat{\mu}(s))(z - s)\begin{pmatrix} \frac{1}{\theta} \\ 0 \end{pmatrix}$ with $B = \begin{pmatrix} \hat{\mu}(s) - d & 0 \\ 0 & -d \end{pmatrix}$.

– Consider the positive definite function

$$V = \frac{1}{2} e_1^t S e_1 = \frac{1}{2}||e_1||_S^2$$

We want to prove that V is bounded. We have:

$$\begin{aligned} \dot{V} &< -\theta\frac{\lambda}{2}||e_1||_2^2 + N(S).N(B).||e_1||_2^2 \\ &+ \frac{1}{\theta}||e_1||_2.N(S).|\mu(s) - \hat{\mu}(s)|.|z - s| \end{aligned}$$

where $N(S)$, $N(B)$ are the induced matrix norm corresponding to the Euclidean one

$$N(M) = \max\{\sqrt{\lambda}, \lambda \in \text{Spect}(M^* M)\}$$

We remark that in our case $N(B)$ is equal to $\hat{\mu}(s) - d$ which is between $\hat{\mu}(\epsilon) - d$ and $\hat{\mu}_{\max} - d$ using lemma 3.4. Moreover, since the states variables are bounded:

$$\dot{V} < \left(-\frac{\theta\lambda}{2} + N(S).N(B)\right)||e_1||_2^2 + \frac{\alpha a x_{\max} N(S)}{\theta}||e_1||_2.$$

As all the norms are equivalent in $\mathbb{R}^n$, we have

$$\gamma_1||e_1||_S \leq ||e_1||_2 \leq \gamma_2||e_1||_S.$$

Hence:

$$\dot{V} < 2\Gamma_1 V + 2\Gamma_2\sqrt{V},$$

with $\Gamma_1 = \gamma_1^2\left(-\frac{\theta\lambda}{2} + N(S).N(B)\right)$ and $\Gamma_2 = \gamma_2\frac{\alpha a x_{\max} N(S)}{\theta}$. Thus:

$$\frac{\dot{V}}{2\sqrt{V}} = \frac{d\sqrt{V}}{dt} < \Gamma_1\sqrt{V} + \Gamma_2.$$

Using the Gronwall lemma:

$$\sqrt{V} < \left(\sqrt{V(0)} + \frac{\Gamma_2}{\Gamma_1}\right)e^{\Gamma_1 t} - \frac{\Gamma_2}{\Gamma_1}.$$

We easily see that for $t \to \infty$, $\sqrt{V}$ is bounded.

– We come back to the initial variable e and we conclude on the convergence. We prove by a simple computation that

$$\frac{m^2}{\theta^4} e^t e < V < \frac{M^2}{\theta^4} e^t e,$$

with m, M positive real constants chosen such that

$$\Delta_\theta^{-1} S\Delta_\theta - \frac{m^2}{\theta^4} Id \text{ positive}$$

$$\Delta_\theta^{-1} S\Delta_\theta - \frac{M^2}{\theta^4} Id \text{ negative.}$$

Finally, we conclude:

$$\sqrt{e^t e} < \left(\frac{M}{m}\sqrt{e^t(0)e(0)} + \frac{\theta^2\Gamma_2}{\Gamma_1}\right)e^{\Gamma_1 t} - \frac{\theta^2\Gamma_2}{\Gamma_1}. \tag{12}$$

When $t \to \infty$, we can easily see that $||e||$ is bounded. More precisely, the limit of the bound:

$$\lim_{t\to\infty}\left(\frac{M}{m}\sqrt{e^t(0)e(0)} + \frac{\theta^2\Gamma_2}{\Gamma_1}\right)e^{\Gamma_1 t} - \frac{\theta^2\Gamma_2}{\Gamma_1} = -\frac{\theta^2\Gamma_2}{\Gamma_1}$$

does not depend on θ when we focus on the expressions of Γ_1 and $\Gamma_2\left(i.e., \frac{\Gamma_2}{\Gamma_1} \sim \frac{K}{\theta^2}\right)$.

Second step

– We switch to an asymptotic observer like taking $k_2 = 0$ when $|\hat{s} - s|$ stays during some time less or equal to ε (ε a fixed small constant). We obtain the new bounded error observer:

$$\begin{aligned} \dot{\hat{s}} &= -\hat{\mu}(s)(\hat{z} - \hat{s}) - d\hat{s} + ds_{\text{in}} - k_1\theta(\hat{s} - s) \\ \dot{\hat{z}} &= -d\hat{z} + ds_{\text{in}} \end{aligned}. \tag{13}$$

The equation of the error becomes:

$$\dot{e} = \begin{pmatrix} -k_1\theta & -\hat{\mu}(s) \\ 0 & 0 \end{pmatrix} e + \begin{pmatrix} -\hat{\mu}(s) - d & 0 \\ 0 & -d \end{pmatrix} e + (\mu(s) - -\hat{\mu}(s))(z - s)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

– Thus, solving the second equation and injecting it in the first one, we obtain:

$$\dot{e_s} = -k_1\theta e_s - \hat{\mu}(s)e_z(0)\mathrm{e}^{-dt} + (\hat{\mu}(s) - d)e_s + (\mu(s) - \hat{\mu}(s))(z - s)$$

$$e_z = e_z(0)e^{-dt}$$

This observer is a kind of asymptotic one since convergence rate of e_z is fixed by the model and is equal to e^{-dt}; moreover, e_z goes asymptotically to zero.

– Let us consider $|e_s|$. It dynamics is given by

$$\dot{|e_s|} = \mathrm{sgn}(e_s)\dot{e_s}.$$

That is:

$$\dot{|e_s|} = \mathrm{sgn}(e_s)(\mu(s) - \hat{\mu}(s))(z - s) + (\hat{\mu}(s) - d - k_1\theta)|e_s| - \mathrm{sgn}(e_s)\hat{\mu}(s)e_z(0)\mathrm{e}^{-dt}.$$

But $\mathrm{sgn}(e_s) \leq 1$ and for all s, $\hat{\mu}(s) - d - k_1\theta$ for θ fixed (large) and $k_1 > 0$; thus, we obtain:

$$\dot{|e_s|} \leq a\alpha x_{\max} + (\hat{\mu}_{\max} - d - k_1\theta)|e_s| + \hat{\mu}_{\max}|e_z(0)|\mathrm{e}^{-dt}.$$

Hence, by Gronwall lemma:

$$\begin{aligned} |e_s| \leq & (|e_s(0)| + \frac{\hat{\mu}_{\max}|e_z(0)|}{\hat{\mu}_{\max} - d - k_1\theta})\mathrm{e}^{(\hat{\mu}_{\max} - d - k_1\theta)t} \\ & + \frac{a\alpha x_{\max}}{-\hat{\mu}_{\max} + d + k_1\theta}\mathrm{e}^{(\hat{\mu}_{\max} - d - k_1\theta)t} \\ & + \frac{\hat{\mu}_{\max}|e_z(0)|}{-\hat{\mu}_{\max} + d + k_1\theta}\mathrm{e}^{-dt} \\ & + \frac{a\alpha x_{\max}}{-\hat{\mu}_{\max} + d + k_1\theta} \end{aligned} \tag{14}$$

The bound of $|e_s|$ 14 depends on k_1 θ meaning that as $k_1\theta$ increases, the bound decreases. Thus, the error e_s is as small as we want with a fixed convergence rate of order $e^{-dt}/(-\hat{\mu}_{\max} + d + k_1\theta)$.

Third step

– Finally, the global error e is as small as we want (*i.e.*, e_z tends to 0 and e_s is as small as we want) with a large convergence rate at the beginning of the integration and with a fixed convergence rate at the end.

Before the switch between the two different observers, one can see that the error of the first converges, when $t \to \infty$, towards a bound independent of the gain θ 12.

To obtain a faster convergence rate than the asymptotic one, the initial error on the unmeasured variable must be bigger than the limit error on this variable (related to 12). Under this condition, the first step, which can be seen as a high gain observer, goes rapidly towards the bound 12; then, when the output error (i.e the difference between the measured and the observed variable) is small enough, we switch to the asymptotic like observer. Let us remark that the switch takes place rapidly because the output error reaches a low value very fast.

One can notice that the final error bound 14 depend on θ, that is to say if θ is large this bound goes to zero, and we go as near as we want: it is the idea of "practical observer" [4].

Remark 35 *All proofs and simulations have been done for fixed d and* s_{in}. *However, it is easy to see that these proofs are valid for* $s_{\mathrm{in}}(t)$ *(i.e., eliminated in error dynamic equations) and* $d(t) > d_{\min} > 0$. *For example, in this case, bound* 14 *is valid with* $d_{\min}$ *instead of d.*

Simulations

We take for parameters values: $s_{\mathrm{in}} = 50$, $d = 0.1$, and the difference a between μ(s) and $\hat{\mu}(s)$ equal to 0.2. Moreover, we choose $k_2 = -1.5$ when the absolute value error between $\hat{s}$ and s is bigger than 0.1 else, we take $k_2 = 0$. The other gains are θ = 3 and $k_1 = 5$.

We take for initial conditions $\hat{s}(0) = 10$, $\hat{z}(0) = 30$ that is to say $\hat{x}(0) = 20$ (Fig. 4).

The peak which appears at the beginning of the simulations provides non positive observed variables; this is the same phenomenon put into relief in high gain observer when gain and output error are large (Fig. 5).

We see that hybrid observer converges faster than asymptotic one; indeed, if we choose $s - \hat{s} = 0.1$ (see the first part of Figure 5), hybrid observer reaches this bound for $t \simeq 10$, asymptotic observer for $t \simeq 20$ and

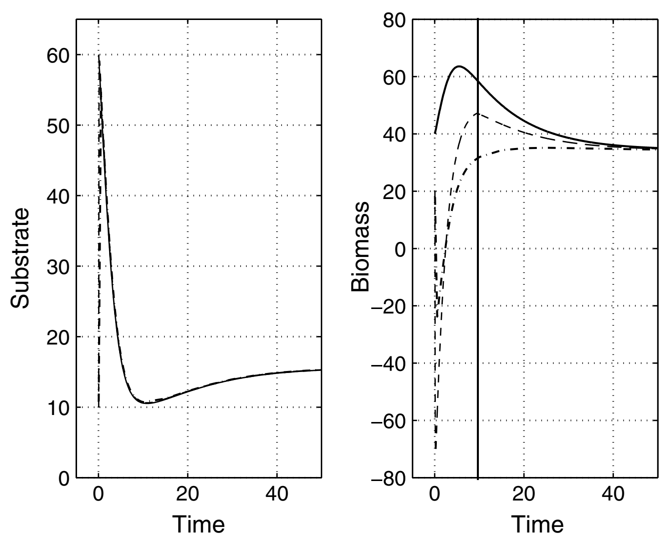

Fig. 4 In *dash line* the hybrid bounded observer, in *dash-dotted* line the asymptotic one, in *plain line* the model

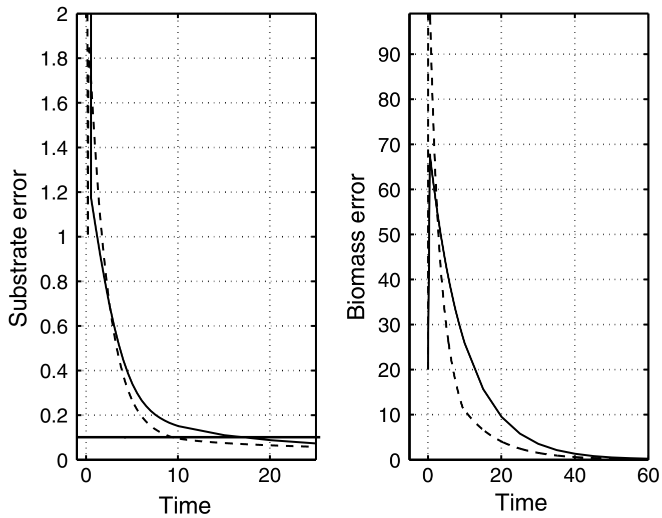

Fig. 5 In *plain line* the biomass error and the substrate error of the asymptotic observer; in *dash line* of the hybrid bounded observer

after this bound hybrid bounded observer is always below asymptotic one (see the second part of Fig. 4).

Conclusion

The purpose of bounded observers is to provide a tool allowing the state variable estimation when the model is poorly known, that is usually the case in biology.

We build observers reconstructing variables with a reasonable error. The convergence rate of one-dimensional observer cannot be improved because we cannot consider the output error as a control parameter. Thus, we build a two-dimensional observer and we obtain a faster convergence rate than the asymptotic observer if the initial error is large enough.

A way to improve the convergence seems to build a more adaptive version of the hybrid observer taking a smooth gain k_2 (depending on a differential equation for example). Some simulations studies seem to support this idea [11].

In this paper, we only consider two dimensional systems, a generalization to higher dynamical system is evidently possible. For example, we can show that for the 4-dimensional system in the canonical form, defined by the following equations, an hybrid observer can be built. Consider the system:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2, x_3) \\ \dot{x}_3 = \hat{f}_3(x, z) + (f_3(x, z) - \hat{f}_3(x, z)) \\ z = -Kz + u \\ y = x_1, \end{cases} \tag{15}$$

with $x_i \in \mathbb{R}$ $i = 1, 2, 3$, $z \in \mathbb{R}$ and $K, u \in \mathbb{R}^+$ fixed constants, and z defined such that:

$$z = \sum_{i=1}^{4} c_i x_i \Leftrightarrow x_4 = \frac{1}{c_4}(z - c_1 x_1 - c_2 x_2 - c_3 x_3),$$

with c_i positive fixed constants.

We assume that $f_3(x,z)$ is partially known and that $\hat{f}_3(x, z)$ is defined such that:

$$\left| f_3(x, z) - \hat{f}_3(x, z) \right| \le a \quad \text{with} \quad a > 0.$$

An hybrid observer can be built:

$$\begin{cases} \dot{\hat{x}}_1 = f_1(\hat{x}_1, \hat{x}_2) - k_1\theta(\hat{x}_1 - x_1) \\ \dot{\hat{x}}_2 = f_2(\hat{x}_1, \hat{x}_2, \hat{x}_3) - k_2\theta^2(\hat{x}_1 - x_1) \\ \dot{\hat{x}}_3 = \hat{f}_3(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{z}) - k_3\theta^3(\hat{x}_1 - x_1) \\ \dot{\hat{z}} = -K\hat{z} + u - k_4\theta^4(\hat{x}_1 - x_1) \\ y = x_1. \end{cases} \tag{16}$$

The main hypotheses are that the error of the model is in the penultimate unmeasured variables (x_3) and the last variable z is a linear combination of the state variables verifying a linear differential equation. There are also more technical hypotheses on the f_i similar to [4].

To prove the convergence towards an error as small as we want, the same steps detailed before can be used.

First, in the hybrid bounded observer, we take $k_4 \neq 0$, we have a kind of high gain observer with a partially known model; then, bounded observer converges towards a fixed bound with a large convergence rate.

Moreover we take $k_4 = 0$ and we have a kind of high gain observer with a bounded error on the penultimate unmeasured variable (x_3) and an asymptotic observer for the last state (z); we can prove using the results of Farza et al. [4] that the hybrid observer converges towards an error as small as we want with a fixed convergence rate (because of the asymptotic observer).

We obtain a better convergence rate than the asymptotic observer. More details can be found in [11].

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