J. C. BERMOND and D. SOTTEAU, PARIS

CYCLE AND CIRCUIT DESIGNS ODD CASE

1. Introduction.

In what follows, n, k, λ will be positive integers with $n \geq k$ and G will always denote a graph with k vertices. Definitions not given here can be found in Berge's book /1/.
Let us denote by
λK_n: the complete multigraph with n vertices and any two distinct vertices joined by exactly λ edges.

λK_n^*: the complete symmetric directed multigraph with n vertices and any two distinct vertices x and y joined by λ arcs (x,y) and λ arcs (y,x).

$\lambda K_{n_1,n_2,\dots,n_h}$: the complete multipartite graph with vertex set $X = \bigcup X_i$, $1 \leq i \leq h$, where the X_i are disjoint sets with $|X_i| = n_i$ and where two vertices, which belong to different sets X_i and X_j are joined by λ edges.

$\lambda K^*_{n_1,n_2,\dots,n_h}$: the complete directed multipartite graph.

C_k: the elementary circuit of length k.

$\vec{C}_k$: the elementary directed circuit of length k.

We shall say that a graph (respectively directed graph) H can be decomposed into subgraphs isomorphic to G, where G is a given graph (respectively directed graph) if we can partition the edges (resp. arcs) of H into subgraphs isomorphic to G.

1.1. An (n,k,λ) G-design is an edge disjoint decomposition of λK_n into subgraphs isomorphic to G. The G-designs have been introduced by Hell et Rosa /12/. In the particular case where G ist the complete graph K_k, an (n,k,λ) K_k-design is nothing else than an (n,k,λ) B.I.B.D. (balanced incomplete block design: see Hall /9/).

1.2. A similar definition holds for directed graphs. If G is a directed graph, with k vertices, an (n,k,λ) G-design is a partition of the arcs of λK_n^* into subgraphs isomorphic to G.

1.3. For results concerning the existence of G-designs, the reader can see, for example, the survey we have done in /6/.

Here we are interested in the particular case where G is equal to C_k or $\vec{C}_k$. An (n,k,λ) C_k-design is also called a B.C.D. (n,k,λ) (balanced circuit design /13/).

1.4. Proposition. Necessary conditions for the existence of an (n,k,λ) C_k-design are

$$\lambda n(n-1) \equiv 0 \pmod{2k}$$
$$\lambda(n-1) \equiv 0 \pmod{2}.$$

1.5. Proposition. A necessary condition for the existence of an (n,k,λ) $\vec{C}_k$-design is

$$\lambda n(n-1) \equiv 0 \pmod{k}.$$

1.6. In this paper we will restrict ourselves to the case k odd. The case k even will be considered in /7/.
For k odd, it has been proved that the necessary conditions of the proposition 1.4 are sufficient for

$k = 3$ (case of Steiner Triple System) see Hanani /10/.
$k = 5$ by Huang and Rosa /13/.
k odd, $n = 2qk + 1$ or $n = 2qk + k$, $\lambda = 1$ by Rosa /16/.

The necessary condition 1.5 of the existence of a $\vec{C}_k$-design is sufficient for $k = 3$, $\lambda = 1$ except $n = 6$ in /2/ and also by Bruck /8/ Mendelsohn /14/.

A short proof can be found in /3 or 4/.

$k = 5$, $\lambda = 1$ in /3/ and /17/ and also by Merriel /15/.
$k = p^{\alpha}$, p an odd prime and $\lambda = 1$ in /17/ and also for $k = 7$ by Merriel /15/.
It has also been shown that an $(n,k,1)$ $\vec{C}_k$-design exists if n is a power of a prime and $n(n-1) \equiv 0 \pmod{k}$ by Hartnell and Milgram /11/ and if $n \equiv 0$ or $1 \pmod{k}$, k odd in /17/.
Finally we note that the asymptotic results obtained by Wilson /19/ prove that the conditions 1.4 and 1.5 are sufficient, for a given k, for n large enough.
Here we shall prove the following

1.7. Theorem. The necessary conditions of propositions 1.4 and 1.5 are sufficient for $k = 3, 5, 7$ except for $n = 6, k = 3, \lambda = 1$.

2. General Lemmas.

The 3 following lemmas are simple consequences of the definitions.

2.1. Lemma. If there exists an (n,k,λ_1) G-design and an (n,k,λ_2) G-design, then there exists an $(n,k,p\lambda_1 + q\lambda_2)$ G-design (p and q are positive integers).

2.2. Lemma. Let H^* be the directed graph obtained from H by associating to each edge of H two opposite arcs. If H can be decomposed into C_k, then H^* can be decomposed into $\vec{C}_k$.

Remark. The contrary is false: for example we shall wee that there exists a $(4,3,1)$ $\vec{C}_3$-design, but by 1.4 there exists no $(4,3,1)$ C_3-design.

2.3. Lemma. If there exists an (n,k,λ) $\vec{C}_k$-design, then there exists an $(n,k,2\lambda)$ C_k-design.

Remark. The contrary is false: there exists a $(6,3,2)C_3$-design (see Hanani /10/), but there exists no $(6,3,1)$ $\vec{C}_3$-design /2/.

2.4. Lemma. If there exists an (n_i,k,λ) G-design for $1 \leq i \leq h$ and if $\lambda K_{n_1,\ldots,n_h}$ (or $\lambda K^*_{n_1,\ldots,n_h}$, if G is a directed graph) can be decomposed into subgraphs isomorphic to G, then there exists a $(\Sigma n_i,k,\lambda)$ G-design.

2.5. Lemma. If there exists an (n_i+1,k,λ) G-design for $1 \leq i \leq h$ and if $\lambda K_{n_1,\ldots,n_h}$ (or $\lambda K^*_{n_1,\ldots,n_h}$ if G is a directed graph) can be decomposed into subgraphs isomorphic to G then there exists a $((\Sigma n_i) + 1,k,\lambda)$ G-design.

Proofs. We shall give the proof of lemma 2.5 in the undirected case. (The proof of lemma 2.4 and of the directed case are similar). Let the vertices of $\lambda K_{(\Sigma n_i)+1}$ be: $X_1 \cup X_2 \cup \ldots \cup X_h \cup \{\infty\}$, with $|X_i| = n_i$, for $1 \leq i \leq h$.

The graph $\lambda K_{(\Sigma n_i)+1}$ is the edge disjoint union of:

- the h complete multigraphs, constructed on the sets $X_i \cup \{\infty\}$ which are isomorphic to λK_{n_i+1} and thus can be decomposed (by hypothesis) into subgraphs isomorphic to G.

- the complete multipartite multigraph with vertex sets $\cup X_i$ which is isomorphic to $\lambda K_{n_1,\ldots,n_h}$ decomposable into subgraphs isomorphic to G by hypothesis.

2.6. <u>Remark.</u> The interest of lemmas 2.4 and 2.5 is that it gives a method of composition which enables us to construct G-designs by knowing the existence of smaller ones. (In the case $G = K_k$, these lemmas are used with different notations by Hanani /10/ and Wilson /19/: indeed a decomposition of $\lambda K_{n_1,\ldots,n_h}$ into K_k is equivalent to the existence of group divisible designs).

We shall apply this method with tripartite graphs (h =3). The following lemma will enable us to construct decompositions of tripartite multigraphs from decomposition of smaller ones.

2.7. Lemma. <u>If</u> $K_{a,b,c}$ <u>and</u> $K_{a,b,c'}$ <u>can be decomposed into subgraphs isomorphic to</u> G, <u>then also</u> $K_{qa,qb,pc+(q-p)c'}$ <u>with</u> p <u>and</u> q <u>positive integers</u>, $0 \le p \le q$.

<u>Proof.</u> Let the vertices of $K_{qa,qb,pc+(q-p)c'}$ be $X \cup Y \cup Z$ with

$$X = \bigcup_{i=1}^{q} X_i,\ |X_i| = a\ (1 \le i \le q);\ Y = \bigcup_{i=1}^{q} Y_i,\ |Y_i| = b\ (1 \le i \le q)$$

and $Z = \bigcup_{i=1}^{q} Z_i$, $|Z_i| = c$, $1 \le i \le p$ and $|Z_i| = c'$ for $p+1 \le i \le q$.

Then $K_{qa,qb,pc+(q-p)c'}$ is the edge disjoint union of the q^2 tripartite graphs constructed respectively on X_i, Y_{i+j}, Z_j with $1 \le i \le q$, $1 \le j \le q$ and where the indices of Y are to be taken modulo q. These q^2 graphs are isomorphic to $K_{a,b,c}$ or to $K_{a,b,c'}$ which can be decomposed into subgraphs isomorphic to G.

A similar lemma holds in the directed case.

2.8. Lemma. <u>If</u> $K^*_{a,b,c}$ <u>and</u> $K^*_{a,b,c'}$ <u>can be decomposed into subgraphs isomorphic to</u> G <u>then also</u> $K^*_{qa,qb,pc+(q-p)c'}$ (p,q <u>positive integers</u> $0 \le p \le q$).

The two following lemmas can be sometimes useful.

2.9. Lemma. Let $h \equiv 1$ or 3 (mod. 6); if K_{n_i} (resp. K_{n_i+1}), $1 \le i \le h$ and K_{n_i,n_j,n_k}, $1 \le i < j < k \le h$, can be decomposed into subgraphs isomorphic to G then also K_n (resp. K_{n+1}) for $n = \sum_{i=1}^{h} n_i$.

Sketch of proof: These lemmas result from the existence of (6t + 1,3,1) or (6t + 3,3,1) Steiner Triple Systems (see /17/ for more details).

3. Existence of C_3, C_5, C_7, $\vec{C}_3$, $\vec{C}_5$, $\vec{C}_7$-Designs.

In this paragraph, k will always denote an odd integer ≥ 3. We shall prove the theorems 1.7 (k = 3, 5, 7): we give first the ideas of the proof and general propositions which are valid for all (odd) $k \ge 5$.
We apply them only to the particular values of k : 5,7 but the methods can be applied to give other results for greater values of k.

3.1. The proofs of theorem 1.7 for k = 5 or 7 are by induction and all of the same kind:

Step 1. First we prove the existence of decomposition of some K_n or K_n^*, for the first values of n in order to start the induction.

Step 2. We prove the existence of decomposition of some $K_{a,b,c}$ or $K^*_{a,b,c}$ which will be needed in the verification.

Step 3. Then we use lemma 2.4 or lemma 2.5 with tripartite graphs (h = 3) or lemma 2.9. We indicate in each case, in a table, which lemma we use and which values of n_i occur in the lemma. The verification that the hypothesis of these lemmas are satisfied follows

- for the K_{n_i} or $K^*_{n_i}$ from step 1 or induction hypothesis.

- for the K_{n_1,n_2,n_3} or $K^*_{n_1,n_2,n_3}$ from lemma 2.7 (or 2.8):

we indicate in the same table the values of the parameters a, b, c, c', p and q used. The $K_{a,b,c}$ and $K_{a,b,c'}$ used in these lemmas can be decomposed by step 2.

For step 1 the following proposition is useful

3.2. Proposition. Let k be an odd integer ≥ 3, then there exists

a) a$(k,k,1)$ C_k-design
b) a$(k,k,1)$ $\vec{C}_k$ -design
c) a$(k+1,k,1)$ $\vec{C}_k$-design
d) a$(2k,k,1)$ $\vec{C}_k$-design for $k \geq 5$.

Proof. a) is well known (see Berge /1/).
b) is a consequence of a) and lemma 2.2.
c) has been proved in /5/: Let $Z_k \cup \{\infty\}$ be the set of vertices of K^*_{k+1} (we recall that Z_k is the additive group of residues modulo k). A decomposition of K^*_{k+1} into $\vec{C}_k$ is given by the following k directed circuits, for $0 \leq i \leq k-1$,
$(\infty,\ldots,\alpha+i,\ k-1-\alpha+i,\ldots,\ \beta+i,k-2-\beta+i,\ldots,\infty)$ with $0 \leq \alpha \leq [k/4]-1$, and $[k/4] \leq \beta \leq (k-3)/2$, and the circuit $(\ldots,\alpha,\alpha+(k+1)/2,\ldots)$ with $0 \leq \alpha \leq (k-1)/2$.
d) There does not exist a $(6,3,1)$ $\vec{C}_3$-design (see /2/). For $k \geq 5$, there exists a $(2k,k,1)$ $\vec{C}_k$-design (/4/): let $Z_{k-1} \cup A_\infty$, with $A_\infty = \{\infty_0, \ldots, \infty_k\}$ be the set of vertices of K^*_{2k}. As the complete graph on A_∞ is isomorphic to K^*_{k+1} and thus is decomposable into $\vec{C}_k$, it suffices to prove that the arcs of K^*_{2k}, which do not join two points of A_∞, can be partitioned into $\vec{C}_k$; that is given by the following $3(k-1)$ directed circuits for $0 \leq i \leq k-2$:

$$(\infty_0, i, \infty_1, k-2+i, \ldots, \alpha+i, k-2+\alpha+i, \ldots, (k-3)/2+i, \infty_0) \text{ with}$$

$1 \leq \alpha \leq (k-5)/2$

$$(i,\ldots,\ \beta+i, \infty_{\beta+1}, \ldots, i) \text{ and } (i,\ldots,k-1-\beta+i,\ \infty_{\beta+(k+1)/2}, \ldots, i) \text{ with}$$

$1 \leq \beta \leq (k-1)/2$.

For step 2 the following propositions are useful:

3.3. Proposition. Let k be an odd integer ≥ 3, then

a) $K_{k,k,k}$ can be decomposed into C_k.
b) $K^*_{k,k,2k}$ can be decomposed into $\vec{C}_k$, for $k \geq 5$.
c) $K^*_{k,2k,2k}$ can be decomposed into $\vec{C}_k$, for $k \geq 5$.
d) $K_{qk,qk,qk}$ can be decomposed into C_k, for $k \geq 5$, q integer.

A proof of this proposition can be found in /17/.

3.4. **Proposition.** <u>For all integers</u> n,p, $n \geq p \geq 1$, $(2p+1)K_{n,n,n}$ <u>can be decomposed into</u> C_{2p+1}.

<u>Proof.</u> Let $X \cup Y \cup Z$ be the set of vertices of $(2p+1)\,K_{n,n,n}$ with $X = \{x_i\}_{0 \leq i \leq n-1}$, $Y = \{y_i\}_{0 \leq i \leq n-1}$, $Z = \{z_i\}_{0 \leq i \leq n-1}$; the $3n^2$ circuits of length $2p+1$ of the decomposition are given by the following circuits $\{C_{i,j}, C'_{i,j}, C''_{i,j}\}$, for $0 \leq i \leq n-1$ and $0 \leq j \leq n-1$:

$$C_{i,j} = (\ldots, x_{i+\alpha}, y_{j+\alpha}, \ldots, z_{i+j}, x_i)$$
$$C'_{i,j} = (\ldots, y_{i+\alpha}, z_{j+\alpha}, \ldots, x_{i+j}, y_i) \qquad 0 \leq \alpha \leq p-1$$
$$C''_{i,j} = (\ldots, z_{i+\alpha}, x_{j+\alpha}, \ldots, y_{i+j}, z_i)$$

3.5. Theorem. <u>There exists an</u> $(n,5,\lambda)$ C_5-<u>design if and only if</u>

$$\lambda n(n-1) \equiv 0 \pmod{10}$$
$$\lambda(n-1) \equiv 0 \pmod{2}$$

<u>There exists an</u> $(n,5,\lambda)$ $\vec{C}_5$ <u>design if and only if</u>

$$\lambda n(n-1) \equiv 0 \pmod{5}.$$

<u>Proof.</u> By lemma 2.1 it suffices to verify the theorem for the minimum values of λ, that is to prove the existence

- of C_5-designs in the following cases:

a.	$\lambda = 1$,	$n \equiv 1, 5$	(mod 10)
b.	$\lambda = 2$,	$n \equiv 0, 6$	(mod 10)
c.	$\lambda = 5$,	$n \equiv 3, 7, 9$	(mod 10)
d.	$\lambda = 10$,	$n \equiv 2, 4, 8$	(mod 10).

- of $\vec{C}_5$-designs in the following cases:

e.	$\lambda = 1$,	$n \equiv 0, 1$	(mod 5)
f.	$\lambda = 5$,	$n \equiv 2, 3, 4$	(mod 5).

In order to minimize the verifications we shall use lemmas 2.2 and 2.3 and thus give a proof of the different cases a, b, c, d, e, f in a convenient order.

a. Existence of an $(n, 5, 1)$ C_5-design for $n \equiv 1$ or $5 \pmod{10}$.
We use the method exposed in 3.1.

Step 1. K_5, K_{11}, K_{21}, K_{25}, K_{85} can be decomposed into C_5. For K_5 see 3.2 a). For the others it can be done by direct construction (see Rosa /16/). For example for K_{11}: if Z_{11} is the set of vertices of K_{11} a decomposition is given by the circuits: $(i, i+2, i+10, i+6, i)$, $0 \leq i \leq 10$.

Step 2. $K_{5,5,5}$, $K_{10,10,10}$, $K_{10,10,20}$, $K_{15,15,15}$, $K_{15,15,5}$, $K_{15,15,25}$ can be decomposed into C_5.

- $K_{5,5,5}$, $K_{10,10,10}$, $K_{15,15,15}$: see proposition 3.3 a), d).

- $K_{10,10,20}$: Let $X \cup Y \cup Z$ be the set of vertices of $K_{10,10,20}$ with $X = \{x_i\}_{0 \leq i \leq 9}$, $Y = \{y_i\}_{0 \leq i \leq 9}$, $Z = \{z_i\}_{0 \leq i \leq 9} \cup \{\infty_i\}_{0 \leq i \leq 9}$.

A decomposition into C_5 is given by the 100 following circuits:

$$(x_{i+j}, y_j, z_{2i+1+j}, y_{j+1}, \infty_i, x_{i+j})_{0 \leq i \leq 4,\ 0 \leq j \leq 9}$$

$$(x_j, z_{2i+1+j}, x_{j+1}, \infty_{i+5}, y_{i+j+1}, x_j)_{0 \leq i \leq 4,\ 0 \leq j \leq 9}.$$

- $K_{15,15,5}$: Let $X \cup Y \cup Z$ be the set of vertices of $K_{15,15,5}$ with $X = \{x_i\}_{0 \leq i \leq 14}$, $Y = \{y_i\}_{0 \leq i \leq 14}$, $Z = \{\infty_i\}_{0 \leq i \leq 4}$.

A decomposition into C_5 is given by the 75 following circuits:

$$(x_i, y_{i+3j+1}, x_{i+1}, y_{i+3j+3}, \infty_j, x_i)_{0 \leq j \leq 4,\ 0 \leq i \leq 14}.$$

- $K_{15,15,25}$: Let $X \cup Y \cup Z$ be the set of vertices of $K_{15,15,25}$ with $X = \{x_i\}_{0 \leq i \leq 14}$, $Y = \{y_i\}_{0 \leq i \leq 14}$, $Z = \{z_i\}_{0 \leq i \leq 14} \cup \{\infty_i\}_{0 \leq i \leq 9}$.

A decomposition into C_5 is given by the following circuits:

$$(x_i, y_{i+1}, x_{i+1}, y_{i+3}, z_i, x_i)_{0 \leq i \leq 14}, (x_i, z_{i+2}, x_{i+1}, z_{i+4}, y_{i+6}, x_i)_{0 \leq i \leq 14}$$

$$(x_i, y_{i+\alpha+9}, z_{i+2\alpha+9}, y_{i+\alpha+3}, \infty_\alpha, x_i)_{0 \leq i \leq 14,\ 0 \leq \alpha \leq 5}, (x_i, z_{i+5}, x_{i+1}, z_{i+7}, y_{i+8}, x_i)_{0 \leq i \leq 14}$$

$(x_i, z_{\alpha+i+10}, x_{i+3}, \infty_{\alpha+6}, y_{\alpha+3+i}, x_i)_{0\le i\le 14, 0\le\alpha\le 2}, (x_i, z_{i+14}, x_{i+1}, \infty_9,$

$y_{i+7}, x_i)_{0\le i\le 14}.$

Step 3.

n		Lemma	$n_1=n_2$	n_3	a=b=c	c'	q	q-p
30t+1	$t\ge 1$	2.5	10t	10t	5	-	2t	0
30t+11	$t\ge 1$	2.5	10(t-1)+10	10(t-1)+20	10	20	t	1
30t+15	$t\ge 0$	2.4	10t+5	10t+5	5	-	2t+1	0
30t+21	$t\ge 2$	2.5	10(t-2)+20	10(t-2)+40	10	20	t	2
90t+25	$t\ge 1$	2.4	30(t-1)+45	30(t-1)+25	15	5	2t+1	2
90t+55	$t\ge 0$	2.4	30t+15	30t+25	15	25	2t+1	1
90t+85	$t\ge 1$	2.4	30(t-1)+75	30(t-1)+25	15	5	2t+3	5

The remaining case $n = 30t+5$ is solved for $t \ge 1$ by lemma 2.9 with $h = 6t+1$, $n_i = 5$ for $1 \le i \le h$.

e. Existence of an $(n,5,1)$ $\vec{C}_5$ -design for $n \equiv 0,1 \pmod 5$.

By lemma 2.2 and case a we know the existence of $(n,5,1)\vec{C}_5$ -designs for $n \equiv 1,5 \pmod{10}$. It remains to prove it for $n \equiv 0,6 \pmod{10}$.

Step 1. K_5^*, K_6^*, K_{10}^*, K_{11}^*, K_{26}^* can be decomposed into $\vec{C}_5$.

- K_5^*, K_{11}^*: case a + lemma 2.2.
- K_6^*, K_{10}^*: see proposition 3.2.c.
- K_{26}^*: K_{26}^* is the edge disjoint union of three complete graphs isomorphic to K_6^*, K_{11}^*, K_{11}^* which can be decomposed into $\vec{C}_5$ and of a complete tripartite graph isomorphic to $K_{5,10,10}^*$ which can be decomposed into $\vec{C}_5$ by proposition 3.3.c.

Step 2. $K_{5,5,5}^*$, $K_{5,5,10}^*$ can be decomposed into $\vec{C}_5$ (proposition 3.3.a, lemma 2.2 and proposition 3.3.b).

Step 3.

n		Lemma	$n_1=n_2$	n_3	a=b=c	c'	q	q-p
30t	$t \geq 1$	2.4	10t	10t	5	-	2t	0
30t+6	$t \geq 1$	2.5	10(t-1)+10	10(t-1)+15	5	10	2t	1
30t+10	$t \geq 1$	2.4	10(t-1)+10	10(t-1)+20	5	10	2t	2
30t+16	$t \geq 0$	2.5	10t+5	10t+5	5	-	2t+1	0
30t+20	$t \geq 0$	2.4	10t+5	10t+10	5	10	2t+1	1
30t+26	$t \geq 1$	2.5	10(t-1)+15	10(t-1)+25	5	10	2t+1	2

b. Existence of an (n,5,2) C_5-design for $n \equiv 0,6 \pmod{10}$:

It results from case e by use of lemma 2.3.

c. Existence of (n,5,5) C_5-design for $n \equiv 3,7,9 \pmod{10}$.

Step 1. $5K_5$, $5K_7$, $5K_9$, $5K_{11}$ can be decomposed into C5:

- $5K_5$, $5K_{11}$: case a + lemma 2.1.
- $5K_7$: Let Z_7 be the set of vertices of $5K_7$; a decomposition into C_5 is given by the following 21 circuits:

$$(i,i+1,i+2,i+3,i+4,i),(i,i+1,i+4,i+6,i+1,i),(i,i+1,i+6,i+2,i+5,i),\quad 0 \leq i \leq 6.$$

- $5K_9$: Let Z_9 be the set of vertices of $5K_9$. A decomposition into C_5 is given by the following 36 circuits:

$$(i,i+1,i+2,i+3,i+4,i),(i,i+2,i+4,i+6,i+8,i),(i,i+3,i+6,i+1,i+4,i),$$
$$(i,i+7,i+2,i+8,i+5,i),\quad 0 \leq i \leq 8.$$

Step 2. $5K_{2,2,2}$, $5K_{2,2,4}$ can be decomposed into C_5

- $5K_{2,2,2}$: see proposition 3.4.
- $5K_{2,2,4}$: Let $X \cup Y \cup Z$ be the set of vertices of $K_{2,2,4}$ with $X = \{x_i\}_{0 \leq i \leq 1}$, $Y = \{y_i\}_{0 \leq i \leq 1}$, $Z = \{z_i\}_{0 \leq i \leq 3}$

then a decomposition of $5K_{2,2,4}$ into C_5 is given by the following 20 circuits:

$$(y_j,z_i,y_{j+1},z_{i+1},x_j,y_j),\ (z_i,x_j,z_{i+1},x_{j+1},y_j,z_i)_{0 \leq i \leq 3,\ 0 \leq j \leq 1}$$
$$(y_0,z_0,y_1,z_1,x_0,y_0)(y_1,z_2,y_0,z_1,x_1,y_1)(z_3,x_0,z_2,x_1,y_0,z_3)$$
$$(z_3,x_1,z_0,x_0,y_1,z_3).$$

Step 3.

n		Lemma	$n_1=n_2$	n_3	a=b=c	c'	q	q-p
6t+5	t ≥ 2	2.5	2(t-2)+4	2(t-2)+8	2	4	t	2
6t+7	t ≥ 1	2.5	2(t-1)+4	2(t-1)+4	2	-	t+1	0
6t+9	t ≥ 1	2.5	2(t-1)+4	2(t-1)+6	2	4	t+1	1

f. Existence of an (n,5,5) $\vec{C}_5$-design for n ≡ 2, 3, 4 (mod 5).

By lemma 2.2 and case c, we know the existence of (n,5,5) $\vec{C}_5$-designs for n ≡ 3,7,9 (mod 10). It remains to prove it for n ≡ 2, 4, 8 (mod 10).

Step 1. $5K_6^*$, $5K_8^*$, $5K_{10}^*$, $5K_{12}^*$, $5K_{14}^*$, $5K_{16}^*$ can be decomposed into $\vec{C}_5$:

- $5K_6^*$, $5K_{10}^*$, $5K_{16}^*$: case a + lemma 2.1.
- $5K_8^*$: Let $Z_7 \cup \{\infty\}$ be the set of vertices of $5K_8^*$. A decomposition into $\vec{C}_5$ is given by the following 56 directed circuits:

(i,i+1,i+3,i+6,i+2,i)(i,i+1,i+3,i+6,i+5,i)(i,i+1,i+4,i+2,i+3,i)

(i,i+4,i+1,i+5,∞,i) (i,i+1,i+3,i+6,∞,i) (i,i+2,i+6,i+4,∞,i)

(i,i+5,i+3,i+2,∞,i) (i,i+6,i+5,i+4,∞,i), $0 \le i \le 6$.

- $5K_{12}^*$: Let $Z_{11} \cup \{\infty\}$ be the set of vertices of $5K_{12}^*$. A decomposition is given by the following 11 x 12 directed circuits: $0 \le i \le 10$

(i,i+1,i+2,i+3,i+4,i)(i,i+2,i+4,i+6,i+8,i)(i,i+3,i+6,i+9,i+1,i)

(i,i+4,i+8,i+1,i+5,i)(i,i+5,i+10,i+4,i+9,i)(i,i+6,i+1,i+7,i+2,i)

(i,i+7,i+3,i+10,i+6,i)(i,i+8,i+5,i+2,∞,i)(i,i+9,i+7,i+5,∞,i)

(i,i+8,i+5,i+3,∞,i)(i,i+10,i+9,i+8,∞,i)(i,i+10,i+3,i+4,∞,i).

- $5K_{14}^*$: Let $Z_{13} \cup \{\infty\}$ be the set of vertices of $5K_{14}^*$. A decompositions is given by the following 14 x 13 directed circuits: $0 \le i \le 12$

(i,i+1,i+2,i+3,i+4,i)(i,i+2,i+4,i+6,i+8,i)(i,i+3,i+6,i+9,i+12,i)

(i,i+4,i+8,i+12,i+3,i)(i,i+5,i+10,i+2,i+7,i)(i,i+6,i+12,i+5,i+11,i)

(i,i+7,i+1,i+8,i+2,i)(i,i+8,i+3,i+11,i+6,i)(i,i+9,i+5,i+3,i+1,i)

(i,i+10,i+7,i+4,∞,i)(i,i+11,i+9,i+7,∞,i)(i,i+12,i+11,i+10,∞,i) (i,i+12,i+9,i+7,∞,i)(i,i+8,i+12,i+2,∞,i).

Step 2. $5K^*_{2,2,2}$, $5K^*_{2,2,4}$ can be decomposed into $\vec{C}_5$ (see step 2 of case c + Lemma 2.2).

Step 3.

n		Lemma	$n_1=n_2$	n_3	a=b=c	c'	q	q-p
6t+6	t ≥ 2	2.4	2(t-2)+6	2(t-2)+6	2	-	t+1	0
6t+8	t ≥ 2	2.4	2(t-2)+6	2(t-2)+8	2	4	t+1	1
6t+10	t ≥ 2	2.4	2(t-2)+6	2(t-2)+10	2	4	t+1	2

d. Existence of an $(n,5,10)C_5$-design for $n \equiv 2, 4, 8 \pmod{10}$.
It results from case f by use of lemma 2.3.

3.6. Theorem. <u>There exists an $(n,7,\lambda)C_7$-design if and only if</u>

$$\lambda n(n-1) \equiv 0 \pmod{14}$$
$$\lambda(n-1) \equiv 0 \pmod{2}.$$

<u>There exists an $(n,7,\lambda)\vec{C}_7$-design if and only if</u>

$$\lambda n(n-1) \equiv 0 \pmod{7}.$$

By lemma 2.1, it suffices to prove the theorem in the following cases:

- for C_7-designs:

a. $\lambda = 1$, $n \equiv 1, 7 \pmod{14}$
b. $\lambda = 2$, $n \equiv 0, 8 \pmod{14}$
c. $\lambda = 7$, $n \equiv 3, 5, 9, 11, 13 \pmod{14}$
d. $\lambda = 14$, $n \equiv 2, 4, 6, 10, 12 \pmod{14}$.

- for $\vec{C}_7$-designs:

e. $\lambda = 1$, $n \equiv 0, 1 \pmod{7}$
f. $\lambda = 7$, $n \equiv 2, 3, 4, 5, 6 \pmod{7}$.

We shall use lemmas 2.2 and 2.3 and give a proof of all these cases in a convenient order.

a. Existence of an $(n,7,1)C_7$-design for $n \equiv 1, 7 \pmod{14}$.

We still use the method exposed in 3.1:

Step 1. K_7, K_{15}, K_{29}, K_{71} can be decomposed into C_7.

- K_7: see proposition 3.2.a.
- K_{15}: Let Z_{15} be the set of vertices of K_{15}. A decomposition into C_7 is given by the 15 following circuits:

$(i,i+1,i+3,i+8,i+14,i+7,i+4,i)$, $\quad 0 \le i \le 14$.

- K_{29}: Let Z_{29} be the set of vertices of K_{29}. A decomposition into C_7 is given by the 58 following circuits:

$(i,i+14,i+27,i+10,i+21,i+2,i+3,i)$, $(i,i+2,i+6,i+11,i+5,i+12,i+20,i)$, $\quad 0 \le i \le 28$.

- K_{71}: Let Z_{71} be the set of vertices of K_{71}. A decomposition into C_7 is given by the following 5 x 71 circuits:

$(i,i+4,i+9,i+15,i+47,i+7,i+50,i)$, $(i,i+7,i+15,i+24,i+51,i+6,i+52,i)$

$(i,i+10,i+21,i+33,i+57,i+9,i+58,i)$, $(i,i+14,i+29,i+45,i+62,i+9,i+51,i)$,
$(i,i+1,i+3,i+6,i+39,i+5,i+41,i)$, $0 \le i \le 70$.

Step 2. $K_{7,7,7}$, $K_{14,14,14}$, $K_{14,14,28}$, $K_{7,7,21}$ can be decomposed into C_7:

- $K_{7,7,7}$, $K_{14,14,14}$ (see proposition 3.3. a, d.).
- $K_{14,14,28}$: Let $X \cup Y \cup Z$ be the set of vertices of $K_{14,14,28}$ with

$$X = \{x_i\}_{0 \le i \le 13},\ Y = \{y_i\}_{0 \le i \le 13},\ Z = \{\infty_i\}_{0 \le i \le 13};$$

then a decomposition into C_7 is given by the following 10 x 14 circuits:

$(x_i, y_{i+\alpha}, \infty_{\alpha+4}, x_{i+1}, \infty_{\alpha+12}, y_{i+\alpha+1}, \infty_{\alpha+20}, x_i)$, $0 \le i \le 13$, $0 \le \alpha \le 7$

$(x_{9+i}, y_{6+i}, \infty_0, x_{1+i}, y_i, \infty_{1+\varepsilon_i}, y_{i+7}, x_{i+9})$ $0 \le i \le 13$; $\varepsilon_i = 1$ if $7 \le i \le 13$

$(x_{8+i}, y_{2+i}, \infty_3, x_i, \infty_{1+\varepsilon_i}, x_{7+i}, y_{3+i}, x_{8+i})$, $0 \le i \le 13$; $\varepsilon_i = 1$ if $7 \le i \le 13$.

- $K_{7,7,21}$: Let $X \cup Y \cup Z$ be the set of vertices of $K_{7,7,21}$ with

$$X = \{x_i\}_{0 \le i \le 6},\ Y = \{y_i\}_{0 \le i \le 6},\ Z = \{\infty_i\}_{0 \le i \le 20}.$$

Then a decomposition into C_7 is given by the following 7 x 7 circuits

$(x_i, y_{i+\alpha}, \infty_\alpha, x_{i+1}, \infty_{\alpha+7}, y_{\alpha+1+i}, \infty_{\alpha+14}, x_i)_{0 \le i \le 6,\ 0 \le \alpha \le 6}$.

Step 3.

n		Lemma	$n_1=n_2$	n_3	a=b=c	c'	q	q-p
42t+1	$t \ge 1$	2.5	14(t-1)+14	14(t-1)+14	7	-	2t	0
42t+15	$t \ge 1$	2.5	14(t-1)+14	14(t-1)+28	14	28	t	1
42t+21	$t \ge 0$	2.4	14t+7	14t+7	7	-	2t+1	0
42t+29	$t \ge 2$	2.5	14(t-2)+28	14(t-2)+56	14	28	t	2
42t+35	$t \ge 0$	2.4	14t+7	14t+21	7	21	t	1

The remaining case n = 42t+7 is solved for $t \ge 1$ by lemma 2.9 with $h = 6t+1$, $n_i = 7$ for $1 \le i \le h$.

e. Existence of an $(n,7,1)\vec{C}_7$-design for $n \equiv 0,1 \pmod 7$.

By lemma 2.2 and case a, we know the existence for $n \equiv 1, 7 \pmod{14}$. It remains to prove it for $n \equiv 0, 8 \pmod{14}$:

Step 1. K_7^*, K_8^*, K_{14}^*, K_{15}^*, K_{36}^* can be decomposed into $\vec{C}_7$:

- K_7^*, K_{15}^*: case a + lemma 2.2.
- K_8^*, K_{14}^*: proposition 3.2.c.
- K_{36}^* is the edge disjoint union of three complete directed graphs isomorphic to K_8^*, K_{15}^*, K_{15}^* and of a tripartite directed graph isomorphic to $K_{7,14,14}^*$ which can be decomposed into $\vec{C}_7$ by proposition 3.3.c.

Step 2. $K_{7,7,7}^*$, $K_{7,7,14}^*$ can be decomposed into $\vec{C}_7$: see proposition 3.3.a + lemma 2.2 and proposition 3.3.b.

Step 3.

n		Lemma	$n_1=n_2$	n_3	a=b=c	c'	q	q-p
42t	$t \ge 1$	2.4	14(t-1)+14	14(t-1)+14	7	-	2t	0
42t+8	$t \ge 1$	2.5	14(t-1)+14	14(t-1)+21	7	14	2t	1
42t+14	$t \ge 1$	2.4	14(t-1)+14	14(t-1)+28	7	14	2t	2
42t+22	$t \ge 0$	2.5	14t+7	14t+7	7	-	2t+1	0
42t+28	$t \ge 0$	2.4	14t+7	14t+14	7	14	2t+1	1
42t+36	$t \ge 1$	2.5	14(t-1)+21	14(t-1)+35	7	14	2t+1	2

b. Existence of an $(n,7,2)$ C_7-design for $n \equiv 0, 8 \pmod{14}$.
It's a consequence of case e by use of lemma 2.3.

c. Existence of an $(n,7,7)C_7$-design for $n \equiv 3,5,9,11,13 \pmod{14}$.

<u>Step 1.</u> $7K_7$, $7K_9$, $7K_{11}$, $7K_{13}$, $7K_{15}$, $7K_{17}$, $7K_{21}$, $7K_{27}$, $7K_{29}$, $7K_{39}$ can be decomposed into C_7:

- $7K_7$, $7K_{15}$, $7K_{21}$, $7K_{29}$: case a + lemma 2.1.

- $7K_{19}$ is the edge disjoint union of three complete graphs isomorphic to $7K_7$ and of a tripartite graph isomorphic to $7K_{7,7,7}$ which can be decomposed into C_7 by proposition 3.4.

- $7K_{27}$ is the edge disjoint union of three complete graphs isomorphic to $7K_9$ and of a tripartite complete graph isomorphic to $7K_{9,9,9}$ which can be decomposed into C_7 by proposition 3.4.

- $7K_{39}$ is the edge disjoint union of three complete graphs isomorphic to $7K_{13}$ and of a tripartite complete graph isomorphic to $7K_{13,13,13}$ which can be decomposed into C_7 by proposition 3.4.

- $7K_9$: Let Z_9 be the set of vertices of $7K_9$. A decomposition is given by the following 4 x 9 circuits of length 7:

$(i,i+1,i+2,i+3,i+4,i+5,i+6,i)$, $(i,i+2,i+4,i+6,i+8,i+1,i+5,i)$

$(i,i+3,i+6,i+1,i+4,i+7,i+2,i)$, $(i,i+4,i+8,i+5,i+1,i+7,i+2,i)$, $0 \le i \le 8$.

- $7K_{11}$: Let Z_{11} be the set of vertices of $7K_{11}$. A decomposition is given by the following 5 x 11 circuits of length 7, $\{C_{\beta,i}\}$ for $1 \le \beta \le 5$, $0 \le i \le 10$:

$$C_{\beta,i} = (\ldots, \alpha\beta+i, \ldots, i)_{0 \le \alpha \le 6}.$$

- $7K_{13}$: Let Z_{13} be the set of vertices of $7K_{13}$. A decomposition is given by the following 6 x 13 circuits $\{C_{\beta,i}\}$ for $1 \le \beta \le 6$, $0 \le i \le 12$,

$$C_{\beta,i} = (\ldots, \alpha\beta+i, \ldots, i)_{0 \le \alpha \le 6}.$$

- $7K_{17}$: Let Z_{17} be the set of vertices of $7K_{17}$. A decomposition is given by the following 8 x 17 circuits of length 7, $\{C_{\beta,i}\}$ for $1 \le \beta \le 8$ and $0 \le i \le 16$,

$$C_{\beta,i} = (\ldots, \alpha\beta+i, \ldots, i)_{0 \le \alpha \le 6}.$$

Step 2. $7K_{4,4,4}$, $7K_{4,4,2}$ can be decomposed into C_7.

- $7K_{4,4,4}$: see proposition 3.4.
- $7K_{4,4,2}$: Let $X \cup Y \cup Z$ be the set of vertices of $7K_{4,4,2}$ with

$$X = \{x_i\}_{0 \leq i \leq 3},\quad Y = \{y_i\}_{0 \leq i \leq 3},\quad Z = \{z_i\}_{0 \leq i \leq 1}.$$

A decomposition of $7K_{4,4,2}$ into C_7 is given by the following circuits:

$(z_i, x_i, z_{i+1}, x_{i+1}, y_{i+3}, x_{i+3}, y_i, z_i)$

$(z_i, y_i, z_{i+1}, y_{i+1}, x_{i+3}, y_{i+2}, x_i, z_i)$ $\quad 0 \leq i \leq 3$

$(z_i, x_i, y_{i+1}, x_{i+1}, y_{i+2}, x_{i+2}, y_i, z_i)$

$(z_i, x_{i+1}, z_{i+1}, x_{i+2}, y_{i+1}, x_i, y_{i+3}, z_i)$ $\quad$ taken twice $0 \leq i \leq 3$.

$(z_i, y_{i+1}, z_{i+1}, y_{i+2}, x_{i+3}, y_{i+3}, x_{i+1}, z_i)$

Step 3.

n		Lemma	$n_1=n_2$	n_3	a=b=c	c'	q	q-p
12t+1	$t \geq 2$	2.5	4(t-2)+8	4(t-2)+8	4	-	k	0
12t+3	$t \geq 4$	2.5	4(t-4)+10	4(t-4)+10	4	2	k+1	5
12t+5	$t \geq 3$	2.5	4(t-3)+16	4(t-3)+8	4	2	k+1	4
12t+7	$t \geq 2$	2.5	4(t-2)+12	4(t-2)+6	4	2	k+1	3
12t+9	$t \geq 2$	2.5	4(t-2)+12	4(t-2)+8	4	2	k+1	2
12t+11	$t \geq 1$	2.5	4(t-1)+8	4(t-1)+6	4	2	k+1	1

f. Existence of an $(n,7,7,)\vec{C}_7$-design for $n \equiv 2, 3, 4, 5, 6 \pmod 7$.

By lemma 2.2 and case c, we know the existence for $n \equiv 3, 5, 9, 11, 13 \pmod{14}$. It remains to prove it for $n \equiv 2, 4, 6, 10, 12 \pmod{14}$.

Step 1. $7K^*_8$, $7K^*_{10}$, $7K^*_{12}$, $7K^*_{14}$, $7K^*_{16}$, $7K^*_{18}$, $7K^*_{20}$, $7K^*_{22}$, $7K^*_{26}$, $7K^*_{28}$, $7K^*_{30}$, $7K^*_{38}$ can be decomposed:

- $7K^*_8$, $7K^*_{14}$, $7K^*_{22}$, $7K^*_{28}$: case a + lemma 2.1.
- $7K^*_{30}$ is the edge disjoint union of three complete directed graph isomorphic to $7K^*_{10}$ and of a tripartite complete directed graph isomorphic to $7K^*_{10,10,10}$ which can be decomposed by proposition 3.4.

- $7K^*_{10}$: Let $Z_9 \cup \{\infty\}$ be the set of vertices of $7K^*_{10}$. Then a decomposition is given by the following 9 x 10 directed circuits:

$(\ldots,\alpha\beta+i,\ldots,\infty,i)_{0\le\alpha\le5}$, $0\le i\le8$, $\beta = 2, 5, 7$

$(\ldots,\alpha\beta+i,\ldots, i)_{0\le\alpha\le6}$, $0\le i\le8$, $\beta = 1, 8, 4$.

(i,i+3,i+6,i+8,i+2,i+5,∞,i), (i,i+6,i+3,i+4,i+2,i+8,∞,i)

(i,i+7,i+5,i+2,i+8,i+3,∞,i), (i,i+8,i+2,i+7,i+3,i+5,∞,i), $0\le i\le8$.

- $7K^*_{12}$: Let $Z_{11} \cup \{\infty\}$ be the set of vertices of $7K^*_{12}$. A decomposition of $7K^*_{12}$ into $\vec{C}_7$ is given by the following 12 x 11 directed circuits:

$(\ldots,\alpha\beta+i,\ldots,i)_{0\le\alpha\le6}$ for $0\le i\le10$, $\beta = 1,2,3,4,10$

$(\ldots,\alpha\beta+i,\ldots,\infty,i)_{0\le\alpha\le5}$ for $0\le i\le10$, $\beta = 5,6,7,8,9$.

(i,10+i,8+i,5+i,1+i,7+i,∞,i),(i,2+i,5+i,10+i,6+i,3+i,∞,i), $0\le i\le10$.

- $7K^*_{16}$: Let $Z_{15} \cup \{\infty\}$ be the set of vertices of $7K^*_{16}$. A decomposition is given by the following 16 x 15 directed circuits: for $0\le i\le14$

$(\ldots,\alpha\beta+i,\ldots,i)_{0\le\alpha\le6}$ for $\beta = 1, 2, 4, 7, 8, 11, 13, 14$,

(i,i+3,i+8,i+14,i+9,i+6,∞,i) taken 5 times,

(i,i+1,i+3,i+7,i+14,i+13,i+6,i+9,i),(i,i+5,i+10,i+4,i+14,i+8,∞,i),

(i,i+10,i+4,i+2,i+11,i+6,∞,i).

- $7K^*_{18}$: Let $Z_{17} \cup \{\infty\}$ be the set of vertices of $7K^*_{18}$. A decomposition is given by the following 18 x 17 directed circuits: for $0\le i\le16$

$(\ldots,\alpha\beta+i,\ldots,i)_{0\le\alpha\le6}$ for $0\le\beta\le11$; $(\ldots,\alpha\beta+i,\ldots,\infty,i)_{0\le\alpha\le5}$ for $12\le\beta\le16$

(i,i+1,i+7,i+14,i+9,i+5,∞,i),(i,i+16,i+14,i+11,i+7,i+2,∞,i).

- $7K^*_{20}$: Let $Z_{19} \cup \{\infty\}$ be the set of vertices of $7K^*_{20}$. A decompsoition is given by the following 20 x 19 directed circuits: for $0\le i\le18$,

$(\ldots,\alpha\beta+i,\ldots,i)_{0\le\alpha\le6}$ for $1\le\beta\le13$, $(\ldots,\alpha\beta+i,\ldots,\infty,i)_{0\le\alpha\le5}$ for $14\le\beta\le18$,

(i,i+5,i+11,i+3,i+15,i+14,∞,i),(i,i+4,i+10,i+7,i+5,i+4,∞,i).

- $7K^*_{26}$: Let $Z_{25} \cup \{\infty\}$ be the set of vertices of $7K^*_{26}$. A decomposition is given by the following 26 x 25 directed circuits: for $0 \leq i \leq 24$

$(\ldots,\alpha\beta+i,\ldots,i)_{0 \leq \alpha \leq 6}$, for $1 \leq \beta \leq 24$, $\beta \neq 5, 10, 15, 20, 21$

$(i,i+5,i+15,i+14,i+4,i+24,\infty,i)$: taken 7 times.

- $7K^*_{38}$: Let $Z_{37} \cup \{\infty\}$ be the set of vertices of $7K^*_{38}$. A decomposition of $7K^*_{38}$ is given by the following 38 x 37 directed circuits: for $0 \leq i \leq 36$

$(\ldots,\alpha\beta+i,\ldots,i)_{0 \leq \alpha \leq 6}$ for $1 \leq \beta \leq 31$, $(\ldots,\alpha\beta+i,\ldots,\infty,i)_{0 \leq \alpha \leq 5}$ for $32 \leq \beta \leq 36$,

$(i,i+36,i+34,i+31,i+27,i+22,\infty,i)$

$(i,i+6,i+18,i+36,i+23,i+16,\infty, i)$.

Step 2. $7K^*_{4,4,4}$, $7K^*_{4,4,2}$ can be decomposed into $\vec{C}_7$: see step 2 of case c + lemma 2.2.

Step 3.

n		Lemma	$n_1=n_2$	n_3	a=b=c	c'	q	q-p
12t	$t \geq 2$	2.4	4(t-2)+8	4(t-2)+8	4	-	k	0
12t+2	$t \geq 4$	2.4	4(t-4)+20	4(t-4)+10	4	2	k+1	5
12t+4	$t \geq 3$	2.4	4(t-3)+16	4(t-3)+8	4	2	k+1	4
12t+6	$t \geq 3$	2.4	4(t-3)+16	4(t-3)+10	4	2	k+1	3
12t+8	$t \geq 2$	2.4	4(t-2)+12	4(t-2)+8	4	2	k+1	2
12t+10	$t \geq 2$	2.4	4(t-2)+12	4(t-2)+10	4	2	k+1	1

Case k = 3.

It is well known that an $(n,3,\lambda)C_3$-design (or $(n,3,\lambda)$B.I.B.D.) exists if and only if $\lambda n(n-1) \equiv 0 \pmod 6$ and $\lambda(n-1) \equiv 0 \pmod 2$ (see /10/).

It has been proved in /2, 3, 8, 14/ that:

3.7. Theorem. An $(n,3,1)\vec{C}_3$-design exists if and only if $n \neq 6$, and $n \equiv 0$ or $1 \pmod 3$.

For $\lambda \geq 2$ we have the theorem:

3.8. Theorem. For $\lambda \geq 2$, an $(n,3,\lambda)\vec{C}_3$-design exists if and only if $\lambda n(n-1) \equiv 0 \pmod 3$.

Proof. By theorem 3.7 and lemma 2.1 it suffices to verify the theorem for $\lambda = 3$, $n \equiv 2 \pmod 3$ and for $\lambda = 2, 3$ and $n = 6$. Lemma 5.3 of /10/ implies that for $n \geq 3$ the edges of K_n can be partitioned into K_3, K_4, K_5, K_6 and K_8. Thus it suffices to prove the existence of a decomposition for $3K_n^*$ $n = 5, 6, 8$ and $2K_6^*$. For $n = 5$, it follows from the existence of an $(5,3,1)C_3$-design.
For $n = 6$ let the vertices be $Z_5 \cup \{\infty\}$: a decomposition of $2K_6^*$ is given by: $(\infty,i,i+1)$; $(\infty,i+1,i)$; $(i,i+2,i+4)$; $(i+4,i+2,i)$ with $0 \leq i \leq 4$ and for $3K_6^*$ by $(\infty,i,i+1)$twice and once $(i,i+2,i+4)$, $(i+4,i+2,i)$, $(i+2,i+1,i)$ with $0 \leq i \leq 4$. For $n = 8$ let the vertices bei $Z_7 \cup \{\infty\}$then a decomposition is given by: 3 times the circuits $(i,i+1,i+3)$ twice $(i,i+3,i+1)$ and once $(\infty,i,i+3)$; $(\infty,i+3,i+1)$; $(\infty,i+1,i)$ where $0 \leq i \leq 6$.

References

/1/ C. Berge
Graphs and Hypergraphs
North Holland, Amsterdam, 1973

/2/ J. C. Bermond
An application of the solution of Kirkman's schoolgirl problem: the decomposition of the symmetric oriented complete graph into 3-circuits, Discrete Math. 8 (1974), 301 - 304

/3/ J. C. Bermond
Decomposition of K_n^* into k-circuits and balanced G-designs, in "Recent advances in graph theory" (ed. M. Fiedler), Proc. Symp. Prague Akademiai Prague, (1975), 57 - 68

/4/ J. C. Bermond
Thesis
University of Paris XI (Orsay) (1975)

/5/ J. C. Bermond and V. Faber
Decomposition of the complete directed graph into k-circuits
J. Combinatorial Theory (B), 21, 1976, 146 - 155

/6/ J. C. Bermond and D. Sotteau
Graph decompositions and G-designs.
Proc. 5th British Combinatorial Conference, Aberdeen July 1975, Congressus Numerantium 15, Utilitas Math. Publ., 53 - 72

/7/ J. C. Bermond, C. Huang and D. Sotteau
Cycle and circuit designs: even case, to appear.

/8/ R. H. Bruck
Existence of the A-maps of a A.K. Dewney and N. Robertson, unpublished manuscript.

/9/ M. Hall Jr.
"Combinatorial Theory"
Blaisdell Waltham, Mass. (1967)

/10/ H. Hanani
Balanced incomplete block designs and related designs
Discrete Math. 11 (1975), 255 - 369

/11/ B. Hartnell and M. Milgram
Decomposition of K_p, for p a prime into k-circuits
"Proc. Journées franco-belges sur les graphes et les hypergraphes", Paris, Mai 1974, Cahiers du C.E.R.O, 17, (Bruxelles) 1975, 221 - 222

/12/ P. Hell and A. Rosa
Graph decompositions, handcuffed prisoners and balanced P-designs, Discrete Math. 2 (1972), 229 - 252

/13/ C. Huang and A. Rosa
Another class of balanced graph designs, balanced circuit designs,
Discrete Mathematics, 12 (1975), 269 - 293

/14/ N. S. Mendelsohn
A natural generalization of Steiner triple systems, in "Computers in number Theory" (eds. A.O.L. Atkin and B.J. Birch)
Academic Press, New York (1971), 323 - 338

/15/ D. Merriel
Partitioning the directed graph into 5-cycles, unpublished manuscript

/16/ A. Rosa
On the cyclic decomposition of the complete graph into polygons with odd number of edges
Casopis Pest. Math. 91 (1966), 53 - 63

/17/ D. Sotteau
Decomposition of K_n^* into circuits of odd length,
Discrete Math. 15 (1976), 185 - 191

/18/ R. M. Wilson
Construction and uses of pairwise balanced designs, in "Combinatorics, Part I" (eds. M. Hall and J. M. Van Lint),
Math. Centre Tracts, 55, Amsterdam (1974), 18 - 41

/19/ R. M. Wilson
Decompositions of complete graphs into subgraphs isomorphic to a given graph, in Proc. 5^{th} British Combinatorial Conference, Aberdeen, July 1975, Congressus Numerantium 15, Utilitas Math., 647 - 659