GRAPH DECOMPOSITIONS AND G-DESIGNS

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ABSTRACT *Let G be a graph (directed or not) with k vertices, then an (n,k,λ) G-design is an edge disjoint decomposition of the complete (directed or not) multigraph λK_n^* or λK_n into subgraphs isomorphic to G . G-designs are a generalization of the well known B.I.B.D., which correspond to the case $G = K_k$.*

The problem is to determine for what values of the parameters n and λ there exists an (n,k,λ) G-design where G is a given graph. First we give a survey concerning this problem ; then we indicate the general methods which are used to prove the existence of G-designs.

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I. DEFINITIONS AND INTRODUCTION

In what follows n,k,λ will be positive integers with $n \geq k$ and G will always denote a graph with k vertices (none of them isolated) and e edges. Definitions not given here can be found in Berge's book [B1]

I.1. If λK_n denotes the complete multigraph with n vertices and any 2 distinct vertices joined by exactly λ edges, then *an (n,k,λ) G-design is an edge disjoint decomposition (that is a partition of the edges) of λK_n into subgraphs isomorphic to G.* (In what follows we will omit 'edge disjoint'). The G-designs have been introduced by P.Hell and A.Rosa [H6]. In the particular case where G is the complete graph K_k, an (n,k,λ) G-design is nothing else than an (n,k,λ) B.I.B.D. (balanced incomplete block design) (see M.Hall [H1]).

If furthermore each vertex belongs to exactly r of these subgraphs then the G-design is said to be *balanced*.

I.2. Remark : If G is regular ($d(x) = c$ for every vertex x of G), it is easy to see that the G-design is balanced (that is for example the case if $G = K_k$).

I.3. Another definition of a G-design (similar to the classical definition of a B.I.B.D.) can be given by using the notion of G-blocks. (See P.Hell and A.Rosa [H6]). In a block considered as a set of elements any two distinct elements are either linked or unlinked. The adjacency matrix $M(B) = (b_{ij})$ of a block B with k elements is a symmetric matrix of order k with zero diagonal , where $b_{ij} = 1$ if the elements i and j are linked in B and 0 otherwise. A block B is said to be *a G-block* if its adjacency matrix is equivalent to that of G. Thus *an (n,k,λ) G-design is an arrangement of n elements into b G-blocks such that every G-block contains k elements (satisfied automatically) and any two distinct elements are linked in exactly λ G-blocks.*

I.4. A similar definition holds for directed graphs. If G is a directed graph with k vertices and λK^*_n denotes the complete directed multigraph with n vertices and any two distinct vertices x and y joined by exactly λ arcs (x,y) and λ arcs (y,x) then *an (n,k,λ) G-design is a partition of the arcs of λK^*_n into subgraphs isomorphic to G.*

If furthermore each vertex belongs to exactly r of the subgraphs then the G-design is said to be *balanced*. In particular if G is "strongly regular" (that is $d^+(x) = d^-(x) = c$ for every vertex x of G) then the G-design is balanced.

I.5. Examples.

a. The projective plane of order 2 is a balanced (7,3,1) K_3-design. A decomposition of K_7 into triangles is given by the following triangles $\{ (i,i+2,i+4),\ 0 \leq i \leq 6 \}$ where the numbers are to be taken modulo 7.

b. Fig. 1. shows an example of a (4,3,1) P_3-design (P_3 being the path with 3 vertices). A (4,3,1) P_3-design can never be balanced : see the necessary condition.

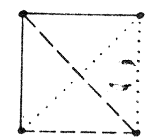


Fig. 1.

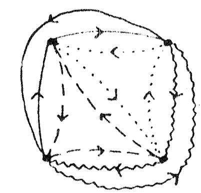


Fig. 2.

c. Fig. 2. shows an example of a (4,3,1) $\vec{C}_3$-design (automatically balanced).

I.6. Remarks. The following propositions are simple consequences of the definitions.

I.6.1. *If there exists an (n,k,λ_1) G-design and an (n,k,λ_2) G-design then there exists an ($n,k,p\lambda_1 + q\lambda_2$) G-design.*

I.6.2. *Let G^* be the directed graph obtained from G by associating to each edge of G two opposite arcs. The existence of an (n,k,λ) G-design is equivalent to the existence of an (n,k,λ) G^*-design.*

I.6.3. *Let $\vec{G}$ be an antisymmetric directed graph obtained from a simple graph G by giving an orientation to the edges of G. If $\vec{G}$ is isomorphic to its opposite $\overleftarrow{G}$, the existence of an (n,k,λ) G-design implies the existence of an (n,k,λ) $\vec{G}$-design.*

The contrary is false : we have seen the existence of a (4,3,1) $\vec{C}_3$-design (example I.5.c.) but there is no (4,3,1) C_3-design (see the necessary conditions).

I.6.4. *Let $\vec{G}$ be an antisymmetric directed graph, and G the underlying undirected graph (obtained by deleting the orientation), the existence of an (n,k,λ) $\vec{G}$-design implies the existence of an $(n,k,2\lambda)$ G-design.*

The contrary is false : there exists a (6,3,2) K_3-design (see M.Hall [H1]) but there exists no (6,3,1) $\vec{C}_3$-design (see [B2]).

I.7. Necessary conditions of existence of an (n,k,λ) G-design :

Proposition : *If there exists an (n,k,λ) G-design (where G is an undirected graph with k vertices and e edges) then*

(i) $\lambda n(n-1) \equiv 0 \quad (\bmod 2e)$

(ii) $\lambda(n-1) \equiv 0 \quad (\bmod d)$

where d is the g.c.d. of the degrees of the vertices of G.

Moreover if the G-design is balanced then

(iii) $\lambda k(n-1) \equiv 0 \quad (\bmod 2e)$.

Proof : Let b be the number of subgraphs (blocks) of the G-design :

$$b = \lambda n(n-1)/2e$$

(i) The number of edges of G must divide the number of edges of λK_n . (We can say also that b must be an integer).

(ii) Let d_i , $1 \leq i \leq k$, be the degrees of the vertices of G ; each vertex x of K_n must appear with degree d_i in α_i subgraphs of the decomposition (for $1 \leq i \leq k$) so that

$$\lambda (n-1) = \sum_{i=1}^{k} \alpha_i d_i$$

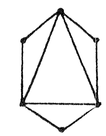
Thus the g.c.d. of the $(d_i)_{1 \leq i \leq k}$ divides $\lambda(n-1)$.

(iii) We have immediately $bk = nr$ and thus $r = \lambda k(n-1)/2e$ must be an integer.

I.8. Remarks :

a. If G is regular then (iii) is equivalent to (ii). (That gives a verification of remark I.2.).

b. (iii) + (i) $\not\Rightarrow$ (ii).

Example : $n = 10$, $k = 6$, $e = 9$, $\lambda = 1$, G =  , $d = 2$.

I.9. If G is a directed graph , we have the following necessary conditions, which proofs (identical to the non directed case) are left to the reader :

Proposition : *If there exists an (n,k,λ) G-design (where G is a directed graph with k vertices and e arcs) then*

(i) $\lambda n(n-1) \equiv 0 \quad (\bmod e)$

(ii) $\lambda(n-1) \equiv 0 \quad (\bmod d^+)$

$\lambda(n-1) \equiv 0 \quad (\bmod d^-)$

where d^+ (resp^t d^-) is the g.c.d. of the out degrees (resp^t in degrees) of the vertices of G.

Moreover if the G-design is balanced

(iii) $\lambda k(n-1) \equiv 0 \quad (\bmod e)$.

I.10. Existence theorem : (R.M.Wilson [W5])

For a given graph G (undirected) and a given λ , the necessary conditions of existence of G-designs (i) and (ii) given in I.7. are sufficient for all sufficiently large integers n .

I.11. Remark : If $G = K_k$ this was stated as conjecture in M.Hall [H1, p.248] and has been proved by R.M.Wilson ([W2] and [W3]). If $\lambda = 1$, this was stated as conjecture by P.Erdös and J.Schönheim in [E1] and proved by them for $k = 2,3,4$ [E1], and proved for every k and $n \equiv 1 \pmod{2e}$ by R.M.Wilson [W3].

I.12. If G is a directed graph, the necessary conditions (i) and (ii) given in I.9. are not sufficient for all graphs G.

R.M.Wilson [W5] has given necessary conditions (more complicated) which are "asymptotically sufficient".

I.13. In what concerns the existence of balanced G-designs, there exist infinite values of n for which the necessary conditions given in I.7. or I.9. are not sufficient. Recently, Wilson [private communication at this conference] has also found necessary and "asymptotically sufficient" conditions for the existence of balanced G-designs.

I.14. In what follows we will first give a survey of the "exact" results known for the existence of G-designs (balanced or not) (part II) and then we will indicate the methods to prove the existence of such G-designs (parts III and IV).

II. SURVEY OF THE RESULTS CONCERNING THE EXISTENCE OF G-DESIGNS

We shall examine successively different classes of graphs G.

II.1. G is the complete graph with k vertices K_k.

In this case, a K_k-design is nothing else than a B.I.B.D. which has been well studied in the litterature. We do not intend to give a survey of this case. The reader interested can find a survey and references on this problem in the recent articles of H.Hanani [H4] and R.M.Wilson [W3].

For example, the necessary conditions

(i) $\lambda n(n-1) \equiv 0 \pmod{k(k-1)}$

(ii) $\lambda(n-1) \equiv 0 \pmod{(k-1)}$

have been shown to be sufficient

- for $k = 3,4,5$ (except for $n = 15$, $\lambda = 2$) and for other cases (see the survey of H.Hanani [H4])
- for n large enough (see R.M.Wilson [W3])

II.2. G is a star S_k (with k vertices and $k-1$ edges).

a. S_k-designs have been studied by P.Cain [C1], C.Huang [H10] and A.Rosa [R2]. C.Huang [H10] has proved that an $(n,k,1)$ S_k-design exists if and only if $n(n-1) \equiv 0 \pmod{2(k-1)}$ and $n \geq 2(k-1)$.

b. Balanced S_k-design have been studied by C.Huang and A.Rosa [H8], and C.Huang [H9] who have proved that a balanced (n,k,λ) S_k-design exists if and only if $\lambda(n-1) \equiv 0 \pmod{2(k-1)}$.

II.3. G is a path P_k (with k vertices and $k-1$ edges).

Balanced P_k-designs have been studied under the name of " handcuffed designs ". Partial results have been obtained by J.F.Lawless [L1,L2], S.H.Hung and N.S.Mendelsohn [H15], P.Hell and A.Rosa [H6] and in [B4]. Only recently C.Huang [H11] has solved completly the problem by proving that a balanced (n,k,λ) P_k-design exists if and only if $\lambda k(n-1) \equiv 0 \pmod{2(k-1)}$.

II.4. G is a directed path $\vec{P}_k$ (with k vertices and $(k-1)$ arcs).

Results concerning the existence of balanced $\vec{P}_k$-designs appear in [B4]. In particular it is proved that a balanced (n,k,λ) $\vec{P}_k$-design exists for $k = 3,4,5,6$ if and only if $\lambda(n-1) \equiv 0 \pmod{k-1}$ except for $k = 3 = n$, λ odd and $k = 5 = n$, $\lambda = 1$.

It is also proved that a balanced $(n,k,1)$ $\vec{P}_k$-design with k even exists if and only if $n-1 \equiv 0 \pmod{k-1}$.

Partial results have been obtained for the existence of balanced $(n,k,1)$ $\vec{P}_k$-design with k odd but a complete solution depends on the proof of the following conjecture :

" For k odd, $k \geqq 5$, there exists a $(k,k,1)$ $\vec{P}_k$-design (which is automatically balanced)". (The case k even is obvious).

It has been pointed out by N.S.Mendelsohn [M1] that the existence is equivalent to the existence of "complete latin square" of order k. The conjecture has been proved by using sequenceable groups for $k = 21$, (N.S.Mendelsohn [M1]), $k = 27$ (A.D.Keedwell [K1]), $k = 39, 55, 57$ (L.L.Wang [W1]) and for $k = 7, 9, 11, 13, 15, 17$ (See [B4,B5]).

For more details concerning this problem and related problems, see the book of J.Denes and A.D.Keedwell ([D1] , chap. 2 and 9)

II.5. G is a cycle C_k (with k vertices and k edges).

A C_k is automatically balanced (see remark I.2.). It has been proved that an $(n,k,1)$ C_k-design exists if

- $k \equiv 0$ (mod 4) and $n = 2pk + 1$ (A.Kotzig [K2])
- $k \equiv 2$ (mod 4) and $n = 2pk + 1$ (A.Rosa [R4])
- k odd and $n = 2pk + 1$ or $n = 2pk + k$ (A.Rosa [R3]).

Necessary and sufficient conditions have been obtained for small values of k. Exactly it has been proved that an (n,k,λ) C_k-design exists if and only if

(i) $\lambda\, n\,(n-1) \equiv 0 \pmod{2k}$

(ii) $\lambda\,(n-1) \equiv 0 \pmod{2}$

- for $k = 4, 5, 6$ by C.Huang and A.Rosa [H12]
- for $k = 4, 5, 6, 7, 8$ in [B5,B6] with different proofs.

The case $k = 3$ is well known, an $(n,3,\lambda)$ C_3-design being nothing else than a Steiner triple system.

II.6. G is a directed circuit $\vec{C}_k$ (with k vertices and k arcs).

It has been proved that an $(n,3,1)$ $\vec{C}_3$-design exists if and only if $n\,(n-1) \equiv 0$ (mod 3) and $n \neq 6$, independently by R.H.Bruck [B7], N.S.Mendelsohn [M2] and in [B2].A short proof can be found in [B3,B4].

It has also been proved that the necessary conditions of existence of an (n,k,λ) $\vec{C}_k$-design

$$\lambda\, n\,(n-1) \equiv 0 \pmod{k}$$

is sufficient

- for k even, $4 \leqq k \leqq 16$ and $\lambda = 1$ in [B4 or B5], except the cases $k = n = 4$, $k = n = 6$;for $k = 4$ see also J.Schönheim [S1] .
- for $k = 5$ and $\lambda = 1$ in [B5,M3]
- for $k = p^{\alpha}$, where p is a prime , and $\lambda = 1$ in [S2] , and in [M3] for $k = 7$.
- for $3 \leqq k \leqq 8$ and $\lambda \geqq 2$ in [B4] and [B6] .

It has also been shown that an $(n,k,1)$ $\vec{C}_k$-design exists if n is a power of a prime and $n(n-1) \equiv 0$ (mod k) (B.Hartnell and M.Milgram [H5] and if $n \equiv 0$ or 1 (mod k) for k odd ([S2]).

II.7. G is a small graph with k vertices, $3 \leqq k \leqq 5$.

An extensive study on the existence of G-designs (balanced or not) for small graphs G (with 3, 4, 5 vertices) has been done by J.Schönheim, A.Rosa, C.Huang and the two authors. The results will appear in forthcoming papers.

II.8. G is a transitive tournament TT3 (with 3 vertices and 3 arcs).

It has been proved in S.H.Hung and N.S.Mendelsohn [H14] that an $(n,3,1)$ TT3-design exists if and only if $n(n-1) \equiv 0$ (mod 3). (For a short proof see [B3 or B4]).

II.9. G is a bipartite graph K_{k_1,k_2} , $k_1 \leqq k_2$ (with $k = k_1 + k_2$ vertices and $k_1 k_2$ edges.).

- If $k_1 = 1$ then K_{k_1,k_2} is a star S_k (see II.2.).
- If $k_1 = k_2 = 2$ then $K_{2,2}$ is a C_4 (see II.5.).
- Elsewhere it has been proved that :

- a balanced $(n,5,\lambda)$ $K_{2,3}$-design exists if and only if $\lambda(n-1) \equiv 0 \pmod{12}$ except for $n = 5$ and $\lambda \equiv 3 \pmod 6$ (C.Huang and A.Rosa [H8]),
- a balanced $(n,6,\lambda)$ $K_{2,4}$-design exists if and only if $\lambda n(n-1) \equiv 0 \pmod{16}$ and $\lambda(n-1) \equiv 0 \pmod 8$ (C.Huang [H9]),
- a balanced $(n,6,\lambda)$ $K_{3,3}$-design exists if and only if $\lambda n(n-1) \equiv 0 \pmod{18}$ and $\lambda(n-1) \equiv 0 \pmod 3$ except for $n = 10$ and $\lambda = 1$, $n = 6$ and $\lambda \equiv 3$ or $15 \pmod{18}$ (C.Huang [H9]).

Remark : It can be noticed also that the concept of resolvable B.I.B.D. (see D.K.Ray Chaudhuri and R.M.Wilson [R1] or H.Hanani, D.K.Ray Chaudhuri and R.M.Wilson [H2]) has been extended to resolvable G-design and studied by P.Hell and A.Rosa [H6] for G a path, and by C.Huang [H13] for G a bipartite graph.

III. METHOD OF DIFFERENCES

This method of direct construction is essentially a generalization of Bose's method of symmetrically repeated differences (see M.Hall [H1]) We suppose that the reader is a little familiar with Bose's method and we shall indicate only the useful lemmas (without proofs) and examples. The reader interested can find more details and proofs in [B4] ,(see also most of the articles given in references : indeed the method is one of the most used).

Let Γ be an additive group. Usually Γ will be Z_n the group of residues modulo n . If G is a graph which vertices are labelled with elements of Γ , we shall denote by $G + g$ the graph having as vertices the elements $x + g$ where x is vertex of G and as edges the pairs $\{ x + g , y + g\}$ where $\{x,y\}$ is an edge of G . If $\{x,y\}$ is an edge of G we put : $d(x,y) = |y-x|$. Note that $d(x + g,y + g) = d(x,y)$. ($d(x,y)$ is sometimes called the edge length between x and y)

The simplest case of the method is given by the following lemma :

III.1. Lemma : *Let $\{G_i = (X_i,E_i)$, $i \in I\}$ a family of graphs which vertices are elements of Γ , isomorphic to G . Then the graphs $\{ G_i + g , g \in \Gamma , i \in I \}$ are the subgraphs of an (n,k,λ) G-design if and only if the family $\{ \pm d(x,y)$, $\{x,y\} \in E_i$, $i \in I \}$ contains every element of $\Gamma - \{0\}$ exactly λ times.*

The G_i are called " base graphs " or " base block ".

III.2. Examples : (a path or cycle is denoted by the sequence of its vertices)

a. Let $\Gamma = Z_5$, $G = P_3$ (path with 3 vertices) , $G_0 = (0,1,3)$.
Then $\{\pm d(e)$, e edge of $G_0\} = \{\pm 1, \pm 2\} = \{1, 4, 2, 3\} = \Gamma - \{0\}$.
So by lemma III.1. there exists a $(5,3,1)$ P_3-design ,the paths of the decomposition being :
$G_0 = (0,1,3)$, $G_0+1 = (1,2,4)$, $G_0+2 = (2,3,0)$, $G_0+3=(3,4,1)$, $G_0+4=(4,5,2)$.

b. Let $\Gamma = Z_4$, $G = C_3$, $G_0 = (0,1,2,0)$.
Then $\{\pm d(e)$, e edge of $G_0\} = \{\pm 1, \pm 1, \pm 2\} = \{1, 3, 1, 3, 2, 2\}$
$= \Gamma - \{0\}$
So there exists a $(4,3,2)$ C_3-design ; the cycles of the decomposition are : $(0,1,2,0)$, $(1,2,3,1)$, $(2,3,0,2)$, $(3,0,1,3)$.

c. Let $\Gamma = Z_7$, $G = P_4$ (path with 4 vertices),
$G_0 = (0,1,2,4)$, $G_1 = (0,3,6,1)$.
Then $\{\pm d(e)$, e edge of $G_0\} = \{\pm 1, \pm 1, \pm 2\} = \{1, 6, 1, 6, 2, 5\}$
$\{\pm d(e)$, e edge of $G_1\} = \{\pm 3, \pm 3, \pm 2\} = \{3, 4, 3, 4, 2, 5\}$
so $\{\pm d(e)$, e edge of G_0 or $G_1\} = 2 (\Gamma - \{0\})$.
So there exists a $(7,4,2)$ P_4-design ; the paths of the decomposition are

$G_0 = (0,1,2,4)$	$G_1 = (0,3,6,1)$
$G_0 + 1 = (1,2,3,5)$	$G_1 + 1 = (1,4,0,2)$
$G_0 + 2 = (2,3,4,6)$	$G_1 + 2 = (2,5,1,3)$
$G_0 + 3 = (3,4,5,0)$	$G_1 + 3 = (3,6,2,4)$
$G_0 + 4 = (4,5,6,1)$	$G_1 + 4 = (4,0,3,5)$
$G_0 + 5 = (5,6,0,2)$	$G_1 + 5 = (5,1,4,6)$
$G_0 + 6 = (6,0,1,2)$	$G_1 + 6 = (6,2,5,0)$

III.3. A similar lemma holds in the directed case by considering, instead of $\pm d(e)$, $d(u) = (y - x)$ for the arc $u = (x,y)$ of G_i.

Example : Let $\Gamma = Z_5$, $G = \vec{C}_4$ (a directed circuit with k vertices is also denoted by the sequence of its vertices), $G_0 = (0,4,1,2,0)$.
Then $\{d(u)\ ,\ u \text{ arc of } G_0\} = \{4, 2, 1, 3\} = \Gamma - \{0\}$.
So there exists a $(5,4,1)$ $\vec{C}_4$-design ; the circuits of the decomposition are : $\{ G_0 + g = (g,4+g,1+g,2+g,g)\ ,\ g \in \Gamma \}$.

III.4. Refinements of the method can be obtained by adjoining to Γ one or several elements ∞ invariant under the action of the group Γ, that is $\infty + g = \infty$ for every g in Γ.

For example, in the directed case we have the following lemma :

III.5. **Lemma** : *Let $\{ G_i = (X_i,U_i)\ ,\ i \in I \}$ a family of directed graphs isomorphic to a directed graph G which vertices are elements of $\Gamma \cup \{\infty\}$. The graphs $\{ G_i + g\ ,\ g \in \Gamma\ ,\ i \in I \}$ are the subgraphs of an (n,k,λ) G-design if and only if*

(i) the family $\{d(u)\ ,\ u \in U_i \cap \Gamma \times \Gamma\ ,\ i \in I\}$ contains every element of $\Gamma - \{0\}$ exactly λ times,

(ii) the graphs $\{G_i\ ,\ i \in I\}$ contains λ arcs having ∞ as initial vertex and λ arcs having ∞ as terminal vertex.

III.6. Example : Let $\Gamma = Z_5$, $G = \vec{C}_4$,

$G_0 = (0,1,3,2,0)$, $G_1 = (\infty,0,1,3,\infty)$, $G_2 = (\infty,0,4,2,\infty)$.

$\{d(u)\ ,\ u \text{ arc of } G_0\} = \{1, 2, 4, 3\}$,

$\{d(u)\ ,\ u \in U_1 \cap Z_5 \times Z_5\} = \{1, 2\}$,

$\{d(u)\ ,\ u\ \ U_2 \cap Z_5 \times Z_5\} = \{4, 3\}$

G_1 and G_2 contain 2 arcs (∞,x) and 2 arcs (x,∞).
So, by lemma III.5., there exists a $(6,4,2)$ $\vec{C}_4$-design ; the circuits of the decomposition are $\{ G_0+g = (g,1+g,3+g,2+g,g)$
$G_1+g = (\infty,g,1+g,3+g,\infty)$
$G_2+g = (\infty,g,4+g,2+g,\infty)\ ,\ g \in \Gamma \}$.

III.7. In some cases, it happens that base graphs are fixed by elements g of the group Γ, that is $G_i + g = G_i$.

If so, let us denote by $\Gamma(G)$ the set of elements g of Γ fixing G. ($\Gamma(G)$ is a subgroup of Γ). We have the following lemma :

III.8. **Lemma** : *Let Γ be a group, $\{ G_i = (X_i,E_i)\ ,\ i \in I \}$ a family of graphs which vertices are elements of Γ, isomorphic to a given graph G. Let $\mathcal{G}_i$ the set of distinct graphs of the family $\{ G_i + g\ ,\ g \in \Gamma \}$. Then the graphs $\{ \mathcal{G}_i\ ,\ i \in I \}$ are the subgraphs of an (n,k,λ) G-design if and only if, for every element g of $\Gamma - \{0\}$,*

$$\sum_{i \in I} \Delta_{G_i}(g) \,/\, |\Gamma(G_i)| = \lambda$$

where $\Delta_{G_i}(g)$ is the number of edges e of G_i such that $d(e) = g$.

Example : Let $\Gamma = Z_{10}$, $G = C_6$, $G_0 = (0,9,1,8,2,7,0)$,
$G_1 = (0,1,9,5,6,4,0)$, then
$\Gamma(G_0) = \{0\}$, $\Gamma(G_1) = \{0, 5\}$, $|\Gamma(G_0)| = 1$, $|\Gamma(G_1)| = 2$,
$\{d(e)\ ,\ e \in G_0\} = \{1, 9, 2, 8, 3, 7, 4, 6, 5, 5, 3, 7\}$,
$\{d(e)\ ,\ e \in G_1\} = \{1, 9, 2, 8, 4, 6, 1, 9, 2, 8, 4, 6\}$,
$\Delta_{G_0}(g) = 1$ for $g = 1, 2, 4, 6, 8, 9$, $\Delta_{G_0}(g) = 2$ for $g = 3, 5, 7$
$\Delta_{G_1}(g) = 2$ for $g = 1, 2, 4, 6, 8, 9$, $\Delta_{G_1}(g) = 0$ for $g = 3, 5, 7$
So by lemma III.8. there exists a $(10,6,2)$ C_6-design ; the cycles of the decomposition are :
$\{ G_0 + g = (g,9+g,1+g,8+g,2+g,7+g,g)\ ,\ g \in \Gamma \} \cup$
$\{ G_1 + g = (g,1+g,9+g,5+g,6+g,4+g,g)\ ,\ g = 0, 1, 2, 3, 4\}$.

IV. METHOD OF COMPOSITION

The methods exposed below are recursive methods that enable us to construct G-designs from smallest one . We give only the principles of the method, the reader interested can find more details in [B3,B4] or in the case of a B.I.B.D. in R.M.Wilson [W3] . Similar ideas are also in papers of P.Hell and A.Rosa [H6] and P.Hell [H7] . The idea of the method is very simple : it is contained in the following lemmas.

We denote by $\lambda K_{n_1,n_2,\dots,n_h}$ the complete multipartite graph with vertex set $X = (\cup X_i)_{1 \leq i \leq h}$, where the X_i are disjoint sets with $|X_i| = n_i$ and where two elements which belong to different sets X_i and X_j are joined by λ edges .

IV.I. Lemma : *If there exists , for each i , an (n_i,k,λ) G-design and if $\lambda K_{n_1,\dots,n_h}$ can be decomposed into subgraphs isomorphic to G then there exists a $(\sum_{i=1}^{h} n_i,k,\lambda)$ G-design.*

Very useful also is the following lemma, a little more tricky :

IV.2. Lemma : *If there exists , for each i , an (n_i+1,k,λ) G-design and if $\lambda K_{n_1,\dots,n_h}$ can be decomposed into subgraphs isomorphic to G then there exists a $(\sum_{i=1}^{h} n_i + 1,k,\lambda)$ G-design.*

Similar lemmas hold in the directed case with $\lambda K^*_{n_1,n_2,\dots,n_h}$ the complete directed multigraph.

IV.3. Remarks :

1. In many cases we will suppose that $n_i = n$ for $1 \leq i \leq h$, and in this case we will write $\lambda K_{n\times h}$ instead of $\lambda K_{\underbrace{n,\dots,n}_{h\ \text{times}}}$

2. Case $G = K_k$ and transversal designs.

A decomposition of $\lambda K_{n\times h}$ into subgraphs isomorphic to K_k is nothing else than a transversal design $T(n,\lambda,k)$ in the notations of H.Hanani [H3] or a $T_0(k,n)$ in the notations of M.Hall's book [H1] p.224 (that is also equivalent to the existence of an orthogonal array or of $k-2$ orthogonal latin squares of order n). For more details the reader can see M.Hall's book [H1] or H.Hanani's survey [H3] or R.M.Wilson [W4] .

The interest of this method is that it is easier to decompose multipartite complete graphs than complete graphs . We shall see how the method can apply with bipartite and tripartite graphs.

IV.4. Composition with bipartite graphs.

The following simple lemma is useful :

IV.4.1. Lemma : *If K_{n_1,n_2} and $K_{n_1,n_2'}$ can be decomposed into subgraphs isomorphic to G then also $K_{pn_1,qn_2+sn_2'}$ with p, q, s integers.*

IV.4.2. Example of application :

Let $G = P_4$ (path with 4 vertices). $K_{3,2}$ and $K_{3,3}$ can be decomposed into P_4's : for example, if the vertices of $K_{3,3}$ are $\{x_0, x_1, x_2\}$ and $\{y_0, y_1, y_2\}$, a decomposition of $K_{3,3}$ is given by the following 3 paths :

(x_0,y_0,x_1,y_2) , (x_1,y_2,x_2,y_0) , (x_2,y_2,x_0,y_1) .

Thus by lemma IV.4.1. $K_{3,3p}$ and $K_{3,3p+2}$ can be decomposed into P_4's. So, by lemma IV.2. with $n_1 = 3$ and $n_2 = 3p$ or $3p+2$, if there exists a $(n_2+1,4,1)$ P_4-design then there exists a $(n_2+4,4,1)$ P_4-design.

Thus in order to prove the existence of an $(n,4,1)$ P_4-design for $n \equiv 0$ or 1 (mod 3) it suffices to prove the existence of a $(4,4,1)$ P_4-design and of a $(6,4,1)$ P_4-design. This can easily be done ; for example we obtain a $(6,4,1)$ P_4-design by applying lemma II.5. (in the undirected case) with $\Gamma = Z_5$ and $G_0 = (\infty,0,1,3)$.

Thus we have the following theorem :

IV.4.3. Theorem : *An $(n,4,1)$ P_4-design exists if and only if $n \equiv 0$ or 1 (mod 3)*

IV.4.4. With the same method, one can solve the existence of P_k-design. That is the method that has been applied in [B5] ,to prove the existence of C_{2k}-designs. For more details see [B4] .

The method of composition with bipartite graphs is very efficient but can apply iff G is a subgraph of bipartite graph.

IV.5. Composition with tripartite graphs.

A similar lemma as IV.4.1. holds. (The proof is a little more complicated, see [B6]).

IV.5.1. Lemma : *If* K_{n_1,n_2,n_3} *and* K_{n_1,n_2,n'_3} *can be decomposed into subgraphs isomorphic to G then also* $K_{qn_1,qn_2,pn_3+(q-p)n'_3}$, *with* p *and* q *integers,* $0 \leqq p \leqq q$.

IV.5.2. Example : We give a sketch of proof of the existence of an (n,5,1) $\vec{C}_5$-design. See [B6] . Different proofs appear in [B3], [S1] and [M3] .

Theorem : *An (n,5,1)* $\vec{C}_5$*-design exists if and only if* $n \equiv 0$ *or* 1 *(mod 5)*

Sketch of proof by induction : $K^*_{5,5,5}$ and $K^*_{5,5,10}$ can be decomposed into $\vec{C}_5$ (see [S1] for such a decomposition).
K^*_5, K^*_6, K^*_{10}, K^*_{11}, K^*_{25} and K^*_{26} can also be decomposed into $\vec{C}_5$ (see [B3])
For the general case, we use the lemmas IV.1. or IV.2. (in the directed case). We indicate below, in ecah case, the lemma and values used. (The verification that the hypothesis of the lemmas are satisfied result from lemma IV.5.1. and induction hypothesis).

$n = 15q$ $q \geqq 1$, lemma IV.1. with $n_1 = 5q$, $n_2 = 5q$, $n_3 = 5q$
$n = 15q + 1$, $q \geqq 1$, lemma IV.2. with $n_1 = 5q$, $n_2 = 5q$, $n_3 = 5q$
$n = 15q + 5$, $q \geqq 1$, lemma IV.1. with $n_1 = 5q$, $n_2 = 5q, n_3=5(q-1)+10$
$n = 15q + 6$, $q \geqq 1$, lemma IV.2. with $n_1 = 5q$, $n_2 = 5q, n_3=5(q-1)+10$
$n = 15q + 10$, $q \geqq 2$, lemma IV.1. with $n_1 = 5q$, $n_2 = 5q, n_3=5(q-2)+10\times 2$
$n = 15q + 11$, $q \geqq 2$, lemma IV.2. with $n_1 = 5q$, $n_2 = 5q, n_3=5(q-2)+10\times 2$

IV.6. It is also interesting to use k-partite graphs $K_{n\times k}$ (where k is the number of vertices of G).

IV.6.1. Indeed, if G is a graph with k vertices i , $1 \leqq i \leqq k$, let $G \otimes S_n$ denote the graph with vertex set X the disjoint union of k independant sets X_i with $|X_i| = n$ and where two points are joined in $G \otimes S_n$ if they belong to two sets X_i and X_j with $\{i,j\}$ an edge of G. ($G \otimes S_n$ is the lexicographic product of G by an independant set of n elements). Then we have the following lemma :

IV.6.2. Lemma : *If there exists a (n,k,λ) G-design and if* $G \otimes S_n$ *can be decomposed into subgraphs isomorphic to G then there exists a decomposition of* $\lambda K_{n\times k}$ *into subgraphs isomorphic to G.*
(See [B3] for more details)

IV.6.3. Decomposition of $G \otimes S_n$ are many times easier to find ; for example it is proved in [B3] that $G \otimes S_n$ can be always decomposed into subgraphs isomorphic to G if $G = C_k, \vec{C}_k, P_k, \vec{P}_k$.

IV.6.4. As consequence of lemma IV.1. and lemma IV.6.2. ,we have:
Lemma a: *If there exists an (m,k,λ) G-design and a (n,k,λ) G-design and if* $G \otimes S_n$ *can be decomposed into subgraphs isomorphic to G then we have an (nm,k,λ) G-design.*

Lemma b : *If there exists an (m,k,λ) G-design and an (n+1,k,λ) G-design and if* $G \otimes S_n$ *can be decomposed into subgraphs isomorphic to G then we have an (nm+1,k,λ) G-design.*

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