On Even Factorisations and the Chromatic Index of the Kautz and de Bruijn Digraphs. [1]

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Abstract

Motivated by the problem of designing large packet radio networks we show that the Kautz and de Bruijn digraphs with in- and out- degree d have arc-chromatic index $2d$. In order to do this, we introduce the concept of even 1-factorisations. An even 1-factor of a digraph is a spanning subgraph consisting of vertex disjoint loops and even cycles; an even 1-factorisation is a partition of the arcs into even 1-factors. We prove that if a digraph admits an even 1-factorisation then so does its line digraph. (In fact, we show that the line digraph admits an even 1-factorisation even under a weaker assumption discussed below.) As a consequence we derive the above property of the Kautz and de Bruijn digraphs relevant to packet radio networks.

1 Introduction.

One method of designing large packet radio networks [8], requires the construction of large digraphs with a given diameter and arc-chromatic index.

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Briefly, the vertices correspond to the users of the network and the arcs to radio channels. The diameter constraint enforces small transmission delay. An arc-coloring with few colors makes it possible to have all transmissions take place in few time slots (a user may be involved in at most one transmission in any one time slot, but all transmissions corresponding to a color class of arcs may take place simultaneously). For more details on this model, as well as some general results, see [8], [4].

There is a very similar problem with well established literature, namely the problem of constructing large digraphs, and graphs, with given diameter and maximum degree [1], [2], [3], [7]. The two best constructions known for general digraphs are the Kautz and de Bruijn digraphs mentioned in the title. We prove in this article that the Kautz and de Bruijn digraphs with in- and out- degree d have the smallest possible arc-chromatic index, namely $2d$. Consequently, they are of prime interest also as large digraphs with given diameter and arc-chromatic index; they substantially improve the bounds given in [8] for packet radio networks. Other constructions of packet radio networks arising from digraphs and graphs with given diameter and maximum degree are given in [4].

Let G be a digraph (possibly containing loops). The length (number of arcs) of a shortest directed path from a vertex u to a vertex v of G is called the *distance* from u to v. (If there is no such path, the distance is considered infinite.) The *diameter* of a digraph G is the maximum distance over all pairs of vertices u and v of G. The *outdegree* of a vertex v in the digraph G is the number of arcs vw in G, and is denoted by $d^+(v)$. The *indegree* $d^-(v)$

is defined analogously. An *arc-coloring* of G is a mapping assigning colors to the arcs of G in such a way that two distinct arcs having a common vertex obtain different colors. The *arc-chromatic index* of G is the minimum number of colors which make an arc-coloring of G possible. A *k-factor* of a digraph G is a spanning subgraph F of G in which the indegree and outdegree of each vertex is k. It is easy to see that a 1-factor of G consists of vertex disjoint directed cycles of G. An *even 1-factor* of G is a 1-factor of G in which all cycles other than loops (i.e., cycles with one arc) have an even number of arcs. Suppose F is a 2-factor of G and m is a function which assigns to each pair (e, v), where e is an arc of F and v is a vertex of e, a value $m(e, v) \in \{0, 1\}$ in such a way that $m(e, v) \neq m(f, v)$ if both e and f begin at v or both e and f end at v. We shall call m the *marking function* and the value $m(e, v)$ the *mark* of e at v. Note that we do not require that $m(e, v) = m(e, u)$ for $e = uv$, i.e., the marks of one arc may be different at the two ends. We shall treat the marks as elements of the additive group Z_2; this will permit us to add the marks and remain in the group. The marking function m induces two partitions of the arc set of F into closed directed walks as follows: In the first partition, denoted by $P(F, m, 0)$, a closed walk is obtained by starting at some unused arc, and after an arc $e = uv$ continuing with the unique arc $f = vw$ with $m(e, v) = m(f, v)$, until the starting arc is encountered again. In the second partition, denoted by $P(F, m, 1)$, a closed walk is obtained in the same way except that after $e = uv$ we take the unique arc $f = vw$ with $m(e, v) \neq m(f, v)$. We say that F is an *even 2-factor* of G if there exists a function m as above, such that all closed walks of both partitions $P(F, m, i)$

$(i = 0, 1)$, other than loops, have an even number of arcs. We say that G has an *even 1-factorisation* if the arcs of G can be partitioned into even 1-factors, and an *even 2-factorisation* if the arcs of G can be partitioned into even 2-factors. The relevance of even 1-factorisations to arc-coloring is made explicit in the following observation:

Proposition 1 *If G has an even 1-factorisation with d even 1-factors, then G has an arc-coloring with $2d$ colors.*

Proof: Each even 1-factor consists of even directed cycles and loops. Using two colours, we can color each even cycle alternating the colours, and each loop with one of the two colours. □

2 Line Digraphs and Even Factorisations.

The *line digraph* of G, denoted by $L(G)$, has as its vertices the arcs of G; there is an arc in $L(G)$ from e to f just if $e = uv$ and $f = vw$. Note that a loop $e = uu$ in G becomes a loop ee in $L(G)$.

Theorem 2 *If G has an even 1-factorisation with d even 1-factors, then its line digraph $L(G)$ also has an even 1-factorisation with d even 1-factors.*

Proof: Let $F_0, F_1, \ldots, F_{d-1}$ be an even 1-factorisation of G. Thus every arc e of G belongs to a unique even 1-factor F_i. To denote this fact we say that e is labeled by i, and write $l(e) = i$. We shall treat the labels $i = 0, 1, \ldots, d-1$ as elements of the additive group Z_d of integers modulo d; this will allow us to perform addition of the labels and remain in the group.

Note that for each $i = 0, 1, \ldots, d-1$, and every vertex v of G, there is a unique arc uv, and a unique arc vw of G, labeled by i. To define an even 1-factorisation of $L(G)$, we first extend the labeling $l : E(G) \mapsto Z_d$ to a labeling $l : E(L(G)) \mapsto Z_d$ by $l(ef) = l(e) + l(f)$, and then let H_i be the subgraph of $L(G)$ formed by all arcs labeled i, for $i = 0, 1, \ldots, d-1$. We claim that $H_0, H_1, \ldots, H_{d-1}$ is an even 1-factorisation of $L(G)$. Since each arc of $L(G)$ has a unique label, it remains to verify that every H_i is an even 1-factor.

For each vertex $e = uv$ of $L(G)$ and each label i there is a unique vertex f of $L(G)$ such that $l(ef) = i$ because the equation $l(e) + x = i$ has a unique solution in the group Z_d, and because there is at v a unique arc $f = vw$ of G with label $l(f) = x$. By a symmetric argument there is a unique vertex f of $L(G)$ such that $l(fe) = i$. Therefore each H_i is a 1-factor.

It remains to show that each directed cycle (other than a loop) of every H_i is even. Let $e_1, e_2, \ldots, e_p$ be the vertices of a directed cycle in some H_i, for $i = 0, 1, \ldots, d-1$. Thus all labels $l(e_j e_{j+1})$ are i, for $j = 1, 2, \ldots, p$ (with subscript addition modulo p). It follows that in G we have the equations $l(e_j) + l(e_{j+1}) = l(e_{j+1}) + l(e_{j+2})$, for $j = 1, 2, \ldots, p$. Hence the labels of the edges e_j in G alternate, i.e., $l(e_1) = l(e_3) = \ldots = a$ and $l(e_2) = l(e_4) = \ldots = b$. If $a \neq b$, then necessarily p must be even. On the other hand, if $a = b$ then $e_1, e_2, \ldots, e_p$ are the edges of a cycle in F_a, and hence p is even by assumption. $\square$

Theorem 3 *If G has an even 2-factorisation with d even 2-factors, then its line digraph $L(G)$ has an even 1-factorisation with $2d$ even 1-factors.*

Proof: Let $F_0, F_1, \ldots, F_{d-1}$ be an even 2-factorisation of G. As above, we label the arcs of G by Z_d, with $l(e) = i$ whenever e belongs to F_i. Let m_i be the marking function associated with the 2-factor F_i. We let $m(e, v)$ equal to $m_i(e, v)$ with $i = l(e)$; in this way marks are defined for all incident vertex-arc pairs. We again extend the labeling l of the arcs of G to a labeling of the arcs of $L(G)$. Suppose $e = uv$ and $f = vw$. We define the label $l(ef)$ to be the ordered pair $< \lambda, \mu > \in Z_d \times Z_2$, where $\lambda = l(e) + l(f)$ and $\mu = m(e, v) + m(f, v)$. We let $H_{<\lambda,\mu>}$ be the subgraph of $L(G)$ formed by all arcs labeled $< \lambda, \mu >$, for $\lambda \in Z_d$ and $\mu \in Z_2$. We claim that the $H_{<\lambda,\mu>}$ form an even 1-factorisation of $L(G)$. As above, it it enough to verify that each $H_{<\lambda,\mu>}$ is an even 1-factor.

For each vertex $e = uv$ of $L(G)$ and each label $< \lambda, \mu >$ there is a unique vertex f of $L(G)$ such that $l(ef) =< \lambda, \mu >$, because the equation $l(e)+x = \lambda$ has a unique solution in Z_d, and because exactly one of the two arcs of F_x beginning at v, say f, satisfies $m(e, v) + m(f, v) = \mu$. By a symmetric argument there is a unique vertex f of $L(G)$ such that $l(fe) =< \lambda, \mu >$. Therefore each $H_{<\lambda,\mu>}$ is a 1-factor.

Let again $e_1, e_2, \ldots, e_p$ be the vertices of a directed cycle in some $H_{<\lambda,\mu>}$, for $\lambda \in Z_d, \mu \in Z_2$. Thus all labels $l(e_j e_{j+1})$ are $< \lambda, \mu >$, for $j = 1, 2, \ldots, p$ (with subscript addition modulo p). Since all $l(e_j e_{j+1}) = \lambda$, we deduce as before that the labels of the edges e_j in G alternate, i.e., $l(e_1) = l(e_3) = \ldots = a$ and $l(e_2) = l(e_4) = \ldots = b$. If $a \neq b$, then necessarily p must be even. Thus assume that $a = b$, i.e., that all e_j are in F_a. Writing $e_j = v_{j-1}v_j$, we also have $m(e_j, v_j) + m(e_{j+1}, v_j) = \mu$ (for all $j = 1, 2, \ldots, p$). Therefore either all $m(e_j, v_j) = m(e_{j+1}, v_j)$ (if $\mu = 0$) or all $m(e_j, v_j) \neq m(e_{j+1}, v_j)$ (if $\mu = 1$). This means that $e_1, e_2, \ldots, e_p$ is one of the closed directed walks of the partition $P(F_a, m_a, \mu)$, and hence $p = 1$ or p is even. $\square$

3 The Kautz and de Bruijn digraphs.

Assume $d \geq 2$. For $D \geq 0$, we define the *de Bruijn digraph* $B(d, D)$ as the digraph whose vertices are all strings of length D over an alphabet of d symbols, in this paper always Z_d, and whose arcs are all strings of length $D+1$ over the same alphabet. The arc $a_1a_2...a_Da_{D+1}$ starts from the vertex $a_1a_2...a_D$ and ends in the vertex $a_2...a_Da_{D+1}$. Note that $B(d, 0)$ is the digraph with one vertex (corresponding to the empty string), and d loops, one for each letter of the alphabet; $B(d, 1)$ is a complete symmetric digraph with d vertices, and a loop at each vertex.

For $D \geq 1$, the *Kautz digraph* $K(d, D)$ is the digraph whose vertices are all those strings of length D over an alphabet of $d+1$ symbols, here $Z_d \cup \infty$, in which consecutive characters are distinct, and whose arcs are all strings of length $D+1$ over the same alphabet with the same property. The arc $a_1a_2...a_Da_{D+1}$ starts from the vertex $a_1a_2...a_D$ and ends in the vertex $a_2...a_Da_{D+1}$. Note that $K(d, 0)$ is undefined, and $K(d, 1)$ is a complete symmetric loopless digraph on $d+1$ vertices. It is easy to see (cf. [3]) that both $B(d, D)$ and $K(d, D)$ have diameter D and are regular of in- and out-degree d. Furthermore, the digraph $B(d, D)$ has d^D vertices and the digraph $K(d, D)$ has $d^D + d^{D-1}$ vertices.

It follows from the definitions that the following is true:

Proposition 4 *For all relevant d and D,*

$$B(d, D) = L(B(d, D-1)),$$

$$K(d, D) = L(K(d, D-1)).$$

□

Corollary 5 *Each $B(d, D)$ has an even 1-factorisation.*

Proof: The proof proceeds by induction on D. It is obvious for $D = 0$, and then it follows by using Theorem 2 and the above proposition. □

Corollary 6 *Let $D \geq 2$ or $D = 1$ and d be odd.*

Then $K(d, D)$ has an even 1-factorisation.

Proof: The proof again proceeds by induction on D. However, we cannot start at $D = 0$, and even for $D = 1$ the Kautz digraph $K(d, 1)$ does not admit an even 1-factorisation when d is even. Indeed, it has an odd number of vertices and no loops; thus each 1-factor must contain an odd cycle of more than one arc. On the other hand, for $D = 1$ and d odd, we can construct an even 1-factorisation of $K(d, 1)$ by starting with the complete undirected graph on $d + 1$ vertices, which is known to have edge-chromatic index d ([6], cf. below). Since $K(d, 1)$ is obtained from it by replacing each edge with the two opposite arcs, we can associate with each color class of such an edge-coloring by d colors a 1-factor of $K(d, 1)$ consisting of directed two-cycles. Thus for d odd, and any D, we obtain an even 1-factorisation of $K(d, D)$ via Theorem 2.

When d is even, we can proceed the same way, as soon as we have constructed an even 1-factorisation of $K(d, 2)$. For this purpose we use Theorem 3. Indeed, the complete undirected graph on $d + 1$ vertices admits a partition of its edge set into hamiltonian cycles [6]. This partition yields a 2-factorisation of $K(d, 1)$ as follows: We replace each undirected edge with the two opposite arcs; thereby every hamiltonian cycle $C = v_0, v_1, \ldots, v_d$ produces a 2-factor F of $K(d, 1)$. We claim that the 2-factor F is even. Indeed, we may define a marking function m in such a way that the two opposite arcs obtain the same marks at the vertex v_0 and obtain different marks at all other vertices, i.e., $m(vv_0, v_0) = m(v_0v, v_0)$ for $v = v_1$ and $v = v_d$, and $m(vv_i, v_i) \neq m(v_iv, v_i)$ for $i \neq 0$ and $v = v_{i-1}$ and $v = v_{i+1}$. Recall that the partition $P(F, m, 0)$ consists of closed walks obtained by following arcs that leave a vertex on the same mark as they entered it. Thus starting with the arc v_0v_1 we pass through all the arcs $v_1v_2, v_2v_3, \ldots$ until the arc v_dv_0; at this point we must follow with the arc v_0v_d and then retrace our steps through the arcs $v_dv_{d-1}, \ldots, v_1v_0$. (The closed walk ends here as the next arc would be the starting arc v_0v_1). Thus $P(F, m, 0)$ consists of a single closed walk, of length $2(d + 1)$. In the same spirit, the partition $P(F, m, 1)$ consists of one closed walk of length four with arcs $v_0v_1, v_1v_0, v_0v_d, v_dv_0$ and $d - 1$ cycles of length two $v_iv_{i+1}, v_{i+1}v_i$ for $i = 1, 2, \ldots, d - 1$. Thus both partitions $P(F, m, i)$ $(i = 0, 1)$ consist of even closed walks, and we obtain an even 2-factorisation of $K(d, 1)$, and hence by Theorem 3 an even 1-factorisation of $K(d, 2)$. □

Corollary 7 *Each $B(d, D)$ with $D \geq 1$ has arc-chromatic index $2d$.* □

Corollary 8 *Each $K(d,D)$ with $D \geq 2$ (or $D = 1$ and d odd) has arc chromatic index $2d$.* $\square$

In some applications it may be useful to know directly which 1-factor contains the arc $a_1 a_2 ... a_{D+1}$. (This will also allow us, via Proposition 1, to directly find the color of each arc in the corresponding arc-coloring.) It follows by unwinding the above induction (as was also observed by L. Goddyn, personal communication) that

Proposition 9 *There is an even 1-factorisation $F_0, F_1, ..., F_{d-1}$ of $B(d,D)$ in which the arc $a_1 a_2 ... a_{D+1}$ belongs to F_i where*

$$i = \sum_{j=0}^{D} \binom{D}{j} \cdot a_{j+1}.$$

Proof: For $D = 0$ the digraph $B(d,0)$ consists of the loops $0, 1, \ldots, d-1$ and we label each loop i by $l(i) = i$, i.e., we let each $F_i = \{i\}$. Now we proceed by induction: Suppose that l is a labeling of $B(d,D)$ where

$$l(a_1 a_2 ... a_{D+1}) = \sum_{j=0}^{D} \binom{D}{j} \cdot a_{j+1}.$$

Then we define as in the proof of Theorem 3 a labeling of $B(d, D+1)$ by $l(a_1, a_2, \ldots, a_{D+1}, a_{D+2}) = l(a_1, \ldots, a_{D+1}) + l(a_2, \ldots, a_{D+2})$. Therefore,

$$l(a_1, a_2, \ldots, a_{D+2}) = \sum_{j=0}^{D} \binom{D}{j} \cdot a_{j+1} + \sum_{j=0}^{D} \binom{D}{j} \cdot a_{j+2} = \sum_{j=0}^{D+1} \binom{D+1}{j} \cdot a_{j+1},$$

using Pascal's equality $\binom{D+1}{j} = \binom{D}{j} + \binom{D}{j-1}$. Letting F_i consist of all arcs labeled i we obtain the desired even 1-factorisation of $B(d,D)$. $\square$

The situation is somewhat less elegant for Kautz digraphs, but a calculation is possible. In particular, we need to use "nice" decompositions of the complete graph on $d+1$ vertices. We have already remarked that for d odd the complete undirected graph with vertices $Z_d \cup \infty$ admits a d-edge-coloring; one such coloring (cf. [6]) assigns to the edge ij the color $i+j$ when neither i nor j is ∞ and assigns to the edge ∞i the color $2i$. We define $\alpha(a,a') = a + a'$ if neither a nor a' is ∞ and $\alpha(a,\infty) = \alpha(\infty, a) = 2a$. (Both $a+a'$ and $2a = a+a$ are computed in the group Z_d.)

Proposition 10 *Let $D \geq 1$, and d be odd. There is an even 1-factorisation $F_0, F_1, ..., F_{d-1}$ of $K(d,D)$ in which the arc $a_1a_2...a_{D+1}$ belongs to F_i where*

$$i = \sum_{j=0}^{D-1} \binom{D-1}{j} \cdot \alpha(a_{j+1}, a_{j+2}).$$

Proof: We again proceed by induction using the labeling in the proof of Theorem 3. For $D=1$, the graph $K(d,1)$ is the complete symmetric digraph on $Z_d \cup \infty$. We let $l(aa') = \alpha(a,a')$. Then all arcs labeled i form an even 1-factor consisting of $\frac{d+1}{2}$ cycles of length two, because of the property of the above coloring of the complete undirected graph. The remainder of the proof is the same as in the preceeding Proposition. □

When d is even, we need a nice partition of the edges of the complete undirected graph with vertices $Z_d \cup \infty$ into $d/2$ hamiltonian cycles $C_1, C_2, \ldots, C_{d/2}$. The following folklore partition cf. [6], will be used (here the subscripts are modulo $d/2$): The edge ∞i belongs to C_i; the edge ij (with neither i nor j equal to ∞) belongs to $C_{\lfloor \frac{i+j}{2} \rfloor}$. To obtain from this partition an even 2-factorisation of $K(d,1)$ we must replace each undirected edge by two opposite arcs (both of the same color as the undirected edge) and we must also specify the marking functions $m_1, m_2, \ldots, m_{d/2}$. We use the marking functions explained in the proof of Corollary 6, where in each hamiltonian cycle C_s we let the vertex ∞ be the vertex v_0 distinguished in the definition of m_s. Specifically, each $m_s(\infty i, \infty) = m_s(i\infty, \infty)$, and each other $m_s(ij, j) \neq m_s(ji, j)$. Now we transform the even 2-factor of $K(d, 1)$ into an even 1-factor of $K(d, 2)$ as explained in the proof of Theorem 3 except we use the label $2\lambda + \mu$ instead of $< \lambda, \mu >$. This means that our labels are in the group Z_d instead of $Z_{d/2} \times Z_2$. Put $V = Z_d \cup \infty$. We define the auxiliary functions $\gamma : V \times V \mapsto Z_{d/2}$ and $\beta : V \times V \times V \mapsto Z_d$ by $\gamma(a, a') = \lfloor \frac{a+a'}{2} \rfloor$ if neither a nor a' is ∞ and $\gamma(a, \infty) = \gamma(\infty, a) = a$, and $\beta(a, a', a'') = 2(\gamma(a, a') + \gamma(a', a'')) + \delta$, where δ is 0 if $a' \neq \infty$ and $a \neq a''$ or $a' = \infty$ and $a = a''$ and is 1 otherwise. Now we obtain (by induction, as above) the following formula:

Proposition 11 *Let $D \geq 2$, and d be even. There is an even 1-factorisation $F_0, F_1, ..., F_{d-1}$ of $K(d, D)$ in which the arc $a_1 a_2 ... a_{D+1}$ belongs to F_i where*

$$i = \sum_{j=0}^{D-2} \binom{D-2}{j} \cdot \beta(a_{j+1}, a_{j+2}, a_{j+3}).$$

□

4 Conclusions.

Theorem 2 can be used with other digraphs as well. In particular, J. Bond [5] has recently given an even 1-factorisation of the graph with 50 vertices from [7], regular of in- and out- degree 2, and diameter 5; using Theorem 2 it follows that for each $D \geq 5$ there exists a graph of diameter D, with $25 \cdot 2^{D-4}$ vertices, regular of in- and out- degree 2, admitting an even 1-factorisation (and hence of arc-chromatic index 4). This is the largest known family of digraphs with arc-chromatic index 4 and diameter D (better than the Kautz or de Bruijn digraphs).

Recall that a digraph with many vertices but small arc-chromatic index and small diameter may be useful for packet radio networks. In [8], [4] one studies the largest number $n_C(f, D)$ of vertices of a digraph with diameter D and arc-chromatic index f. We can thus interpret our results (Corollary 6 and the above remark) as lower bounds on the function $n_C(f, D)$ (for $D \geq 2$ and $D \geq 5$ respectively):

$$n_C(2q, D) \geq q^D + q^{D-1}$$

$$n_C(4, D) \geq 25 \cdot 2^{D-4}.$$

These are the best known bounds on $n_C(f, D)$ for even f.

Finally we remark that the underlying graphs of the Kautz and de Bruijn digraphs, known as the *Kautz and de Bruijn graphs*, obtain, in the coloring implied by Corollaries 8 and 7, an edge coloring with Δ (the maximum degree) colors. Such graphs are called *of class 1.*

Corollary 12 *The Kautz graphs (other than the even complete graph) and the de Bruijn graphs are of class 1.*

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Fouquet and M.L. Yu, who independently proposed a tentative solution, i.e., constructed an even 1-factorisation of $K(d, 2)$. Our method, Theorem 3, has the advantage of explaining how a non-solution for $D = 1$ leads to a solution for $D = 2$.

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