THE DIAMETER OF GRAPHS - A SURVEY

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§0. Introduction

The diameter of a graph is the maximal distance between pairs of vertices. Thus a graph G has diameter D if any two vertices are at distance at most D and there are two vertices at distance D. Graphs of small diameter are of interest as examples of telecommunication networks. The aim of this review is to present some of the most important results concerning the diameter and to indicate areas for future research. The extremal problems concerning the diameter are presented in detail in Bollobás (1978, Ch.IV), so we shall concentrate on very recent results.

Many of the problems we are going to discuss were stimulated by various applications in mind - even more, some of the problems were first posed and tackled by engineers. Perhaps the most important problem studied so far is the following. Given integers n, D, D', Δ, k and ℓ, determine or estimate the minimal number of edges in a graph G of order n with the following properties:

(i) G has maximal degree at most Δ,

(ii) the diameter of G is at most D,

(iii) if G' is obtained from G by suppressing any k of the vertices or any ℓ of the edges, the diameter of G' is at most D'.

It is worth noting that though we asked for the minimum of the number of edges, the problem includes the question whether or not there are graphs satisfying the conditions.

The conditions in the problem above are dictated by rather natural restrictions and requirements on the communication network we intend to build: we know that there are n centres to join and it is not feasible to join a centre to more than Δ other centres; we wish to keep the diameter small to guarantee communications between any two centres without passing through many intermediate centres, we wish to make our network reliable in the sense that if any k of the centres (or any ℓ of the lines) are out of order, the remaining centres can still communicate with each other fairly fast; and, finally, in order to minimize the cost, we wish to achieve all these with the aid of as few edges as possible.

Having formulated this single but rather big problem we hasten to add that at the moment we are very far from a satisfactory solution and it seems unreasonable to expect a complete solution with an exact value and a concrete graph for every choice of the parameters. Perhaps it is better to think of this problem as a union of a number of smaller problems, for the nature of the difficulty depends on the domain of our parameters. In fact, various aspects of this problem arose separately and sound more natural in other formulations: Elspas (1964), Erdős, Rényi and Sós (1966) and Murty and Vijayan (1964) initiated essentially disjoint areas of research. For example, Elspas (1964) asked for what Δ and D there are graphs of order n with maximal degree at most Δ and diameter at most D. Equivalently, what is $n(D,\Delta)$, the maximal value of n for which there is a graph of order n, maximal degree Δ and diameter at most D? Though many papers have been published on this problem, no essential progress had been made until very recently. In §1 we shall indicate some of the most important results in this area.

The problems concerning the reliability of a communication network were posed by Murty and Vijayan (1964). Here D, D', k and ℓ are given, and one is interested in the number of edges for large values of n, without any restriction on the maximal degree. Though not much has happened since Bollobás (1978, Ch.IV), in §2 we briefly mention some of the results since an important part of the main problem is still wide open.

It is perhaps a little surprising that results about random graphs can give information about extremal problems concerning the diameter. In fact, some of the best bounds on the function $n(D,\Delta)$ mentioned earlier are proved with the aid of random graphs. However, as we shall see in §3, we are mostly interested in random graphs for their own sake and the information on $n(D,\Delta)$ is more or less incidental.

The last two sections concern problems of different type. For what values of n can the complete graph K^n be factored into $F_1, F_2, \dots, F_k$ such that for each i, $1 \le i \le k$, the factor F_i is of diameter at most D_i? The best outstanding problem was solved recently by Znám (1981) - we present it together with a review of some of the older results in §4. The last section contains miscellaneous results.

Before proceeding to the main part of the paper let us note a number of basic properties of the diameter. Suppose G has diameter D. Then there is a vertex $x \in G$ such that $\max_{y \in G} d(x,y) = D$. In particular, if $V_k = \Gamma_k(x)$ denotes the set of vertices at distance exactly k from x then $V(G) = \bigcup_{k=0}^{D} V_k$ is a partition of the vertex set into $D+1$ non-empty sets $V_0, V_1, \dots, V_D$ such that $V_0 = \{x\}$ and no edge joins V_i to V_{i+j} for $i = 0,1,\dots,D-2$ and $j \ge 2$. This shows that if G is a maximal graph of diameter D then $V(G)$ has a partition $V(G) = \bigcup_{k=0}^{D} V_k$ into non-empty sets such that $|V_0| = 1$ and a vertex $y \in V_i$ is joined to a vertex $z \in V_j$ iff $|i-j| \le 1$.

Some immediate consequences of this remark were noted by Ore (1968) and Bosák, Rosa and Znám (1968). If G has diameter $D \ge 2$ then its size $e(G)$ and maximal degree $\Delta(G)$ satisfy

$$e(G) \le D + \frac{(n-D-1)(n-D+4)}{2}$$

and

$$\Delta(G) \le n-D+1 .$$

Furthermore, as pointed out by Watkins (1967), if G is k-connected (so that each class V_i, $1 \le i \le D-1$, has to have at least k vertices) then

$$n \ge k(D-1) + 2 .$$

These remarks indicate that in some sense the problems concerning graphs of diameter at least D are not particularly exciting. Of course this does not mean that all results of this type are easy to obtain, since the actual calculations may present grave difficulties as shown by many of the papers about graphs of large diameter: Amar, Fournier and Germa (1981), Goldberg (1965,1966), Hirschberg and Wong (1979), Klee and Quaife (1976,1977), Klee (1980), Kramer and Kramer (1970), Moon (1965) and Myers (1980a, 1980b, 1981). However, the point remains that the diameter is guaranteed to be large, say at least D, if there is a single partition $V(G) = \bigcup_{k=0}^{D} V_k$ with $V_k \ne \emptyset$ and no edge joining V_i to V_j for $i \le j-2$. On the other hand, in order to ensure that the diameter is small, say at most D, we must guarantee that for every vertex $x \in G$ we have $\bigcup_{k=0}^{D} \Gamma_k(x) = V(G)$. In particular, there is no useful characterization of the minimal graphs of diameter D. Therefore the deeper results about the diameter concern graphs of diameter at most D and in the paper we focus our attention on these papers.

§1. Large graphs of given diameter and maximal degree

At most how many vertices can a graph have if its maximal degree is Δ and diameter is at most D? As in the introduction, we denote this maximum by $n(D,\Delta)$. Clearly $n(D,2) = 2D+1$, as shown by the cycle C^{2D+1}, so in the sequel we shall assume that $\Delta \ge 3$. Since in a graph of maximal degree Δ there are at most $\Delta(\Delta-1)^{k-1}$ vertices at distance $k \ge 1$ from a vertex

$$n(D,\Delta) \le 1 + \Delta \frac{(\Delta-1)^D - 1}{\Delta-2} = n_o(D,\Delta) . \qquad (1)$$

If we have equality for a certain pair (D,Δ) then a graph of order $n(D,\Delta) = n_o(D,\Delta)$, maximal degree Δ and diameter D is said to be a Moore graph of degree Δ and diameter D. Unfortunately there are very few Moore graphs: it has been proved by Hoffmann and Singleton (1960), Damerell (1973) and Bannai and Ito (1973) that if there is a Moore graph of degree $\Delta \ge 3$ and diameter D then $D = 2$ and $\Delta = 3,7$ or 57. In fact Moore graphs do exist for $D = 2$ and $\Delta = 3,7$, but it is not known whether there is a Moore graph of diameter 2 and degree 57. (For a discussion of this and related problems see Biggs (1974), and Bollobás (1978, Ch.III) and (1979, Ch.VIII).)

In practical applications, and also in extremal graph theory, it is not very important to know whether or not equality can hold in (1). It would be considerably more exciting to know the order of $n(D,\Delta)$. The least we would like to know from above is whether $n(D,\Delta)$ is usually close to $n_o(D,\Delta)$ or not. A recent result in this direction is due to Erdős, Fajtlowicz and Hoffmann (1980): with the exception of the 4-cycle C^4 there are no graphs of diameter 2 with maximal degree Δ and order Δ^2. Thus if Δ is even and at least 4 then $n(2,\Delta) \le \Delta^2-1 = n_o(2,\Delta) - 2$. To illustrate how little we do know about the bound for $n(D,\Delta)$, we pose the following two questions.

Is it true that for every $c \in \mathbb{N}$ there are D and Δ such that

$$n(D,\Delta) < n_o(D,\Delta) - c \text{ ?}$$

Is it true that if $D_k \to \infty$ and $\Delta_k \to \infty$ then

$$\lim_{k\to\infty} \frac{n(D_k,\Delta_k)}{n_o(D_k,\Delta_k)} = 1 \text{ ?}$$

In connection with the second question it is worth noting that we do not even know whether or not

$$n(D_k,\Delta_k)/n_o(D_k,\Delta_k)$$

tends to a limit whenever $D_k \to \infty$ and $\Delta_k \to \infty$ or D_k is fixed and $\Delta_k \to \infty$.

In view of the lack of results that improve substantially the trivial upper bound for $n(D,\Delta)$, we shall concentrate on lower bounds. This is exactly the problem that is interesting in practical applications: one needs graphs showing that $n(D,\Delta)$ is large, rather than theorems stating that it cannot be too large. In fact the main role of an upper bound on $n(D,\Delta)$ is to put an end to our search for graphs showing that $n(D,\Delta)$ is large. The first lower bounds were given by Elspas (1964), Akers (1965), Friedman (1966), (1971), Korn (1967), Storwick (1970) and Memmi and Raillard (1981). Asymptotically the best they could do was

$$\liminf_{\Delta\to\infty} n(D,\Delta)/\Delta^{2D/3} > 0 .$$

Recently this bound has been improved substantially by Bermond, Delorme and Farhi (1981a, 1981b), Delorme (1981) and Delorme and Farhi (1981).

In order to give a lower bound for $n(D,\Delta)$ we have to show the existence of a graph with many vertices, small diameter and small maximal degree or, even better, we can construct such a graph. In this section we look at the available constructions and in §3 we use random graphs to show the existence of appropriate graphs.

There are rather general constructions giving examples of graphs with many symmetries that have at least $c(D)\,\Delta^D$ vertices, diameter D and maximal degree Δ, where $c(D) > 0$ is independent of Δ. Here we describe one of these constructions, for other constructions and a more detailed treatment of the topic we refer the reader to Delorme and Farhi (1981).

Let $h \geq 2$, $k \geq 2$, $m \geq 2$, $E = \{1,2,\dots,h\}$ and let $\mathbb{Z}_m$ be the cyclic group of integers modulo m. We shall define a graph with vertex set $V = \mathbb{Z}_m \times E^k$ by joining a vertex $(\alpha;a_1,a_2,\dots,a_k)$ to all vertices in the set

$\{(\alpha+1;a_2,a_3,\dots,a_k,\lambda):\ \lambda=1,\dots,h\}\cup\{(\alpha-1;\mu,a_1,a_2,\dots,a_{k-1}):\mu=1,\dots,h\}$.

Theorem 1.1. The graph above has mh^k vertices, it is vertex transitive with degree $\Delta = 2h$ and its diameter is

$$D = k + \max_{\alpha\in\mathbb{Z}_m} \min\{|\alpha-k|,|\alpha+k|\} . \qquad \square$$

Here $|a| = \min\{|a'| : a' \equiv a \pmod m\}$. Let us illustrate the theorem with a concrete example. With $m = 5$, $h = 3$ and $k = 9$ we obtain a regular graph of degree 6, diameter 10 and order 98415. Thus this improves greatly the value 9465 given by Storwick (1970) and the order 17496 found by Memmi and Raillard (1981). In fact graphs of this kind were first considered by Memmi and Raillard (1981) for the particular case $k = m-1$. They also conjectured that in their graphs the connectivity was equal to the degree. This conjecture was proved by Amar (1981) for all graphs given above thereby showing that in some sense these graphs are as reliable as possible.

If we take $m = 1$ in the construction above, the graphs we obtain are not simple. However, by deleting loops and multiple edges we arrive at a graph of maximum degree $\Delta = 2h$ on $h^k = (\Delta/2)^k$ vertices that has diameter $D = k$. In particular, with $\Delta = 10$, $D = 9$, one obtains a graph of maximal degree 10, diameter 9 and order 1953125. (The best value before this result was 92378.) It is worth emphasizing the bound on $n(D,\Delta)$ given by these graphs.

Theorem 1.2. If $\Delta \geq 2$ is even then

$$n(D,\Delta) \geq (\Delta/2)^D . \qquad \square$$

Delorme and Farhi (1981) gave similar constructions by joining vertices as in the graphs associated to projective planes. Other constructions are in Delorme (1981). Among others, Delorme constructs a graph of degree $3(q+1)$, diameter 6 and order $18(q^2+q+1)^3$, where q is a prime power.

Bermond, Delorme and Farhi (1981a,b) present a different kind of construction which gives good results for small values of D and Δ. This construction is based on a new product of graphs.

Let $G = (X,E)$ and $G' = (X',E')$ be graphs. Give the edges of E an arbitrary orientation and denote by U the set of arcs (oriented edges) obtained. Finally, for every arc $xy \in U$ (edge oriented from x to y) let f_{xy} be a permutation of X'. Now define the product $G * G'$ as follows. The vertex set is $X \times X'$. A vertex (x,x') is joined to a vertex (y,y') if

$$x = y \quad \text{and } x'y' \in E'$$

or

$$xy \in U \quad \text{and } f_{xy}(x') = y' .$$

Note that $G \times G'$ is obtained from the union of $|X|$ copies of G', indexed by the vertices of G, by adding for every edge $xy \in E$ a perfect matching from a copy indexed by x to a copy indexed by y. This perfect matching is a function of the edge xy. It is important to keep in mind that the product depends on these matchings, that is on the permutations f_{xy}, though the notation $G * G'$ suppresses this fact.

For example, the Petersen graph is $K^2 * C^5$ with $f_{12}(x') = 2x' \pmod 5$, where the notation is self-explanatory. If we take $f_{xy}(x') = x'$ for every arc xy then we obtain the Cartesian sum, and in this case the diameter of $G * G'$ is the sum of the diameters of G and G'. However, more appropriate choices of f_{xy} may give rise to graphs $G * G'$ with small diameters. For example, Bermond, Delorme and Farhi (1981a) prove the following result.

Theorem 1.3. Let $m = 2a^2+2a+1$ and let the cycle C^m have vertex set $\mathbb{Z}_m$ with i joined to $i+1$ and $i-1$. Let G be a graph of maximal degree Δ and diameter D, $D \geq 2$, with a given orientation U. Set $f_{xy}(x') = (2a+1)x'$ for any arc xy of U. Then $G * G'$ has maximal degree $\Delta+2$ and diameter $D+a$. The theorem is also true for $D = 1$, $a = 1$.

Note that $K^3 * C^5$ is a graph of degree 4, diameter 2 and order 15. In fact this graph is extremal: $n(2,4) = 15$. As another example, take for G the graph on 24 vertices with degree 5 and diameter 2. (This graph itself was constructed as a product by Bermond, Delorme and Farhi (1981b).) Then $G * C^5$ has degree 7, diameter 3 and order 120. (The best graph known before had 80 vertices.)

Products were used by Delorme (1981) and Bermond, Delorme and Farhi (1981a,b) in connection with graphs constructed with the aid of finite geometries. For example the following result gives a rather good lower bound for $n(3,\Delta)$, much better than Theorem 1.2.

Theorem 1.4. If q is an odd prime and $\Delta = 3(q+1)/2$ is an integer then there is a graph on $(8\Delta^3 - 12\Delta^2 + 18\Delta)/27$ vertices having diameter 3 and maximal degree Δ. □

Delorme (1981) made use of polarity to construct graphs of diameter at most 6. For example, a special case of his general construction gives a graph of degree 4, diameter 5 and order 364. Furthermore, Delorme (1981) showed how to obtain graphs of large order and diameter 7 by inserting vertices into a graph of diameter 6.

In conclusion we give two tables about the best known values of $n(D,\Delta)$. Almost all these bounds are due to Bermond, Delorme and Farhi.

2	3	4	5	6	7	8	9	10
1	$\frac{8}{27}$	$\frac{3}{16}$	$\frac{1}{32}$	$\frac{2}{81}$	$\frac{1}{2^7}$	$\frac{3}{2^8}$	$\frac{3}{2^{10}}$	$\frac{3}{2^{10}}$

Table 1. Lower bounds for $\lim_{\Delta \to \infty} \frac{n(D,\Delta)}{\Delta^D}$ for $D = 2,\ldots,10$.

Degree \ Diameter	3	4	5	6	7	8	9	10
3	20	30	52	126	142	208	240	480
4	40	95	364	728	782	1230	1932	2560
5	60	170	386	2730	2912	4368	5824	7776
6	105	312	680	7812	10920	16380	25480	98415
7	120	340	1248	8024	31248	51870	95550	204750
8	192	800	1820	39216	39902	148428	273420	1310720
9	585	1170	3200	74898	156864	235296	531216	1600000
10	593	1620	4680	132860	299542	745104	1953125	12252303

Table 2. Lower bounds for $n(D,\Delta)$.

§2. Reliable networks of small diameter

Most of the results in this section are contained in Ch.IV of Bollobás (1978) so our main aim here is to draw attention to the fact that the area is far from being exhausted. Let n, D, D' and s be natural numbers, $s < n$, $2 \le D \le D' \le n-1$. Denote by $V(n,D,D',s)$ the set of graphs of order n and diameter at most D such that the deletion of any s of the vertices results in a graph of diameter at most D'. Thus graphs belonging to $V(n,D,D',s)$ represent networks of small diameter whose diameter remains fairly small even if s of the vertices are suppressed (say they are "out of order"). As mentioned in the Introduction, one is specially interested in those members of $V(n,D,D',s)$ that have few edges. Put

$$f(n,D,D',s) = \min\{e(G) : G \in V(n,D,D',s)\}.$$

A similar graph and a similar function can be defined for edge deletions but we shall not discuss them here.

The study of the class $V(n,D,D',s)$ and the function $f(n,D,D',s)$ was proposed by Murty and Vijayan (1964) and many results have been obtained by Murty (1968a, 1968b, 1968c), Bollobás (1968a, 1968b, 1976a, 1976b), Bondy and Murty (1972), Bollobás and Eldridge (1976), Caccetta (1976a, 1976b, 1977, 1978, 1979a, 1979b, 1979c) and Bollobás and Erdős (1975). Most of these papers concern the case $s = 1$ and graphs of diameter at most 4, in fact especially graphs of diameter 2. One is particularly interested in the behaviour of $f(n,D,D',s)$ as $n \to \infty$ and D, D' and s are fixed.

It turns out that if D' is considerably larger than D then the order of $f(n,D,D',s)$ is determined by D and s and it seems likely that if D' is not much larger than D then the order of $f(n,D,D',s)$ depends on D' and s. For simplicity we emphasize the case $s = 1$. The following is a special case of a result in Bollobás (1976a).

Theorem 2.1. If $D \ge 2$ and $D' \ge 2D-2$ then

$$\lim_{n\to\infty} \frac{f(n,D,D',1)}{n} = \frac{D}{D-1}. \qquad \square$$

Let us observe that it is trivial to construct a graph G showing that $f(n,D,D',1)$ is at most $\frac{D}{D-1}n-2$. Write $n = 2+(D-1)k+\ell$ where $0 \le \ell \le D-2$. Join two vertices by k disjoint paths of length D and by one path of length $\ell+1$. Then $G \in V(n,D,D',1)$ and $e(G) \le Dn/(D-1)-2$.

The theorem above has an extension, due to Bollobás (1976a), to all values of D and s, provided D' is sufficiently large. For simplicity we state it only for even values of D.

Theorem 2.2. Given $m \ge 1$ and $s \ge 2$ there is a constant $C(s,m)$ such that if D' is sufficiently large then

$$\frac{n}{2}\left\{s+1+\frac{s-1}{s^m-1}\right\} - C(s,m) \le f(n,2m,D',s) \le \frac{n}{2}\left\{s+1+\frac{s-1}{s^m-1}\right\}. \qquad \square$$

If D' is not much larger than D we know considerably less about the function $f(n,D,D',s)$. Even the case $s = 1$ is very far from being settled. However, we have the following conjecture.

Conjecture 2.3. If $D \ge 2$ then

$$\lim_{n\to\infty} \frac{f(n,D,D,1)}{n} = \frac{\lfloor D/2\rfloor+1}{\lfloor D/2\rfloor}. \qquad \square$$

This conjecture is motivated by graphs similar to the ones described above. For example if $n = \ell\lfloor D/2\rfloor+2$ and G consists of ℓ independent paths of length $\lfloor D/2\rfloor$ joining two vertices then $G \in V(n,\lfloor D/2\rfloor+1, 2\lfloor D/2\rfloor,1) \subset V(n,D,D,1)$ and $e(G) = \ell(\lfloor D/2\rfloor+1)$. The conjecture is further supported by the fact that in Bollobás (1976b) it is proved for $D \le 4$.

§3. The diameter of random graphs

Random graphs are studied partly for their own sake and partly because they can be used to tackle standard questions in graph theory. (See Bollobás (1979, Ch.VII) for an introduction to random graphs.) As far as the diameter is concerned, the situation is rather cheerful: we know a fair amount about the diameter of random graphs and in certain problems the random graphs take us closest to a solution.

It is important to emphasize that we consider random labelled graphs. We shall be concerned with two models. $G(n,m)$ is the space of all graphs with a fixed set of n labelled vertices having exactly m edges and $G(n,r\text{-reg})$ is the space of all r-regular graphs with a fixed set of n labelled vertices. In both models any two graphs have the same probability.

It turns out that in both models most graphs have small diameters: most graphs are such that their orders give fairly good lower bounds for $n(D,\Delta)$. In the model $G(n,m)$ we are forced to choose m fairly large, at least about $\frac{1}{2}n \log n$, since otherwise most graphs in the model are disconnected, and then the diameter is fairly small, almost as small as one would hope for a graph of maximal degree $2m/n$. On the other hand, most graphs in $G(n,r\text{-reg})$ are connected (though, by definition, they have only $rn/2$ edges) and provide good bounds for $n(D,\Delta)$ with Δ fixed and D large.

The first results concerning the diameter of graphs in $G(n,m)$ are due to Erdős and Rényi (1960), Moon and Moser (1966) and Korshunov (1971). More recently Klee and Larman (1981) studied those functions $m(n)$ for which almost every graph in $G(n,m)$ has diameter D, where D is a fixed natural number. The results of Klee and Larman were extended considerably by Bollobás (1981a) who obtained essentially best possible results about the distribution of the diameters of random graphs, including the case when $D = D(n)$ increases (fairly slowly) with n. Here we state only the following corollary of the main result in Bollobás (1981a).

Theorem 3.1. Suppose the functions $D = D(n) \geq 3$ and $m = m(n)$ satisfy

$$(\log n)/D - 3 \log \log n \to \infty,$$
$$(2m)^D n^{-D-1} - 2 \log n \to \infty \quad \text{and}$$
$$(2m)^{D-1} n^D - 2 \log n \to -\infty.$$

Then almost every graph in $G(n,m)$ has diameter D. □

Since in the range given in the theorem almost every graph has maximal degree at most $(1+\varepsilon)\frac{2m}{n}$, one obtains the following rather crude consequence of the result above.

Corollary 3.2. Suppose $0 < \varepsilon < 1$ and the sequences (D_k), (Δ_k) are such that

$$D_k^4 \leq \Delta_k \quad \text{and} \quad D_k \to \infty.$$

Then if k is sufficiently large,

$$n(D_k,\Delta_k) \geq \frac{((1-\varepsilon)\Delta_k)^{D_k}}{2 D_k \log \Delta_k}.$$ □

Note that if $(2-\varepsilon)^D/(D \log \Delta) \to \infty$, for example if $\Delta = D^5$ or $\Delta = \lfloor (3/2)^D \rfloor$, then this corollary gives a much better bound for $n(D,\Delta)$ than Theorem 1.2.

The diameter of random regular graphs was studied by Bollobás and de la Vega (1981). Again we give only one of the main results.

Theorem 3.3. Let $r \geq 3$ and $\varepsilon > 0$ be fixed and define $D = D(n)$ as the least integer satisfying

$$(r-1)^{D-1} \geq (2+\varepsilon) rn \log n.$$

Then almost every r-regular graph of order n has diameter at most D. □

It is easy to extract from this a bound on $n(D,\Delta)$ in a certain range.

Corollary 3.4. Suppose $\varepsilon > 0$ and $\Delta \geq 3$ are fixed. Then if D is sufficiently large,

$$n(D,\Delta) \geq \frac{1-\varepsilon}{2\Delta D \log(\Delta-1)} (\Delta-1)^{D-1}.$$ □

Note that, once again, the bound above is considerably better than the one given by Theorem 1.2, provided D is sufficiently large. Putting it rather crudely, Corollary 3.4 asserts that for every $\Delta \geq 3$ there is a constant $c_\Delta > 0$ such that

$$n(D,\Delta) > c_\Delta (\Delta-1)^D/D$$

for every D.

§4. Factorization into graphs of small diameter

Given a sequence $D_1, D_2, \ldots, D_k$ of integers, $D_i \geq 2$, for what values of n does K^n have a factorization into $F_1, F_2, \ldots, F_k$ such that each F_i has diameter at most D_i? It is easily seen that there is such an n and Bosák, Rosa and Znám (1968) pointed out that if K^n has such a factorization then so has $K^{n'}$ for $n' > n$. Denote by $f(D_1, D_2, \ldots, D_k)$ the minimal integer n for which K^n has such a factorization. Furthermore, put $f_k(D) = f(D_1, D_2, \ldots, D_k)$ where $D_1 = D_2 = \ldots = D_k = D$.

The function $f(D_1, D_2, \ldots, D_k)$ has been studied by a number of authors, including Bosák, Rosa and Znám (1968), Bosák, Erdős and Rosa (1971), Sauer (1970), Palumbiny (1972, 1973), Sauer and Schaer (1973), Bosák (1974), Erdős, Sauer, Schaer and Spencer (1975), Tomova (1975, 1977) and a survey of the topic is given in Bollobás (1978, Ch. IV, §4). It so happens that due to these efforts perhaps the most interesting questions have been solved.

The function $f(D_1, \ldots, D_k)$ has been determined by Palumbiny (1973) if each D_i is at least 3.

<u>Theorem 4.1</u>. If $D_1, \ldots, D_k \geq 3$ and $k \geq 2$ then

$$f(D_1, D_2, \ldots, D_k) = 2k .$$

To see that f is at least $2k$ all one has to note is that each factor F_i must be connected so must have a least $n-1$ edges. Hence $k(n-1) \leq \frac{1}{2}n(n-1)$. On the other hand K^{2k} is easily factorized into k factors of diameter 3.

The problem becomes considerably harder if for some i we have $D_i = 2$. In fact, a breakthrough was achieved by Sauer (1970). The result of Sauer was improved by Bosák (1974) to $6k-50 \leq f_k(2) \leq 6k$. The lower bound was improved a little by Bollobás (1978) and then Znám (1981) showed that the upper bound is the correct value if k is large.

<u>Theorem 4.2</u>. If k is sufficiently large then $f_k(2) = 6k$. □

§5. Miscellaneous topics

A graph G has diameter at least 3 if and only if it contains two vertices, say x and y, such that $xy \notin E(G)$ and $\Gamma(x) \cap \Gamma(y) = \emptyset$. These conditions mean that $\bar{G}$ contains a double star: $xy \in E(\bar{G})$ and in $\bar{G}$ every vertex is joined to at least one of x and y. Hence diam $G \geq 3$ iff $|G| \geq 2$ and $\bar{G}$ contains a double star. This was observed by Bloom, Kennedy and Quintas (1981), who also deduced some consequences of this observation concerning graphs of diameter 3 and at least 4.

The diameters of graphs of special structures, including polytopes, semirings and semigroups have been studied by Klee (1967), Barnette (1974), Nakassis (1976) and Bosák (1964).

A graph is said to be <u>geodetic</u> if for any two vertices there is a unique shortest path joining them. The problem of characterizing geodetic graphs was posed by Ore (1962) and is still unsolved. However, there is a huge literature on the subject, especially on geodetic graphs of diameter 2: Plesnik (1977), Stemple and Watkins (1968), Bosák, Kotzig and Znám (1968), Bose and Dowling (1971), Lee (1977), Stemple (1974, 1979), Plesnik and Znám (1974), Parsons (1975), Zelinka (1975) and Bosák (1978). Among others it is proved that a graph with exactly one cutvertex is a geodetic graph of diameter 2 if and only if every block of the graph is complete. The <u>friendship graphs</u> are rather restricted kinds of geodetic graphs of diameter 2. A graph is said to be a <u>friendship graph</u> if any two different vertices have exactly one common neighbour. Erdős, Rényi and Sós (1966) proved that a friendship graph has an odd number of vertices, say $2\ell+1$, and it consists of ℓ triangles having one vertex in common. This theorem has been reproved a number of times and various extensions of it have been studied: Wilf (1971), Longyear and Parsons (1972), Kotzig (1975), Skala (1972), , Wallis (1970), Bose and Shrikhande (1970), Rudvalis (1971), Parsons (1975) and Sudolsky (1978).

For example, given $m \geq 1$ and $k \geq 1$, a graph is said to be an (m,k)-graph if any m-tuple of its vertices has exactly k common neighbours. Thus a $(2,1)$-graph is exactly a friendship graph. In fact, it so happens that only the $(2,k)$-graphs are of interest since Sudolsky (1978) proved that for $m \geq 3$ and $k \geq 1$ the only (m,k)-graph with more than m vertices is the complete graph K^{m+k}.

The extension of the concept of geodetic graphs to oriented and partially-oriented graphs has been studied by Bosák, Kotzig and Znám (1968), Bosák (1971), Plesnik and Znám (1974), Plesnik (1977) and Gassman, Entringer, Gilbert, Lonz and Vucenic (1975). Entringer conjectured that every graph has an orientation in which for any two vertices x,y there is at most one shortest directed path from x to y, but this was disproved by Bollobás (1981b).

A problem closely related to the function $n(D,\Delta)$ was proposed and studied by Erdős and Rényi (1962). Let D, Δ and n be natural numbers, $D < n$ and $\Delta < n$. Suppose there is a graph of order n with maximal degree at most Δ and diameter at most D. Denote by $e_D(n,\Delta)$ the minimal number of edges of such a graph. What can one say about $e_D(n,\Delta)$? The main results so far concern the case $D \leq 3$: Erdős and Rényi (1962) Erdős, Rényi and Sós (1966) and Bollobás (1971) ; these and related results are presented in Bollobás (1978, Ch.IV). Recently, the case $D = 2$ has been settled more or less completely by Pach and Surányi (1981). They defined a function $g(c)$ on $(0,1)$ and proved that $\lim_{n\to\infty} e_2(n,\lfloor cn \rfloor)/n = g(c)$ for every c, $0 < c < 1$, except for $c = c_1, c_2, \ldots,$ where (c_k) is a sequence tending to 0.

A graph is said to be diameter edge-critical if the omission of any edge increases the diameter. Similarly a graph is diameter vertex critical if the omission of any vertex increases the diameter. These graphs have been studied by Glivjak (1968, 1975a,b, 1976), Glivjak, Kys and Plesnik (1969a and 1969b), Glivjak and Plesnik (1969, 1970, 1971) and Plesnik (1975a, 1975b). Perhaps the most intriguing conjecture in this area is due to Plesnik (1975a) and Murty and Simon : a diameter edge-critical graph of order n and diameter 2 has at most $\lfloor n^2/4 \rfloor$ edges. The complete bipartite graph $K(\lfloor n/2 \rfloor, \lceil n/2 \rceil)$ shows that the bound $\lfloor n^2/4 \rfloor$ is best possible. Caccetta and Häggkvist (1979) came tantalizingly close to proving this conjecture when they showed that the number of edges is at most $\frac{1+\sqrt{5}}{12} n^2 < 0.27n^2$. Caccetta and Häggkvist proposed an extension of this conjecture to graphs of higher diameter. However, their conjecture was refuted by Krishnamoorthy and Nandakumar (1981) who gave another conjecture.

Plesnik (1975a) proved that a diameter edge-critical graph of order n and diameter at least 2 has minimal degree at most $\lfloor n/2 \rfloor$. Once again the graph $K(\lfloor n/2 \rfloor, \lceil n/2 \rceil)$ shows that this bound is best possible. However, for $D \geq 3$ very little is known about $\max_G \delta(G)$, where G runs over the diameter edge-critical graphs of order n and diameter D.

Akiyama, Ando and Avis (1981) studied equi-eccentric graphs of diameter 2. A graph G is said to be equi-eccentric if $\max_{y \in G} d(x,y) = \operatorname{diam} G$ for every vertex x. It is not known at least how many (resp. at most how many) edges there are in an equi-eccentric graph of diameter D and order n.

We conclude with a remark on orientations of graphs. Let $f(D)$ be the smallest integer such that every 2-edge connected graph of diameter D has an orientation of directed diameter at most $f(D)$. Chvátal and Thomassen (1978) proved that $\frac{1}{2} D^2 + D \leq f(D) \leq 2D^2 + D$. They also showed that it is NP-complete to decide if a graph has an orientation of directed diameter 2. Katona and Szemerédi (1967) showed that a graph of order n with fewer that $\frac{n}{2} \log_2 \frac{n}{2}$ edges has no such orientation.

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