A Broadcasting protocol in Line Digraphs *

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Abstract

We propose broadcasting algorithms for line digraphs in the telephone model. The new protocols use a broadcasting protocol for a graph G to obtain a broadcasting protocol for the graph L^kG, the graph obtained by applying k times, the line digraph operation to G. As a consequence improved bounds for the broadcasting time in De Bruijn, Kautz and Wrapped Butterfly digraphs are obtained.

1 Introduction

Considerable attention has been recently devoted to the dissemination of information in parallel computing, since the time needed to disseminate in-

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formation among the processors in a network often determines the running time of the whole algorithm. Broadcasting is the process in which a node of the network (the originator) sends one piece of information to all members of the network by placing calls over the communication lines of the network. This is to be completed as quickly as possible subjected to the constraints of the considered model. This problem has been widely studied both for its theoretical and practical interest for different network configurations and under different models (see [9, 12] for recent surveys or the book [14]).

In this paper we are mainly concerned with the 1–port telephone model, that is, each call involves only two vertices of the network and requires one unit of time (round); moreover we require that a vertex can participate in at most one call per unit of time by either sending or receiving a call and we restrict links to be one-way, that is, information will always flow in the same direction.

Following the literature we model the network topology by a digraph $G = (X, U)$; a vertex $x \in X$ corresponds to a node in the network, and an arc $(x, y) \in U$, corresponds to a link in which x is the sender and y is the receiver. $b(G, x)$ represents the minimum number of time units (number of rounds) necessary to complete broadcasting from vertex x; $b(G)$ is defined as the maximum of $b(G, x)$ taken over all vertices x of G. It is well known that for any digraph G with n vertices $b(G) \geq \lceil \log_2 n \rceil$ and that the problem of determining $b(G)$ is a NP–hard problem. Values of $b(G)$ are known for many usual interconnection networks; however for Butterfly digraphs and related networks the order is still unknown.

In this paper we propose a general methodology for designing broadcasting algorithms in line digraphs and iterated line digraphs. In Section 3.1 we give a simple protocol for broadcasting in LG (the graph obtained by applying the line digraph operation to the graph G) using a broadcast protocol in G where G is a regular digraph with degree d; if broadcast in G can be performed in time t, then the algorithm performs broadcasting in LG in time $t + \lceil \log_2 d \rceil + 1$. In section 3.2 we show that if G satisfies an additional property, that we call A, then there exists a broadcasting protocol in LG that runs in time $t + \lfloor \log_2 d \rfloor + 1$.

By iterating the algorithms of Sections 3.1 and 3.2 we obtain protocols for iterated line digraphs ($L(L^{n-1}G) = L^n G$). In Section 4.1 we show an improved algorithm for broadcasting in L^2G using a broadcasting protocol in G. For some values of d, the direct derivation of the protocol for L^2G

has running time $t + 2\lfloor \log_2 d \rfloor + 1$ if G satisfies property A. In Section 5 we extend the results to $L^k G$. For example we obtain for $d = 2^\alpha$, $\alpha > 1$, a broadcasting algorithm in $L^3 G$ running in time $t + 3\log_2 d + 1$. Finally, in Section 6, we sketch the application of these ideas to the p–port model.

The iteration of the line digraph operation is a good method to obtain large digraphs with fixed degree and diameter. Besides, many other good properties are observed in such digraphs when used as models of communication networks ([3, 11]). In fact some of the best known families of digraphs are indeed line digraphs: De Bruijn, Kautz,Wrapped Butterfly, among others.

Property A is satisfied in particular if G is itself a line digraph. So we obtain new and better protocols for De Bruijn, Kautz and Butterfly networks which improve the best current known bounds both for digraphs ([3], [11]) and graphs ([2]).

2 Notation and Preliminaries

We refer to [7, 14] for basic definitions concerning graphs and digraphs and we recall basic definitions and properties of line digraphs (see also [8, 10]).

Given a directed graph $G = (X, U)$, if (x, y) is an arc, then x is *adjacent to* y and y is *adjacent from* x. We also say that arc (x, y) is *incoming to* y and *outgoing from* x, and that arc $u = (x, y)$ is adjacent to arc $v = (y, z)$. We denote by $d^+(x)$ ($d^-(x)$) the number of vertices adjacent from x (adjacent to x); a digraph is said to be *d-regular* if $d^+(x) = d^-(x) = d$ for any vertex x.

A *dipath* of *length* h from a vertex x to a vertex y is a sequence of vertices $x = x_0, x_1, \ldots, x_{h-1}, x_h = y$ where (x_i, x_{i+1}) is an arc of G. A digraph is *strongly connected* if, for any couple of vertices x, y, there exists a dipath from x to y. The length of a shortest dipath from x to y is the *distance* from x to y. Its maximum value over all couples of vertices is the *diameter* of the digraph denoted $D(G)$.

Given a digraph $G = (X, U)$ the *line digraph* operation allows to define a new digraph LG whose vertex set is in one to one correspondence with the set of arcs of G. The vertex u of LG representing the arc $u = (x, y)$ is adjacent to the vertex representing the arc v if and only if $v = (y, z)$, that is, arcs of LG represent dipaths of length 2 of G. It can be easily shown that if G is d–regular with n vertices, then LG is d–regular with dn vertices; furthermore if G is a strongly connected digraph different from a directed cycle, then the diameter of LG is the diameter of G plus one. We denote by L^kG the graph $L(L^{k-1}G)$.

Let G be a digraph with n vertices and LG the corresponding line digraph. The arcs of LG can be partitioned into n complete bipartite digraphs isomorphic to $\overrightarrow{K}_{d,d}$ that are in one to one correspondence with vertices of G. If x is a vertex of a digraph G, then we denote by B_x the corresponding bipartite digraph in LG (one subset consist of the vertices repressenting incoming arcs to x, and the other of those repressenting outgoing arcs from x). Note also that each vertex of LG belongs to exactly to two of such bipartite digraphs: in one case with in-degree 0 and in the other with out-degree 0.

An *arc–labeling* for a digraph G is a labeling of its arcs such that any two arcs incident to the same vertex have different labels and any two arcs incident from the same vertex have also different labels. If G is d-regular then it is always possible to obtain such an arc–labeling with d labels. Since we always consider d-regular digraphs we simply say an arc–labeling instead of an arc–labeling with d labels. Given a digraph with an arc–labeling, we identify the labels with the elements of $\mathbf{Z}_d$ and we refer to the arcs incoming to or outgoing from the same vertex by their labels.

The De Bruijn digraph $B(d,D)$ is a d-regular digraph with diameter D defined as follows: vertices are the strings of length D, $z_1z_2\dots z_D$, where $z_i \in \mathbf{Z}_d$, $i=1,2,\dots,D$, and vertex $z_1z_2\dots z_D$ is adjacent to vertices $z_2\dots z_D\alpha$, $\alpha \in \mathbf{Z}_d$. In [8] it is shown that $B(d,D) = L^{D-1}K_d^+$, where K_d^+ denotes the complete symmetric digraph on d vertices with an additional loop in each vertex. It is also easy to see that $K_d^+ = L(dK_1^+)$, where dK_1^+ is a digraph consisting of a single vertex with d loops. Thus we can also write $B(d,D) = L^D dK_1^+$.

The Kautz digraph $K(d,D)$ is defined analogously: vertices are all strings of length D, $z_1z_2\dots z_D$, $z_i \in \mathbf{Z}_{d+1}$, $i=1,2,\dots,D$, such that two consecutive symbols cannot be equal. Vertex $z_1z_2\dots z_D$ is adjacent to vertices $z_2\dots z_D\alpha$, $\alpha \in \mathbf{Z}_{d+1}$, $\alpha \neq z_D$. The digraph $K(d,D)$ is d-regular and has diameter D; moreover $K(d,D) = L^{D-1}K_{d+1}^*$, where K_{d+1}^* denotes the complete symmetric digraph on $d+1$ vertices [8].

The Directed Wrapped Butterfly $\overrightarrow{WBF}(d,n)$ has as vertex set the couples (x,l) where x is a string of length n $(x_{n-1}x_{n-2}\dots x_0)$, $x_i \in \mathbf{Z}_d$, $i = 0,1,\dots,n-1$, and $l \in \mathbf{Z}_n$. Vertex $(x_{n-1}x_{n-2}\dots x_0, l)$ is adjacent to vertices $(x_{n-1}\dots x_{l+1}\alpha x_{l-1}\dots x_0, l+1)$, $\alpha \in \mathbf{Z}_d$. In [4] it is shown that $\overrightarrow{WBF}(d,n) =$

$L^n dC_n$, where dC_n denotes the graph obtained from the directed cycle with n vertices by replacing each arc with d parallel arcs.

3 Broadcasting in line digraphs

In this section we will show that, given a broadcasting algorithm in a digraph G running in time t, it is possible to construct a broadcasting protocol in LG that runs in time $t + \lceil \log_2 d \rceil + 1$. The running time is $t + \lfloor \log_2 d \rfloor + 1$ if G satisfies some additional properties.

3.1 A broadcasting protocol in a line digraph

A broadcasting algorithm in a digraph $G = (X, U)$ with originator r can be described via the broadcast tree $T(r)$ with root r and vertex set X. $T(r)$ contains the arcs through which the information has been broadcasted; since each vertex in $T(r)$ has in-degree 1, then $T(r)$ is an arborescence.

Any broadcasting algorithm P with originator r induces a partial broadcasting in LG with originator any vertex in LG representing an arc (r', r) , which informs those vertices of LG corresponding to the arcs of the broadcast tree of G. If P is a broadcasting protocol running in time t in G, then at time t, for any x, there is at least one vertex $u = (\cdot, x)$ in LG, where $\cdot$ stands for an undetermined symbol, that has been informed (namely vertex corresponding to (r', r) and the vertices that correspond to the arcs of the broadcast tree $T(r)$).

Let us choose an arc-labeling for the digraph G, that is a labeling of the vertices of LG. During round $t + 1$, for any x in G, the informed vertex $u = (\cdot, x)$ in LG sends the information, denoted by `inf`, to vertex $(x, \cdot)$ of label 1. Since the labeling of the arcs is an arc–labeling in G, for all x', vertices $(\cdot, x')$ of label 1 have been informed. At round $t+2$, all vertices $(\cdot, x)$ of label 1 send `inf` to vertices $(x, \cdot)$ of label 2. Thus at the end of this round all vertices of LG with labels 1 or 2 will know `inf`; so they will be able to send `inf` to vertices of LG of labels 3 and 4 during round $t + 3$. Following this pattern, we claim that at the end of round $t + k$, vertices of LG with label i, $1 \leq i \leq 2^{k-1}$, will know `inf`. The proof is by induction on k; it is true for $k = 1$ and $k = 2$. Suppose it is true at time $t + k$, then at time $t + k + 1$ for any x the vertex $(\cdot, x)$ of LG of label i informs the vertex $(x, \cdot)$

of label $i + 2^{k-1}$. In this way all vertices with label i, $2^{k-1} + 1 \leq i \leq 2^k$, get the information. Since G is d-regular at round $t + \lceil \log_2 d \rceil + 1$, the protocol is completed. So we can state the following theorem.

Theorem 1 *Given a regular digraph G with degree d, such that broadcast in G can be completed in t rounds, then there exists a broadcast protocol in LG that runs in time $t + \lceil \log_2 d \rceil + 1$.*

3.2 An improved protocol in a line digraph

In the previous protocol we have not used the facts that `inf` arrived at vertex x of G on some arc of a specific label and that `inf` might arrive before time t. For example, suppose that it arrives to an arc $(\cdot, x)$ not of label 1, then before round $t + 2$ the information `inf` has reached two vertices of LG. Therefore, at round $t + 2$, `inf` could have been sent on two vertices of LG instead of only one; analogously if `inf` reaches x at time $t - h$ in the successive h rounds `inf` can be sent on h new arcs.

In the sequel we obtain an improved broadcasting protocol in LG if G admits a suitable arc–labeling. To this aim let us consider the broadcasting tree $T(r)$ given by a broadcasting protocol P with originator vertex r and let us label its arcs in the following way: for each vertex, the last arc used to send `inf` is labeled 0, the penultimate one is labeled -1 (where we perform addition modulo d), the preceding one is labeled with -2 and so on. We will call such a labeled tree a *labeled broadcasting tree*. If it is possible to make an arc–labeling of the digraph G in which the labels of the arcs of $T(r)$ are the ones in the labeled broadcasting tree, we say that this arc–labeling is *consistent* with the labeling of the broadcasting tree. We say that a vertex of G is *of kind* k (in protocol P) if it has received `inf` though an arc of label $-k$ (modulo d).

We note the following immediate lemma.

Lemma 1 *If a vertex is of kind k, then it has been informed in protocol P at time $t - k$ or before.*

Definition 1 *A d–regular digraph G satisfies property A if there exists a broadcasting protocol such that, for any vertex r, there exists an arc–labeling of G that is consistent with the labeling of the broadcasting tree $T(r)$.*

Lemma 2 *If G is a line digraph then it satisfies property A.*

Proof: If G is a line digraph, due to the partition of the arcs into complete bipartite digraphs, it is possible to label both all its arcs by arc–labeling independently the bipartite subdigraphs.

Let $T(r)$ be the broadcasting tree associated to the broadcasting protocol P in G and for a given bipartite digraph B_x of the decomposition of G, let V_1 and V_2 be the partition of the vertices of B_x. For each vertex y of G there exists exactly one arc entering in y that belongs to $T(r)$ (the arc through which `inf` has arrived). Let $\delta_T(x)$ be the out-degree of x in T (that is the number of vertices which have been informed by x). Now we rank the vertices of V_1 and V_2 as follows.

Choose a vertex x in V_1, with $\delta_T(x) = \delta \geq 1$; label this vertex x_0 and label its out-neighbors in V_2 with $y_0, y_1, \ldots, y_{\delta-1}$ in the order they have been informed, i.e. according to their kind (the out-neighbor of kind 0 is labeled y_0, the one of kind 1, y_1, and so on. In general suppose that at some step of the algorithm we have used all labels till y_i for vertices in V_2.

If $i < d-1$, then choose a vertex x in X with out-degree $\delta_T(x) = \delta_i \geq 1$; label this vertex x_{i+1} and its out-neighbors y_{i+j}, $1 \leq j \leq \delta_i$, according to their kind. If $i = d-1$, then there might be vertices in V_1 that have not been labeled; we label them in any order using the remaining labels. Note that the labeling is not unique (in fact it depends on the order in which vertices in V_1 have been chosen).

Let us now consider the following arc–labeling of G: arc (x_i, y_j) has label $(i-j) \bmod d$. It is immediate to see that this arc–labeling is consistent with the labeling of $T(r)$; therefore the lemma is proved. □

Lemma 3 *Given a d–regular digraph G that satisfies property A and a broadcasting protocol P running in G in time t, then there exists a protocol in LG such that at time $t+h$*

i) for a vertex of kind k, $-k \in \{1, 2, \ldots, 2^h - 1\}$, all its outgoing arcs have been informed;

ii) for a vertex of kind k, $-k \notin \{1, 2, \ldots, 2^h - 1\}$, all the incoming arcs of labels j, $1 \leq j \leq 2^h - 1$, have been informed.

Proof: The proof is by induction on h and uses the following protocol Q in LG based on protocol P. Until time t, Q behaves like P but with the

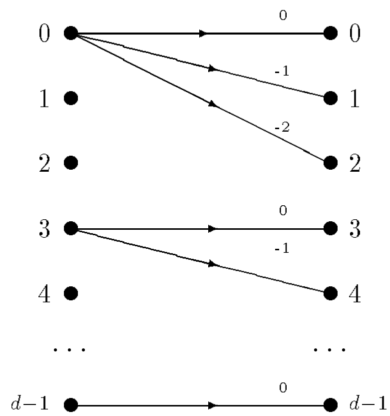

Figure 1: Labeling of the vertices of a bipartite digraph.
Drawn arcs are those in the broadcasting tree T of G

following modification: Let x be a vertex of G of kind k, that, by lemma 1, has received inf at time less than or equal to $t-k$. If we consider protocol P two possible cases arise: either x has sent inf on at least k arcs or x has sent inf to k' arcs, $k'<k$, and it has been idle for at least $k-k'$ rounds before t. In the modified protocol that we use until time t we will use these $k-k'$ rounds to send the information to $k-k'$ arcs in such a way that, at time t, x has sent the information through arcs $0,-1,\ldots,-k+1$.

The specification of protocol Q is completed as follows. At round $t+1$ each vertex $(\cdot,x)$ of LG that corresponds to an arc of the broadcasting tree sends inf to its neighbor of label 1. At the end of this round, if $x\in V(G)$ is of kind $d-1$ then $(\cdot,x)$ has informed all its neighbors in LG (in rounds before t, the vertices corresponding to arcs of label $-j$, $j=0,1,\ldots,d-2$, and, at round $t+1$, arc of label $1=-(d-1)$). If x is of kind k different from $d-1$ then there exist at least two vertices of LG, say (y_1,x) and (y_{-k},x) that have received inf (these vertices correspond to arcs with labels 1 and $-k$); in this case these two vertices can send inf to the two vertices $(x,\cdot)$ with labels 2 and 3 during round $t+2$; note that in the previous protocol at round $t+2$, inf was sent to only one vertex $(x,\cdot)$.

Now we can start the induction. The lemma is satisfied for $h=0$ and $h=1$. Suppose the lemma is true for h_0 and let us consider vertex x of kind k at round $t+h_0+1$. If $-k \in \{1,2,\ldots,2^{h_0}-1\}$, then the lemma is true by induction. If $-k \notin \{1,2,\ldots,2^{h_0}-1\}$, then x has received, at the end of round $t+h_0$ `inf`, on at least 2^{h_0} arcs: arc of label $-k$ (before time $t-k$) and arcs of label j, $j=1,2,\ldots,2^{h_0}-1$ (at rounds $t,t+1,\ldots,t+h_0$). Therefore x can send `inf` on 2^{h_0} arcs of label j, $j=2^{h_0},\ldots,2^{h_0+1}-1$. So if $-k \in \{1,2,\ldots,2^{(h_0+1)}-1\}$ all outgoing arcs of x have been informed; otherwise the set of informed arcs is the claimed one. □

The proof of the following theorem is an immediate consequence of the lemma and of the fact that $t+\lceil\log_2(d+1)\rceil = t+\lfloor\log_2(d)\rfloor+1$.

Theorem 2 *Given a d–regular digraph G that satisfies property A and such that broadcast can be performed in time t, then there exists a protocol in LG that runs in time $t+\lfloor\log_2(d)\rfloor+1$.*

4 Broadcasting in L^2G

Let G be a d-regular digraph with $d=2^\alpha(1+\beta)$, $0\leq\beta<1$, that satisfies property A, and let us assume that there exists a broadcasting algorithm running in time t. By applying twice the protocol for broadcasting in a line digraph (Theorem 2), we obtain a broadcasting algorithm in L^2G running in time $t+2\alpha+2$. The aim of this section is to design a broadcasting protocol in L^2G directly from the protocol in G running in time $t+2\alpha+1$; this is possible if β is not too large.

Recall that, given a graph $G=(X,U)$, the vertex set of L^2G is the set of all possible dipaths of length 2 in G, and that broadcasting in L^2G is equivalent to broadcast the information through all paths of length 2 in G, or equivalently, every arc of G informs every other adjacent arc in G. In the proofs of this section we will use this latter approach.

Observe that, in the protocol for LG described in the previous section, most of the vertices are idle during the last unit of time. This happens because all their neighbors have been already informed and it should not make sense to send again the information. Nevertheless, if we iterate the process in order to obtain a broadcasting algorithm in L^2G, each arc must send `inf` to any other adjacent arc and, therefore, the fact described before represents a waste of time.

In order to obtain the claimed result we will use the protocol in $L(G)$ up to time $t+\alpha$ when the information has passed through almost every arc in G (let us recall that the broadcasting time is $t+\alpha+1$. By specifying which arc informs any other arc we are able to complete the broadcasting in $L^2(G)$ in $\alpha+1$ additional rounds.

4.1 The design of a protocol from small values of d

Lemma 4 *Given a d–regular graph G that satisfies property A and a broadcast protocol in G that runs in time t, then there exists a broadcast protocol in L^2G such that, at time $t+\alpha$, $\alpha = \lfloor \log_2 d \rfloor$, the information is arrived to:*

- *vertices that correspond to pairs of arcs of G with labels $(i, i+2^j)$, where $1 \le i \le 2^{\alpha-1}-1$ and $\lfloor \log_2 i \rfloor < j \le \alpha - 1$.*
- *vertices that correspond to a not completely specified pair of arcs with labels $(\cdot, 2^j)$, where $0 \le j \le \alpha - 1$ and $\cdot$ denotes the label of some arc (depending on the broadcast algorithm in G).*

Proof: We use protocol Q described in Lemma 3 in the following precise way. We know that, for any vertex $x \in G$, at time $t+h$, the information `inf` has arrived on all the arcs i, $1 \le i \le 2^h - 1$, and on arc $-k$ (if x is a vertex of kind k). At time $t+h+1$, we impose that, if it has not already been done, arc i sends the information to $i+2^h$ and arc $-k$ to 2^h.

Note that if $-k \in \{1, 2, \ldots, 2^h - 1\}$ there is no conflict as arc $-k$ has already sent `inf` to 2^h and $2^h + k'$ with $k' = -k \bmod d$, $k' \ge 0$. So the lemma follows by induction and considering the case $h = \alpha = \lfloor \log_2 d \rfloor$. □

Let us call the above protocol a *weak protocol*. Furthermore, if we suppose that the undetermined $\cdot$ is 0 we say that the protocol is a *strong protocol*. In this case arc labeled 0 has informed arcs labeled 2^j, $0 \le j \le \alpha - 1$, or equivalently all the pairs of arcs $(0, 2^j)$ have been informed.

Let D_α be the set of values of d, with $\lfloor \log_2 d \rfloor = \alpha$, such that any weak protocol obtained at time $t+\alpha$ can be completed in a full protocol for L^2G in $\alpha+1$ more steps. Since we can always find a weak protocol in time $t+\alpha$ for any d in D_α, it follows that if $d \in D_\alpha$ then there exists a protocol in L^2G running in time $t + 2\alpha + 1$.

Analogously, let E_α be the set of values of d with $\lfloor \log_2 d \rfloor = \alpha$, such that any strong protocol obtained at time $t + \alpha$ can be completed as in a full protocol for L^2G in $\alpha + 1$ more steps. Notice that $D_\alpha \subseteq E_\alpha$.

Lemma 5 *$4 \in D_2$ and $5 \in E_2$.*

Proof: In the following tables we show an algorithm for completing the protocol in L^2G. In the tables columns represent successive rounds and rows represent labels of arcs incident to any vertex of G. At each round it is shown what must be done by each incoming arc. For instance in Table 4.1(a) at time $t + 3$ all arcs inform arc labelled 0; thus arc labelled 0 will have been informed by three different arcs and at time $t + 4$ it can inform 3 different arcs (with labels 1, 2, 3).

Recall that, by Lemma 4, at time less than or equal to $t+2$ arcs labelled $1, 2$ and 3 have been informed and moreover the incoming arc labelled 1 has informed the outgoing arc with label 3.

	$\leq t+2$	$t+3$	$t+4$	$t+5$
0			1,2,3	0
1	3	0	1	2
2		0	2	1,3
3		0	3	1,2

a) table for $4 \in D_2$

	$\leq t+2$	$t+3$	$t+4$	$t+5$
0	1,2	4	3	0
1	3	4	2	0,1
2		2	1,4	0,3
3		3	1,4	0,2
4			2,3	0,1,4

b) table for $5 \in E_2$

Table 1: Proof of Lemma 5

□

An analogous lemma can be stablished for larger values of d. The proof is straightforward by looking at Table 2.

Lemma 6 *$10 \in D_3$ and $11 \in E_3$.*

	$\leq t+3$	$t+4$	$t+5$	$t+6$	$t+7$
0			8,9	0,1,2,4	3,5,6,7
1	3,5	8	6	0	1,2,4,7,9
2	6	9	5	0	1,2,3,4,7,8
3	7	0	3	1,2,4	5,6,8,9
4		8	6	0	1,2,3,4,5,7,9
5		9	5	0,2,3,4	1,6,7,8
6		0	3	1,2,4,7	5,6,8,9
7		7	5,6	0,4	1,2,3,8,9
8			0,9	1,2,4,7	3,5,6,8
9			0,8	1,2,4,7	3,5,6,9

a) table for $10 \in D_3$

	$\leq t+3$	$t+4$	$t+5$	$t+6$	$t+7$
0	1,2,4	8	9	10	0,3,5,6,7
1	3,5	9	10	0,8	1,2,4,6,7
2	6	10	8	0,9	1,2,3,4,5,7
3	7	8	10	0,1,2	3,4,5,6,9
4		9	8	0,1,2	3,4,5,6,7,10
5		10	9	0,1,2	3,4,5,6,7,8
6		7	3,1	0,4,5	2,6,8,9,10
7		6	4,2	0,3,5	1,7,8,9,10
8			3,5	2,4,6,7	0,1,8,9,10
9			4,6	1,3,5,7	0,2,8,9,10
10			5,7	1,2,4,6	0,3,8,9,10

b) table for $11 \in E_3$

Table 2: Proof of Lemma 6

Lemma 7 *1. If $d \in D_\alpha$ then $2d \in D_{\alpha+1}$.*

2. If $d \in D_\alpha$ and $d+1 \in E_\alpha$ then $2d+1 \in D_{\alpha+1}$.

3. If $d \in E_\alpha$ then $2d \in E_{\alpha+1}$.

Proof: Let $d \in D_\alpha$, thus $\alpha = \lfloor \log_2 d \rfloor$ and $\alpha+1 = \lfloor \log_2 2d \rfloor = \lfloor \log_2 (2d+1) \rfloor$.

1) Let us recall that there exists a weak protocol for G with degree $2d$ as it is explained in the protocol for LG. Let us consider the subdigraphs G_e and G_o induced by the even and odd arcs of G respectively. The protocol for G_e (where we assume $i' = 2i$) is exactly a weak protocol for G_e (with degree d) completed in time $t + \alpha + 1$. Indeed the information is arrived on all the arcs labelled $2i$, $2 \leq 2i \leq 2^{\alpha+1} - 1$, that is, $1 \leq i' \leq 2^\alpha - 1$, and arc labelled i' has informed arcs labelled $i' + 2^j$ with $\lfloor \log_2 i' \rfloor + 1 < j \leq \alpha - 1$. Similarly the protocol induced in G_o (where we identify $2i+1$ with i'') is a strong protocol for G_o.

The protocol in G_e and in G_o can be completed in $\alpha + 1$ steps more since $d \in D_\alpha$. Thus at time $t + 2\alpha + 2$ all vertices in L^2G of the form $(2i, 2j)$ and $(2i+1, 2j+1)$, $0 \leq i, j \leq d-1$ know the information. Using an extra step $(2i, 2j)$ can inform $(2j, 2i+1)$ and $(2i+1, 2j+1)$ can inform $(2j+1, 2i)$ completing the protocol in time $2(\alpha+1)+1$. Thus $2d \in D_{\alpha+1}$.

2) Let us consider subdigraphs G_e and G_o as in the case before, but relabelling arc $2d$ as $2d+1$, that is, considering $2d$ as if it were an odd number, and thus belonging to G_o. The protocols induced in G_e and G_o are a weak protocol and a strong protocol respectively. Since $d \in D_\alpha$ and $d+1 \in E_\alpha$ it is possible for the vertices of the form $(2i, 2j)$, $0 \leq i, j \leq d-1$, and $(2i+1, 2j+1)$, $0 \leq i, j \leq d$ to be informed at time $2\alpha + 2$. Nevertheless let us change a bit the protocol in G_o in order that for each j, $0 \leq j \leq d-1$, one of the informed vertices during last step, say $(2i+1, 2j+1)$ is replaced by the vertex $(2i+1, 2j)$, that is arcs with labels $2i+1$ informs arcs labelled $2j$ instead of informing the ones labelled $2j+1$.

Notice that at time $2\alpha + 2$ arcs in G_e have been informed by $d+1$ other arcs, that is $(2i, 2j)$ know the information for all $i, j \leq d-1$ as well as $(2k+1, 2j)$ for some k. Thus $(2j, 2i+1)$ can get the information in the next step for all $i \leq d, j \leq d-1$.

Moreover, at time $2\alpha + 2$, all arcs in G_o have been informed at least d times, that is, for every $j \leq d$ there are at least d vertices of the form $(2i+1, 2j+1)$ which are informed. Therefore they can inform the $d-1$ remaining vertices of the form $(2j+1, 2i)$, $0 \leq i \leq d-1$ which were not informed at time $2\alpha + 2$ and the vertex $(2i+1, 2j+1)$ which was neither informed at that time. Hence the protocol can be completed in $2(\alpha+1)+1$ steps, and so $2d+1 \in D_{\alpha+1}$.

3) This case is analogous to Case 1 and it is omitted. □

Lemmas 5, 6 and 7 imply that given d, $d = 2^\alpha(1+\beta)$, and $\alpha \geq 3$, then $d \in D_\alpha$ if $\beta < 3/8$ and $d \in E_\alpha$ if $\beta \leq 3/8$.

For $\alpha \leq 2$ we use a more accurate analysis to obtain a broadcast protocol for L^2G that runs in time $t + 2\alpha + 1$, when $d = 2, 5$.

Lemma 8 *Given a d–regular line digraph with $d = 2$ or $d = 5$ there exists a protocol in L^2G running in time $t + 3$, respectively $t + 5$.*

Proof: Let us consider the case $d = 2$. We have to distinguish the kindness of the vertices.

Case a. If the vertex is of kind 0, then table 3.a shows the solution.

Case b. If the vertex is of kind 1 and if the incoming arc of label 0 was an outgoing arc of a vertex of kind 0, then by case a it has received twice the information at time $t + 2$ and so can send it twice at time $t + 3$.

Case c. If the vertex is of kind 1 and if the incoming arc of label 0 was an outgoing arc of a vertex of kind 1, then it has received the information at time less than or equal to t, and thus it can send it at time units $t + 1$ and $t + 2$.

If the incoming arc has label 1, we proceed the same way in both cases b and c as shown in the table 3.

case	in. arc	$\leq t$	$t+1$	$t+2$	$t+3$
a	0		1	0	1
(kind 0)	1			0	1
b	0				0,1
(kind 1)	1	0	1		
c	0		0	1	
(kind 1)	1	0	1		

Table 3: Proof of Lemma 8

An analogous reasoning can be done for the case $d = 5$, looking at the table in the Appendix. □

It is possible to improve the value of $\beta = 3/8$ mentioned above by providing tables analogous to the ones of Lemmas 5 and 6 for larger values of d. For instance, in the Appendix, Tables 9 and 10 show that $44 \in D_5$ and $45 \in E_5$, which lead, by means of the previous results, to the value $\beta = 13/32$.

By computer search we also found that $180 \in D_7$ and $181 \in E_7$ and thus we can state the following Theorem:

Theorem 3 *Given a d–regular digraph G, $d = 2^\alpha(1+\beta)$ and $\beta < 53/128$, that satisfies property A and such that broadcast can be performed in time t then there exists a protocol in L^2G that runs in time $t + 2\lfloor \log_2(d) \rfloor + 1$.*

4.2 Applicability of the method

We now show that if $\beta > \sqrt{2} - 1$ then the proposed method cannot be applied. Since the method uses for the first t rounds a broadcast protocol in G, it follows that only n vertices in L^2G are informed at time t. Since at each subsequent round the number of vertices knowing the information might at most double, it follows that, at time $t + 2\alpha + 1$, the number of vertices being informed is at most $n2^{2\alpha+1}$. On the other hand the number of vertices in L^2G is $n2^{2\alpha}(1+\beta)^2$ that is greater than $n2^{2\alpha+1}$ if $\beta > \sqrt{2} - 1$. Notice that $53/128 = 0.41406..$ is very close to $\sqrt{2} - 1 = 0.41421..$

5 Broadcasting in $L^k(G)$

First of all let us recall that a broadcasting agorithm in L^kG is equivalent to a protocol in G in which a piece of information passes through all possible paths of length k of G. In this section we will only consider digraphs satisfying property A.

The results of the previous section can be extended by showing that, for some values of d, it is possible to construct a protocol in L^kG running in time $t + k\lfloor \log_2 d \rfloor + h$, $h < k$ directly from the broadcasting algorithm in G.

To this end, let us we extend the notion of weak and strong protocols given in Section 4 as follows: let $D_\alpha^{(k,h)}$ be the set of values of d with $\lfloor \log_2 d \rfloor = \alpha$ such that any weak protocol can be completed in a full protocol for L^kG in $(k-1)\alpha + h$ steps more. Analogoulsy we define the set $E_\alpha^{(k,h)}$ in the case of a strong protocol.

The next result extends Lemma 7.

Lemma 9 *If $d \in D_\alpha^{(k,h)}$ and $b(B(2^s, k-1)) = (k-1)s$, then $2^s d \in D_{s+\alpha}^{(k,h)}$.*

Proof: Given a digraph G with degree $2^s d$ and with an arc–labeling given by property A, let us consider the subdigraphs $G_0, \ldots, G_{2^s-1}$, where G_i is the subdigraph induced by arcs of G with label $j \equiv i \pmod{2^s}$. Notice that G_i is d–regular.

It is not dificult to see that the partial protocol induced in G_0 is exactly a weak protocol for G_0 running in time $t + \alpha + s$, while the protocols induced in G_i, $i \neq 0$, are strong protocols for G_i running in the same time. Since $d \in D_\alpha^{k,h}$ (and thus $d \in E_\alpha^{k,h}$), the protocol in which the information passes though all paths of length k in the subdigraphs G_i can be completed in $(k-1)\alpha + h$ rounds. Hence at time $t + k\alpha + s + h$ all paths of G having the sequence of labels $x_0^i x_1^i \cdots x_{k-1}^i$ with $x_j^i \equiv i \pmod{2^s}$ have been informed.

During the next rounds, the rest of the paths of G must be informed. This last part of the protocol is equivalent to complete a broadcasting algorithm in $B(2^s, k)$ in which all vertices with labels $xx \cdots x$, $x \in \mathbf{Z}_{2^s}$, are informed.

It is shown in [15] the existence of a vertex transitive homomorphism from $B(d, D)$ to $B(d, D-1)$ mapping vertices with labels $xx \cdots x$, $x \in \mathbf{Z}_d$ onto vertex $0 \cdots 0$.

The reader can prove that this last part of the protocol is equivalent to a broadcasting algorithm in $B(2^s, k-1)$, being $0 \cdots 0$ the originator. If $b(B(2^s, k-1)) = (k-1)s$, the total runing time of our protocol will be $t + k(s + \alpha) + h$, and thus $2^s d \in D_{s+\alpha}^{(k,h)}$ (see [13] for more details). □

Lemma 10 *The following relations hold:* $4 \in D_2^{3,1}$, $8 \in D_3^{3,1}$ *and* $16 \in D_4^{3,1}$.

Proof: We will show the case $4 \in D_2^{3,1}$. In order to prove the other relations we used the same pattern with the aid of a computer.

Designing a protocol on $L^3 G$ is equivalent to make a piece of information pass through all possible paths of lengh 3 in G, or what is equivalent, that information arrived to any pair of arcs is sent to all the arcs adjacent from it. The initial situation is that at time t information arrived to all vertices in G.

Let us start with a weak protocol on $L^2 G$, and observe that, when an arc with label i informs an arc with label j, there is a path of length 2 with labels ij to which the information has arrived. Therefore this pair can start informing the adjacent arcs from it.

In any case, if arc labelled i informed by an arc with indetermined label $(\cdot)$ informs an arc labeled j, the pair ij becomes informed by some indetermined arc, i.e. there is a triple $\cdot ij$ informed. Thus, the next time ij is informed, we cannot count two triples ending in ij are informed, but only one.

	$\leq t+2$	$t+3$	$t+4$
0			(0),(3)
1	3	0	1
2		0	2
3		(3)	(2)

Table 4: A protocol equivalent to the one in L^2G

This considerations are taken into account in Tables 4 and 5. In Table 4 it is shown a protocol in which pairs of informed arcs are taken into account, as in the case of L^2G. When an arc i is informed by a determined arc j, the pair ij is considered to be informed, and thus it begins to be considered in Table 5. From that round on, the actions of arc j are written in Table 4 enclosed into brackets, meaning that those actions are considered in Table 5.

In Table 5 it is shown using small numbers the number of times pair of labels ij has been informed (i.e. the number of triples ending in ij), and thus, the number of triples beginning with labels ij that can be informed during the next round. The number appears enclosed into brackets whenever the triple informed is the undetermined $\cdot ij$.

Looking at the tables, the protocol is finished at time $t+7 = t+3\alpha+1$, and hence $4 \in D_2^{3,1}$.

□

Thus, using the fact that $b(B(8,2)) = 3$ (see [11]), we can state the following result:

Theorem 4 *Let G be a line digraph with degree $d = 2^\alpha$, $\alpha > 1$, such that there exists a broadcasting algorithm running in time t, then there exist a broadcasting algorithm in L^3G running in time $t+3\alpha+1$.*

An example for L^3G has been shown, but it is possible to get better results for higher number of iterations with the aid of a computer. Table 6 shows the results we obtained up to date. The notation $A_\alpha^{(k,h)}$ stands for

	$t+3$	$t+4$	$t+5$	$t+6$	$t+7$
00			$_{1}$ 1	$_{1}$ 2	$_{2}$ 0,3
10		$_{(1)}$ 0	$_{(1)}$ 2	$_{(1)}$ 3	$_{1}$ 1
20		$_{(1)}$ 3	$_{(1)}$ 1	$_{(1)}$ 2	$_{(1)}$ 0
30				$_{2}$ 0,1	$_{3}$ 2,3
01				$_{2}$ 0,1	$_{3}$ 2,3
11			$_{(1)}$ 2	$_{(1)}$ 3	$_{2}$ 0,1
21				$_{1}$ 2	$_{3}$ 0,1,3
31			$_{1}$ 2	$_{1}$ 1	$_{3}$ 0,3
02				$_{1}$ 2	$_{3}$ 0,1,3
12				$_{2}$ 1,3	$_{3}$ 0,2
22			$_{(1)}$ 1	$_{(1)}$ 3	$_{2}$ 0,2
32			$_{1}$ 3	$_{2}$ 1,2	$_{2}$ 0
03			$_{1}$ 0	$_{1}$ 1	$_{2}$ 2,3
13	$_{(1)}$ 3	$_{(1)}$ 2	$_{(1)}$ 0	$_{(1)}$ 1	$_{1}$
23				$_{1}$ 0	$_{3}$ 1,2,3
33		$_{1}$ 1	$_{1}$ 2	$_{1}$ 3	$_{2}$ 0

Table 5: Ending the protocol in L^3G

the values of d which are not proved to belong to $D_{\alpha}^{(k,h)}$ but for which it is possible to construct an algorithm in L^kG running in time $t + k\alpha + h$.

A similar reasoning to the one in Section 4.2 leads us to an upper bound for the values of β for which it is possible to obtain this improvement. Namely Given a d-regular line digraph G with $d = 2^{\alpha}(1 + \beta)$, $0 \leq \beta < 1$, with a broadcasting algorithm running in time t, it is possible to construct a broadcasting algorithm in L^kG by inducing the algorithm in G onto L^kG that runs in time $t+k\alpha+h$ only if $1+\beta \leq 2^{\frac{h}{k}}$. Equivalently, we cannot hope to get an algorithm in L^kG running in time $t+k\alpha+h$ if $\frac{h}{k} \geq \log_2(1+\beta) = \log d - \alpha$.

Nevertheless to get this bound for the number of iterations we considered that the number of informed vertices at most double at each round, not taking into account that the degree is finite and that after d rounds some vertices will send infomation to no more neighbors. The problem of bounded degree graphs is considered in [1] leading to a more accurate upper bound

d	belongs to	d	belongs to
2	$A_1^{(2,1)}$	10	D_3
3		11	$E_3,\ D_3^{(3,2)}$
4	$D_2,\ D_2^{(3,1)},\ A_2^{(4,1)}$	12	$D_3^{(3,2)}$
5	$E_2,\ A_2^{(2,1)}$	44	D_5
6	$D_2^{(4,3)}$	45	E_5
7		180	D_7
8	$D_3,\ D_3^{(3,1)},\ D_3^{(4,1)}$	181	E_7
9	$D_3,\ D_3^{(3,1)}$		

Table 6: Results obtained with the aid of a computer

for the possible values of $\frac{h}{k}$ specially for small values of d ([13]).

As an example, if we wanted to do better for $d = 2$ we should obtain an algorithm in L^9G running in time $t + 13$ ($h = 4$), leading to $s_d = 1.444$ (see Section 5.1). Unfortunatelly, we have not been able to get such a good result.

5.1 Broadcasting in some families of digraphs

We consider the following families of iterated line digraphs: De Bruijn $B(d, D)$, Kautz $K(d, D)$ and Directed Wrapped Butterfly $\overrightarrow{WBF}\ (d, n)$. Since dK_1^+, K_{d+1}^* and dC_n (the first cases) satisfy property A (see [13]) then all these families satisfy property A as well, and the following corollary easily follows from the previous sections.

Corollary 1 *Let $d = 2^\alpha(1 + \beta)$ such that $d \in D_\alpha^{(i,h)}$ for $i = 1, \dots, k$. Let $s_d = \alpha + \frac{h}{k}$. Broadcasting in De Bruijn, Kautz and Wrapped Butterfly digraphs can be performed in time*

1. $b(B(d, D)) \leq Ds_d$
2. $b(K(d, D)) \leq Ds_d + 1$
3. $b(\overrightarrow{WBF}\ (d, n)) \leq ns_d + n - 1.$

Table 7 shows the value of s_d for small values of d. In that table we have indicated also the lower bounds ([1]) as well as the previous upper bounds for De Bruijn digraphs ([3, 11]) and graphs ([2]). As one can see the new upper bounds are very close to the lower bounds.

d	2	3	4	5	6	7	8	9	10	11	12
Previous bound	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6	6
Undirected case	1.5	2	2.5	2.8	3	3.28	3.5	3.67	3.8	3.9	4
New result	1.5	2	2.25	2.5	2.75	3	3.25	3.33	3.5	3.66	3.66
Lower bound	1.441	1.80	2.11	2.38	2.62	2.82	3.01	3.17	3.32	3.46	3.59

Table 7: Values of s_d for the digraph $B(d, D)$.

6 Broadcasting protocols under the p–port model

The methods used in this paper are designed for the telephone model, i.e. each call involves only two vertices, one sender and one receiver. It can be however easily extended to the p–port model in which a vertex cand send a piece of information to p neighbors at the same time.

As an illustration we show some results given for the p–port model, ommiting their proofs since they are easy generalizations of the ones used for the 1–port telephone model.

Proposition 1 *Given a regular digraph G with degree d, such that broadcast in G under the p–port model, $p \leq d$ can be completed in t rounds, then there exists a broadcast protocol in LG also under the p–port model that runs in time $t + \lceil \log_{p+1} \frac{d}{p} \rceil + 1$.*

If G satisfy property A, the result can be improved as follows:

Proposition 2 *Given a d–regular digraph G that satisfies property A and such that broadcast can be performed in time t under the p–port model, $p \leq d$, then there exists a protocol in LG under the p–port model that runs in time $t + \lceil \log_{p+1}(d+1) \rceil$.*

Note that the running time in both cases is equal if p is large, as it is shown in the following remark

Remark 1 *Let* $d = (p+1)^\alpha(1+\beta)$*,* $0 \leq \beta < p$*. The running time given in Propositions 1 and 2 is the same whenever* $\beta \leq p-1$*.*

Analogous results can be obtained for inducing protocols in L^kG. For instance, we show an upper bound similar to the one shown in Section 4.2:

Proposition 3 *Given a d-regular line digraph* G *with* $d = (p+1)^\alpha(1+\beta)$*,* $0 \leq \beta < p$*, with a broadcasting algorithm running in time* t *under the* p*–port model, it is possible to construct a broadcasting algorithm in* L^kG *by inducing the algorithm in* G *onto* L^kG *that runs in time* $t + 2\alpha + h$ *only if* $1 + \beta \leq (p+1)^{\frac{h}{k}}$*.*

Let us end this section pointing out that the results given for the p–port model are easily extendable to directed hypergraphs. In [5] it is defined, for directed hypergraphs, a similar operation to the line digraph. It is not difficult to see that the ideas of this paper allow to construct broadcasting protocols for directed line hypergraphs under the usual models. For instance the time given in Proposition 1 also holds for hypergraphs.

7 Conclusions

We have described a constructive method to design good broadcasting protocols in line digraphs and in iterated line digraphs. The broadcasting time given by this method is in most cases better than with any other method known up to date, even considering the underlying graph of the line digraph.

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Appendix: Tables used in the proofs

kind	incoming arc	outgoing from kind	$\leq t$	$t+1$	$t+2$	$t+3$	$t+4$	$t+5$
0	0	*		1	2	4	0	3
	1	*			3	4	0	1,2
	2	*				2	0,1	3,4
	3	*				3	0,4	1,2
	4	*					0,1	2,3,4
1	0	*					0	1,2,3,4
		0						0,1,2,3,4
	1	*			3	4	0	1,2
	2	*				2	0,1	3,4
	3	*				3	0,4	1,2
	4	*	0	1	2	4	3	
2	0	*					0	1,2,3,4
		0						0,1,2,3,4
	1	*			3	4	0	1,2
	2	*				2	0,1	3,4
	3	*	0,4	1	2	3		
	4	*					0,4	1,2,3
3	0	*					0	1,2,3,4
		0						0,1,2,3,4
	1	*			3	4	0	1,2
	2	*	0,4,3	1	2			
	3	*				2	0,1	3,4
	4	*					0,4	1,2,3
4	0	*					0	1,2,3,4
		0						0,1,2,3,4
	1	*	0,4,3,2	1				
	2	*				2	0,4	1,3
	3	*				3	0,1	2,4
	4	*					0,4	1,2,3
		4				4	0	1,2,3

The proof follows using this table as for the case $d = 2$. * stands for all kind of vertices from which the arc comes, except the ones splicity specified, if any.

Table 8: Proof of Lemma 8

arc	$\leq t+5$	$t+6$	$t+7$	$t+8$	$t+9$	$t+10$	$t+11$
0						$_{1}$ 32	$_{43}$ *
1	3,5,9,17	26	15	37	0	16	$_{38}$ *
2	6,10,18	27	12	38	21	0	$_{38}$ *
3	7,11,19	$_{1}$ 34	$_{1}$ 17	$_{3}$ 13-15	$_{6}$ 22-27	$_{12}$ S,28-33	$_{18}$ *
4	12,20	28	41	39	22	0	$_{38}$ *
5	13,21	$_{1}$ 35	$_{1}$ 18	$_{3}$ 14-17	$_{6}$ 23-28	$_{12}$ S,29-34	$_{19}$ *
6	14,22	$_{1}$ 36	$_{1}$ 19	$_{3}$ 15-18	$_{6}$ 24-29	$_{12}$ S,30-35	$_{19}$ *
7	15,23	$_{1}$ 37	$_{1}$ 20	$_{3}$ 17-19	$_{6}$ 25-30	$_{12}$ S,31-36	$_{19}$ *
8	24	33	13	40	23	0	$_{38}$ *
9	25	$_{1}$ 39	$_{1}$ 21	$_{3}$ 18-20	$_{6}$ 26-31	$_{12}$ S,32-37	$_{20}$ *
10	26	$_{1}$ 40	$_{1}$ 22	$_{3}$ 19-21	$_{6}$ 27-32	$_{12}$ S,33-38	$_{20}$ *
11	27	$_{1}$ 41	$_{1}$ 23	$_{3}$ 20-22	$_{6}$ 28-33	$_{12}$ S,34-39	$_{20}$ *
12	28	$_{1}$ 42	$_{1}$ 24	$_{3}$ 21-23	$_{6}$ 29-34	$_{12}$ S,35-40	$_{20}$ *
13	29	$_{1}$ 43	$_{1}$ 25	$_{3}$ 22-24	$_{6}$ 30-35	$_{12}$ S,36-41	$_{20}$ *
14	30	$_{1}$ 29	$_{1}$ 26	$_{3}$ 23-25	$_{6}$ 31-36	$_{12}$ S,37-41,17	$_{20}$ *
15	31	$_{1}$ 30	$_{1}$ 27	$_{3}$ 24-26	$_{6}$ 32-37	$_{12}$ S,38-42,5	$_{20}$ *
16		31	14	33	24	0	$_{39}$ *
17		$_{1}$ 32	$_{1}$ 28	$_{3}$ 25-27	$_{6}$ 33-38	$_{12}$ S,39-3	$_{21}$ *
18		$_{1}$ 31	$_{1}$ 29	$_{3}$ 26-28	$_{6}$ 34-39	$_{13}$ S,40-6	$_{20}$ *
19		$_{1}$ 32	$_{1}$ 30	$_{3}$ 27-29	$_{6}$ 35-40	$_{13}$ S,41-7	$_{20}$ *
20		$_{1}$ 33	$_{1}$ 31	$_{3}$ 28-30	$_{6}$ 36-41	$_{13}$ S,42-9	$_{20}$ *
21		$_{1}$ 34	$_{1}$ 32	$_{3}$ 29-31	$_{6}$ 37-42	$_{13}$ S,43-10	$_{20}$ *
22		$_{1}$ 35	$_{1}$ 33	$_{3}$ 30-32	$_{6}$ 38-43	$_{13}$ S,3-11	$_{20}$ *
23		$_{1}$ 36	$_{1}$ 34	$_{3}$ 31-33	$_{6}$ 39-3	$_{13}$ S,5-12	$_{20}$ *
24		$_{1}$ 37	$_{1}$ 35	$_{3}$ 32-34	$_{6}$ 40-5	$_{13}$ S,6-13	$_{20}$ *
25		$_{1}$ 38	$_{1}$ 36	$_{3}$ 33-35	$_{6}$ 41-6	$_{13}$ S,7-14	$_{20}$ *
26		$_{1}$ 39	$_{2}$ 37,17	$_{3}$ 34-36	$_{6}$ 42-7	$_{13}$ S,9-15	$_{19}$ *
27		$_{1}$ 40	$_{2}$ 38,18	$_{3}$ 35-37	$_{6}$ 43-9	$_{13}$ S,10-17	$_{19}$ *
28		$_{1}$ 41	$_{2}$ 39,19	$_{3}$ 36-38	$_{6}$ 3-10	$_{13}$ S,11-18	$_{19}$ *
29		$_{1}$ 42	$_{2}$ 40,20	$_{3}$ 37-39	$_{6}$ 5-11	$_{13}$ S,12-19	$_{19}$ *
30		$_{1}$ 43	$_{2}$ 41,21	$_{3}$ 38-40	$_{6}$ 6-12	$_{13}$ S,13-20	$_{19}$ *
31		$_{1}$ 38	$_{3}$ 42,22	$_{4}$ 39-41	$_{7}$ 7-13	$_{14}$ S,14-21	$_{20}$ *
32			$_{2}$ 43,23	$_{3}$ 40-42	$_{6}$ 9-14	$_{13}$ S,15-22	$_{20}$ *
33			$_{2}$ 3,24	$_{3}$ 41-43	$_{7}$ 10-15,25	$_{13}$ S,17-23	$_{19}$ *
34			$_{2}$ 5,25	$_{3}$ 42-3	$_{7}$ 11-17,26	$_{13}$ S,18-24	$_{19}$ *
35			$_{2}$ 6,42	$_{3}$ 43-5	$_{7}$ 12-18,27	$_{13}$ S,19-25	$_{19}$ *
36			$_{2}$ 7,43	$_{3}$ 3-6	$_{7}$ 13-19,28	$_{13}$ S,20-25,9	$_{19}$ *
37			$_{2}$ 9,3	$_{3}$ 5-7	$_{7}$ 14-20,29	$_{13}$ S,21-26,10	$_{19}$ *
38			$_{2}$ 10,5	$_{3}$ 6-9	$_{7}$ 15-21,30	$_{13}$ S,22-27,11	$_{19}$ *
39			$_{2}$ 11,6	$_{3}$ 7-10	$_{7}$ 17-22,31	$_{13}$ S,23-28,12	$_{19}$ *
40			$_{2}$ 12,7	$_{3}$ 9-11	$_{7}$ 18-23,32	$_{13}$ S,24-29,13	$_{19}$ *
41			$_{2}$ 13,9	$_{4}$ 10-12,34	$_{7}$ 18-24	$_{13}$ S,25-30,14	$_{19}$ *
42			$_{2}$ 14,10	$_{4}$ 11-13,35	$_{7}$ 19-25	$_{13}$ S,26-31,15	$_{18}$ *
43			$_{2}$ 15,11	$_{4}$ 12-14,36	$_{7}$ 20-26	$_{13}$ S,27-32,17	$_{18}$ *

$S = \{0,1,2,4,8,16\}$. The intervals a-b exclude the elements of S and they must be understood cyclic mod 44. For instance 41-5 stands for the set $\{41,42,43,3,5\}$. * represents the rest of the arcs. The small numbers at the left of each cell stand for the number of times a vertex has received the information from known vertices.

Table 9: $44 \in D_5$.

arc	$\leq t+5$	$t+6$	$t+7$	$t+8$	$t+9$	$t+10$	$t+11$
0	1,2,4,8,16	26	6	10	5	9	$_{44}$ *
1	3,5,9,17	$_{1}$ 27	$_{1}$ 28	$_{2}$ 29,30	$_{5}$ 31-35	$_{11}$ 36-1 ,0	$_{22}$ *
2	6,10,18	$_{1}$ 28	$_{1}$ 29	$_{2}$ 30,31	$_{5}$ 32-36	$_{11}$ 37-2 ,0	$_{22}$ *
3	7,11,19	$_{1}$ 29	$_{1}$ 30	$_{2}$ 31,32	$_{5}$ 33-37	$_{11}$ 38-3 ,0	$_{22}$ *
4	12,20	$_{1}$ 30	$_{1}$ 31	$_{2}$ 32,33	$_{5}$ 34-38	$_{12}$ 39-4 ,0, 5	$_{22}$ *
5	13,21	$_{1}$ 31	$_{1}$ 32	$_{2}$ 33,34	$_{5}$ 35-39	$_{12}$ 40-5 ,0, 8	$_{23}$ *
6	14,22	$_{1}$ 32	$_{1}$ 33	$_{3}$ 34,35,11	$_{5}$ 36-40	$_{11}$ 41-6 ,0	$_{22}$ *
7	15,23	$_{1}$ 33	$_{1}$ 34	$_{3}$ 35,36,12	$_{5}$ 37-41	$_{11}$ 42-7 ,0	$_{22}$ *
8	24	$_{1}$ 34	$_{1}$ 35	$_{3}$ 36,37,13	$_{5}$ 38-42	$_{12}$ 43-8 ,0,16	$_{23}$ *
9	25	$_{1}$ 35	$_{1}$ 36	$_{3}$ 37,38,14	$_{5}$ 39-43	$_{12}$ 44-9 ,0,17	$_{23}$ *
10	26	$_{1}$ 36	$_{1}$ 37	$_{3}$ 38,39,15	$_{6}$ 40-44, 4	$_{11}$ 1-10,0	$_{22}$ *
11	27	$_{1}$ 37	$_{1}$ 38	$_{3}$ 39,40,16	$_{6}$ 41-1 , 8	$_{11}$ 2-11,0	$_{22}$ *
12	28	$_{1}$ 38	$_{1}$ 39	$_{3}$ 40,41,17	$_{6}$ 42-2 , 9	$_{11}$ 3-12,0	$_{22}$ *
13	29	$_{1}$ 39	$_{1}$ 40	$_{3}$ 41,42,18	$_{6}$ 43-3 ,16	$_{11}$ 4-13,0	$_{22}$ *
14	30	$_{1}$ 40	$_{1}$ 41	$_{3}$ 42,43,19	$_{6}$ 44-4 ,17	$_{11}$ 5-14,0	$_{22}$ *
15	31	$_{1}$ 41	$_{1}$ 42	$_{3}$ 43,44,20	$_{6}$ 1-5 ,18	$_{11}$ 6-15,0	$_{22}$ *
16		$_{1}$ 42	$_{1}$ 43	$_{3}$ 44,1 ,21	$_{6}$ 2-6 ,19	$_{12}$ 7-16,0,26	$_{23}$ *
17		$_{1}$ 43	$_{1}$ 44	$_{3}$ 1 ,2 ,22	$_{6}$ 3-7 ,20	$_{12}$ 8-17,0,27	$_{23}$ *
18		$_{1}$ 44	$_{1}$ 1	$_{3}$ 2 ,3 ,23	$_{6}$ 4-8 ,21	$_{12}$ 9-18,0,28	$_{22}$ *
19		$_{1}$ 32	$_{1}$ 2	$_{3}$ 3 ,4 ,24	$_{6}$ 5-9 ,22	$_{12}$ 10-19,0,29	$_{22}$ *
20		$_{1}$ 33	$_{1}$ 3	$_{3}$ 4 ,5 ,25	$_{6}$ 6-10,23	$_{12}$ 11-20,0,30	$_{22}$ *
21		$_{1}$ 34	$_{1}$ 4	$_{3}$ 5 ,6 ,26	$_{6}$ 7-11,24	$_{12}$ 12-21,0,31	$_{22}$ *
22		$_{1}$ 35	$_{1}$ 5	$_{3}$ 6 ,7 ,27	$_{6}$ 8-12,25	$_{12}$ 13-22,0,32	$_{22}$ *
23		$_{1}$ 36	$_{1}$ 6	$_{3}$ 7 ,8 ,28	$_{6}$ 9-13,32	$_{12}$ 14-23,0,33	$_{22}$ *
24		$_{1}$ 37	$_{1}$ 7	$_{3}$ 8 ,9 ,29	$_{6}$ 10-14,33	$_{12}$ 15-24,0,34	$_{22}$ *
25		$_{1}$ 38	$_{1}$ 8	$_{3}$ 9 ,10,30	$_{6}$ 11-15,34	$_{12}$ 16-25,0,35	$_{22}$ *
26		$_{1}$ 39	$_{2}$ 9,7	$_{3}$ 10,11,31	$_{6}$ 12-16,35	$_{11}$ 17-26,0	$_{22}$ *
27		$_{1}$ 40	$_{2}$ 10,8	$_{3}$ 11,12,32	$_{6}$ 13-17,36	$_{11}$ 18-27,0	$_{22}$ *
28		$_{1}$ 41	$_{2}$ 11,9	$_{3}$ 12,13,33	$_{6}$ 14-18,37	$_{11}$ 19-28,0	$_{22}$ *
29		$_{1}$ 42	$_{2}$ 12,10	$_{3}$ 13,14,34	$_{6}$ 15-19,38	$_{11}$ 20-29,0	$_{22}$ *
30		$_{1}$ 43	$_{2}$ 13,11	$_{3}$ 14,15,35	$_{6}$ 16-20,39	$_{11}$ 21-30,0	$_{22}$ *
31		$_{1}$ 44	$_{2}$ 14,12	$_{3}$ 15,16,36	$_{6}$ 17-21,40	$_{11}$ 22-31,0	$_{22}$ *
32			$_{2}$ 15,13	$_{3}$ 16,17,37	$_{6}$ 18-22,41	$_{12}$ 23-32,0, 2	$_{23}$ *
33			$_{2}$ 16,14	$_{3}$ 17,18,38	$_{6}$ 19-23,42	$_{12}$ 24-33,0, 3	$_{23}$ *
34			$_{2}$ 17,15	$_{3}$ 18,19,39	$_{6}$ 20-24,43	$_{12}$ 25-34,0, 6	$_{23}$ *
35			$_{2}$ 18,16	$_{3}$ 19,20,40	$_{6}$ 21-25,44	$_{12}$ 26-35,0, 7	$_{23}$ *
36			$_{2}$ 19,17	$_{3}$ 20,21,41	$_{6}$ 22-26, 1	$_{12}$ 27-36,0,10	$_{22}$ *
37			$_{2}$ 20,18	$_{3}$ 21,22,42	$_{6}$ 23-27, 2	$_{12}$ 28-37,0,11	$_{22}$ *
38			$_{2}$ 21,19	$_{3}$ 22,23,43	$_{6}$ 24-28, 3	$_{12}$ 29-38,0,12	$_{22}$ *
39			$_{2}$ 22,20	$_{3}$ 23,24,44	$_{6}$ 25-29, 4	$_{12}$ 30-39,0,13	$_{22}$ *
40			$_{2}$ 23,21	$_{3}$ 24,25, 1	$_{6}$ 26-30, 5	$_{12}$ 31-40,0,14	$_{22}$ *
41			$_{2}$ 24,22	$_{3}$ 25,26, 2	$_{6}$ 27-31, 6	$_{12}$ 32-41,0,15	$_{22}$ *
42			$_{2}$ 25,23	$_{3}$ 26,27, 3	$_{6}$ 28-32, 7	$_{12}$ 33-42,0,43	$_{22}$ *
43			$_{2}$ 26,24	$_{3}$ 27,28, 4	$_{6}$ 29-33, 8	$_{12}$ 34-43,0,44	$_{23}$ *
44			$_{2}$ 27,25	$_{3}$ 28,29, 5	$_{6}$ 30-34, 9	$_{12}$ 35-44,0, 1	$_{23}$ *

The intervals *a*-*b* exclude the zero. and they must be understood cyclic mod 45. * represents the rest of the arcs. The small numbers at the left of each cell stand for the number of times a vertex has received the information from known vertices.

Table 10: $45 \in E_5$