BALANCED CYCLE AND CIRCUIT DESIGNS: EVEN CASES

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1. Introduction.

A k-cycle (or k-circuit) is a set of k distinct elements, $C = (c_1, c_2, \ldots, c_k)$, such that the two elements c_i, c_{i+1}, $i = 1, 2, \ldots, k$ and $k+1 \equiv 1$ are linked by an edge (or by an arc from c_i to c_{i+1}) while any other two elements of C are unlinked. For undefined terms see [1].

A balanced cycle design, BCD (v,k,λ) (or a balanced circuit design, BCD* (v,k,λ)) is an arrangement of v elements into b k-cycles (or k-circuits) such that each element occurs in the same number, say r, of k-cycles (or k-circuits) and any two distinct elements x and y are linked in exactly λ k-cycles (or linked by an arc from x to y in λ k-circuits and by an arc from y to x in λ k-circuits as well) [7].

A BCD (v,k,λ) (or a BCD*(v,k,λ)) is also called a (v,k,λ) C_k - design (or a (v,k,λ) $\vec{C}_k$ - design) [4], and is essentially an edge- (or arc-) disjoint decomposition of the complete (or complete directed) multigraph with multiplicity λ, λK_v (or λK_v^*) into subgraphs isomorphic to a cycle C_k (or a circuit $\vec{C}_k$) of length k.

It is easy to show that the number of k-cycles in a BCD (v,k,λ) is $b = \frac{\lambda v(v-1)}{2k}$ and the number of k-circuits in a BCD*(v,k,λ) is $b = \frac{\lambda v(v-1)}{k}$. We have (see [4]),

PROPOSITION 1.1. The necessary conditions for the existence of a BCD (v,k,λ) are $v \geq k$, $\lambda v(v-1) \equiv 0 \pmod{2k}$ and $\lambda(v-1) \equiv 0 \pmod 2$.

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PROPOSITION 1.2. The necessary conditions for the existence of a $BCD^*(v,k,\lambda)$ are $v \geq k$ and $\lambda v(v-1) \equiv 0 \pmod{k}$.

In this paper we assume that k is even. The case where k is odd is considered in [5]. Partial results concerning the case k even have been obtained (see the survey in [4]). We are interested in proving:

CONJECTURE I. Let k be even. The necessary conditions of Propositions 1.1 and 1.2 are sufficient except for

(i) $v = k = 4$, λ odd, in Proposition 1.2

and (ii) $v = k = 6$, $\lambda = 1$, in Proposition 1.2.

In fact, we have only been able to reduce the problem to the verification of a finite number of cases (for a given k). We prove also the existence of such designs for some modulo classes and small values of k.

2. General Constructions.

We list several lemmas which are useful in the construction of BCD's and BCD*'s. Their proofs are either obvious, or given elsewhere and hence are omitted here.

LEMMA 2.1. If a $BCD(v,k,\lambda_1)$ and a $BCD(v,k,\lambda_2)$ exist, then there exists a $BCD(v,k,p\lambda_1 + q\lambda_2)$, where p and q are non-negative integers.

Let $\lambda K_{v_1,v_2}$ denote the complete bipartite graph with vertex set $X_1 \cup X_2$ with $|X_i| = v_i$, $i = 1, 2$, $X_1 \cap X_2 = \emptyset$ and any vertex in X_1 is joined by λ edges to every vertex of X_2.

LEMMA 2.2. If $\lambda K_{v_1,v_2}$ can be decomposed into cycles of length k and if there exists a $BCD(v_i,k,\lambda)$ for $i = 1, 2$, then there exists a $BCD(v_1 + v_2,k,\lambda)$.

LEMMA 2.3. If $\lambda K_{v_1,v_2}$ can be decomposed into cycles of length k and if there exists a $BCD(v_i + 1,k,\lambda)$, $i = 1, 2$, then there exists a $BCD(v_1 + v_2 + 1, k, \lambda)$.

Similar results hold for BCD*'s by replacing k-cycles by k-circuits and $\lambda K_{v_1,v_2}$ by complete directed bipartite graph $\lambda K^*_{v_1,v_2}$. Relations between BCD's and BCD*'s are given in the following lemmas.

LEMMA 2.4. If there exists a $BCD(v,k,\lambda)$, then there exists a $BCD^*(v,k,\lambda)$.

Remark. The converse of the statement is not necessarily true, for example, a $BCD^*(8,8,1)$ exists ([3]) but a $BCD(8,8,1)$ does not (the necessary conditions are not satisfied).

LEMMA 2.5. If there exists a $BCD^*(v,k,\lambda)$, then there exists a $BCD(v,k,2\lambda)$.

Remark. The converse is again not true, for example, a $BCD(4,4,2)$ exists but a $BCD^*(4,4,1)$ does not (see [3]).

In order to apply Lemmas 2.2 and 2.3 we need the following results which have been obtained in [11].

LEMMA 2.6. K_{v_1,v_2} is decomposable into cycles of length $2n$ if and only if v_1 and v_2 are even, $v_1,v_2 \geq n$ and $v_1 v_2 \equiv 0 \pmod{2n}$.

LEMMA 2.7. $K^*_{v_1,v_2}$ is decomposable into circuits of length $2n$ if and only if $v_1,v_2 \geq n$ and $v_1 v_2 \equiv 0 \pmod{n}$.

All these lemmas enable us to construct BCD's and BCD*'s from smaller ones; to solve the existence problem for small values of v, we use a direct construction which is analagous to R. C. Bose's method of symmetrically repeated differences.

In this paper, the elements of a BCD (v,k,λ), D, are represented by residue classes modulo n, where $n = v$ or $v - 1$. In the latter case, the extra element is represented by ∞. The set of the n elements is denoted by Z_n.

Let A be an automorphism of D. Two cycles C_i and C_j are said to be in the same orbit if $A^p(C_i) = C_j$ for some $p \geq 1$. An orbit can be represented by any one of its cycles, which will be called a <u>base cycle</u> and the <u>order</u> of the base cycle is the cardinality of the orbit it belongs to. Hence a collection of base cycles, one from each orbit, determines the whole design when automorphism A is applied. A BCD is said to be cyclic if A consists of a single cycle of length v, without loss of generality, let $A = (0\ 1\ 2 \ldots (v-1))$.

<u>Remark</u>. We can define base circuits similarly.

To construct base cycles or base circuits, we need families of differences. We will first consider the directed case. If c_i and c_j are linked in a circuit C by an arc from c_i to c_j, then $d_{ij} = c_j - c_i$ is called the difference between c_i and c_j (it is actually the arc from c_i to c_j). If $c_j = \infty$, then $d_{ij} = \infty$ and if $c_i = \infty$ then $d_{ij} = -\infty$. An element d of $\{1, 2, \ldots, n-1\} \cup \{+\infty, -\infty\}$ is said to occur p times as a difference in a base circuit C of order m if pn/m elements d_{ij} in C are of the value d. Notice that d_{ij}'s are taken modulo n and the element $+\infty$ or $-\infty$ can appear at most once as a difference d_{ij} in a circuit.

The family of base circuits C_i, $i \in I$ determine the whole design if for each d in $\{1, 2, \ldots, n-1\}$ or $\{1, 2, \ldots, n-1\} \cup \{+\infty, -\infty\}$, depending on whether $n = v$ or $n = v-1$, $\sum_{i \in I} p_i^d = \lambda$, where p_i^d denotes the number of times the element d occurs in the circuit C_i.

To construct a base circuit C_i, it is sufficient to find

<u>either</u> a family $D = \{d_1, d_2, \ldots, d_k\}$, with $d_i \in \{1, 2, \ldots, n-1\}$ such that

(1) $\sum_{i=1}^{k} d_i = 0$ and $\sum_{i=\alpha}^{\beta} d_i \neq 0$ for $1 \leq \alpha < \beta \leq k$,

$(\alpha, \beta) \neq (1, k)$

and in this case, the base circuit can easily be constructed, for example

$$C = (0, d_1, d_1+d_2, \ldots, d_1+d_2+d_3+ \ldots + d_{k-1});$$

<u>or</u> a family $D = \{d_1, d_2, \ldots, d_{k-2}\}$, with $d_i \in \{1, 2, \ldots, v-2\}$ such that

(2) $\sum_{i=\alpha}^{\beta} d_i \neq 0$ for $1 \leq \alpha < \beta \leq k-2$

and in this case, $C = (\infty, 0, d_1, d_1+d_2, \ldots, d_1+d_2+\ldots+d_{k-2})$.

For the undirected case, the method is similar; as the arc from c_i to c_j and the arc from c_j to c_i are not distinguishable, the difference between c_i and c_j is defined to be

$$d_{ij} = \min\{|c_i - c_j|, n - |c_i - c_j|\}.$$

d_{ij} is sometimes called the <u>edge length</u> of the edge joining c_i and c_j.

As we deal with collections of elements rather than with sets and we want to retain the multiplicities, we will denote by $D_1 \cup D_2$ the result of adjoining the elements of D_1 to those of D_2 with total multiplicities retained. In particular, sD will denote a collection of elements of D with multiplicities being increased s fold. Finally, we denote by I_t the set $\{1, 2, \ldots, t\}$.

3. <u>The Construction of BCD*(v,k,λ), k even.</u>

In this section, we give the constructions of balanced circuit designs, BCD*(v,k,λ) for some modulo classes of v and even k.

We obtain from Proposition 1.2,

LEMMA 3.1. Necessary conditions for the existence of a $BCD^*(v,k,\lambda)$ are $v \geq k$ and that there exist positive integers x,y,z such that $xyz = k$, $x|\lambda$, and $v \equiv \ell y \pmod{yz}$ where $0 \leq \ell < z$ and if $\ell > 0$ $\ell y \equiv 1 \pmod{z}$.

Proof. If there exists a $BCD^*(v,k,\lambda)$, then $b = \lambda v(v-1)/k$ must be integral. Let x be the g.c.d. of λ and k, and put $\lambda = \lambda' x$, $k = k'x$. Then $v(v-1)/k'$ is an integer. Let y be the g.c.d. of v and k', put $k = xyz$, $v = yzt + \ell y$ with $t \geq 0$, $0 \leq \ell < z$, then

$$z|(yzt + \ell y - 1) \quad \text{if} \quad \ell > 0 .$$

We will give the proof of the following theorem later.

THEOREM 3.2. If there exists a $BCD^*(v_o,k,\lambda)$ for $v_o = \overline{v} + pyz$, $k = xyz$ and $\lambda = x$, where $0 \leq p \leq x-1$, then there exists a $BCD^*(v,k,\lambda)$ for $v = \overline{v} + qyz$, $k = xyz$ and $\lambda \equiv 0 \pmod{x}$ for any $q \geq 0$; k is even.

An important consequence of this theorem is that the verification of Conjecture I (in directed case) is reduced to a finite number of cases, namely,

COROLLARY. The necessary condition for the existence of a $BCD^*(v,k,\lambda)$ in Lemma 3.1 is also sufficient for k even if there exists a $BCD^*(v_o,k,\lambda)$ for $v_o = (x+p)yz + \ell y$, $k = xyz$, $\lambda = x$ where $0 \leq p < x$, $0 \leq \ell < z$ and if $\ell > 0$, $\ell y \equiv 1 \pmod{z}$.

We have the following lemma [3].

LEMMA 3.3. There exists a $BCD^*(k+1,k,1)$ for any integer k.

LEMMA 3.4. If there exists a $BCD^*(v,k,x)$, then there exists a $BCD^*(v+k,k,x)$, where k is even.

Proof. Applying Lemma 2.1 to the result of Lemma 3.3, a $BCD^*(k+1,k,x)$ exists. By hypothesis, a $BCD^*(v,k,x)$ exists. Then apply Lemma 2.3 for the directed case with $v_1 = v-1$, $v_2 = k$, $\lambda = x$; since $K^*_{v_1,v_2}$ can be decomposed into circuits of length k (Lemma 2.7), a $BCD^*(v+k,k,x)$ exists.

We can now return to Theorem 3.2.

Proof of Theorem 3.2. Assume that a $BCD^*(\overline{v} + pyz,xyz,x)$ exists for $0 \leq p \leq x-1$, then a $BCD^*(\overline{v} + pyz + k, xyz, x)$ exists also by Lemma 3.4. Now put $q = sx + p$ where $s \geq 0$, then we can prove the existence of a $BCD^*(v,xyz,x)$, where $v = \overline{v} + qyz = \overline{v} + sk + pyz$, by induction on s. Lemma 2.1 is used to prove cases where $\lambda > x$.

<u>Note</u>. Unless otherwise stated, k is always even in this section, furthermore, we only construct designs with minimal λ, in view of Lemma 2.1.

THEOREM 3.5. There exists a $BCD^*(tyz+1,xyz,\lambda)$ for any $t \geq x$ and $\lambda \equiv 0 \pmod{x}$.

If there exists a $BCD^*(tyz+1,xyz,x)$ the number of k-circuits is $b = vt$, where $v = tyz + 1$; hence we will construct t base circuits C_i, $i = 1, 2, \ldots, t$, each of order v. In fact, we will look for t collections of k elements each,

$$D_i = \{d_{i_1}, d_{i_2}, \ldots, d_{i_k}\}, \quad i = 1, 2, \ldots, t$$

such that the elements satisfy condition (1), that is

(3) $$\sum_{j=1}^{k} d_{ij} = 0$$

(4) $$\sum_{j=\alpha}^{\beta} d_{ij} \neq 0 \quad \text{for} \quad 1 \leq \alpha < \beta \leq k, \ (\alpha,\beta) \neq (1,k)$$

and, in addition,

(5) $$\bigcup_{1 \leq i \leq t} D_i = x\, I_{v-1}$$

Consider the case when v is odd. Put

$$D' = \{1, 2, \ldots, (v-1)/2, 1, 2, \ldots, (v-1)/2, \ldots\ldots 1, 2, \ldots, (v-1)/2\} ,$$

that is, x copies of $I_{(v-1)/2}$ in the given order. Now let S_1 contain the first $k/2$ elements of D', S_2 contain the following $k/2$ elements and so on, so S_t contains the last $k/2$ elements. Then reorder each S_j so that its elements are in a strictly increasing order, that is,

$$S_j = \{a_{j_1}, a_{j_2}, \ldots, a_{j_{k/2}}\} \text{ with}$$

$$a_{j_1} < a_{j_2} < \ldots < a_{j_{k/2}} .$$

For $1 \le j \le t$, put

$$D_j = \{a_{j_1}, -a_{j_2}, a_{j_3}, -a_{j_4}, \ldots, \pm a_{j_{k/2}}, \pm a_{j_{k/2-1}}, \mp a_{j_{k/2-2}},$$

$$\ldots, a_{j_2}, -a_{j_1}, \mp a_{j_{k/2}}\} ,$$ where $+a_{j_{k/2}}$ and hence upper signs are used when $k \equiv 2 \pmod 4$ and $-a_{j_{k/2}}$ and hence lower signs are used when $k \equiv 0 \pmod 4$.

It is easy to see that the D_j's satisfy condition (3) and (5) since

$$-a_{ji} \equiv n - a_{ji} .$$

To verify that the D_j's satisfy condition (4) we use the following property: let $b_1 < b_2 < \ldots < b_s$ be s elements of Z_n. Then $b_1, b_1 - b_2, b_1 - b_2 + b_3, \ldots, b_1 - b_2 + b_3 - \ldots \pm b_s$ ($+b_s$ if s is odd, $-b_s$ if s is even) are all different from zero and their absolute value is strictly less than b_s.

For the case where v is even, let

$$D' = \{1,2,\ldots,\tfrac{v}{2},\ldots,1,2,\ldots,\tfrac{v}{2},\ldots,1,2,\ldots,\tfrac{v}{2}-1,\ldots,1,2,\ldots,\tfrac{v}{2}-1\}$$

that is $\frac{x}{2}$ copies of $I_{v/2}$ following by $\frac{x}{2}$ copies of $I_{v/2-1}$ in the given order. Then we construct t sets S_j and t sets D_j exactly as in the odd case. Hence a BCD*(tyz+1,xyz,x) exists and the proof is complete.

<u>Remark</u>. 1) The design BCD*(tyz+1,xyz,λ), $\lambda \equiv 0 \pmod x$, constructed in the last proof is cyclic.

2) We constructed a design for any $t \ge x$ although Theorem 3.2 and its Corollary imply that it suffices to prove the existence for t, where $x \le t \le 2x-1$.

THEOREM 3.6. A BCD*(tyz,xyz,λ) exists for all $t \ge x$ and $\lambda \equiv 0 \pmod x$, except that a BCD*(4,4,λ) with λ odd and a BCD*(6,6,1) do not exist.

Proof. The existence of a BCD*(tyz,xyz,x) implies that we must have $b = x(tyz)(tyz-1)/xyz = t(v-1)$ k-circuits. Hence we will construct t base circuits, each of order $v-1$. Represent the elements of the design by $Z_{v-1} \cup \{\infty\}$.

Again, we will look for x sets $D_j = \{d_{j_1}, d_{j_2}, \ldots, d_{j_{k-2}}\}$, for $j = 1, 2, \ldots, x$, which satisfy

$$(4') \qquad \sum_{i=\alpha}^{\beta} d_{ji} \ne 0 \quad \text{for } 1 \le \alpha < \beta \le k-2$$

and for $t-x$ sets, $D_j = \{d_{j_1}, d_{j_2}, \ldots, d_{j_k}\}$, where $j = x+1, x+2, \ldots t$ which satisfy (3), (4) and

$$(5') \qquad \bigcup_{1 \le j \le t} D_j = x I_{v-2} .$$

When v is even, put

$$D' = \{1,2,\ldots,(v-2)/2,1,2,\ldots,(v-2)/2,\ldots,1,2,\ldots,(v-2)/2\} ,$$

that is, x copies of $I_{(v-2)/2}$ in the given order.

When v is odd, let D' consist of $\frac{x}{2}$ copies of $I_{(v-1)/2}$ followed by $\frac{x}{2}$ copies of $I_{(v-3)/2}$.

Now let F_1 contain the first $\frac{k}{2}-1$ elements of D' and F_2 contain the next $\frac{k}{2}-1$ elements of D' and so on, for $F_3, F_4, \ldots, F_x$; then, let S_{x+1} contain the following $\frac{k}{2}$ elements of D' and so on, and finally S_t contains the last $\frac{k}{2}$ elements of D'. From the sets S_j, $x+1 \le j \le t$, we obtain the sets D_j which satisfy conditions (3) and (4) using the same method as in the proof of Theorem 3.5.

Case 1: x even. In this case, we can partition the sets F_i, $1 \le i \le x$ into consecutive pairs, $F_{2j-1} \cup F_{2j}$, $1 \le j \le \frac{x}{2}$. If an element a occurs twice in a union, then replace one of them by $v-1-a$, and then reorder the elements in a strictly increasing order; that is, $(F_{2j-1} \cup F_{2j}) = \{a_{j_1}, a_{j_2}, \ldots, a_{j_{k-2}}\}$ with $a_{j_1} < a_{j_2} < \ldots < a_{j_{k-2}}$. Then for $1 \le j \le \frac{x}{2}$, put

$$D_{2j-1} = \{a_{j_1}, -a_{j_2}, a_{j_3}, \ldots, -a_{j_{k-2}}\}$$

$$D_{2j} = \{-a_{j_1}, a_{j_2}, -a_{j_3}, \ldots, a_{j_{k-2}}\}$$

Again it is easy to see that the D_j's $1 \le j \le x$ satisfy conditions (4') and that D_j, $1 \le j \le t$, satisfy (5').

Example. A BCD*(12,8,4) exists. Here $v = 12$, $k = 8$, $x = 4$ and $t = 6$.

$D' = \{1,2,3,4,5,1,2,3,4,5,1,2,3,4,5,1,2,3,4,5\}$

$F_1 = \{1,2,3\}$, $F_2 = \{4,5,1\}$, $F_3 = \{2,3,4\}$, $F_4 = \{5,1,2\}$

$S_5 = \{3,4,5,1\}$ and $S_6 = \{2,3,4,5\}$.

$F_1 \cup F_2 = \{1,2,3,4,5,1\}$, $(F_1 \cup F_2)' = \{1,2,3,4,5,10\}$,

Hence $D_1 = \{1,-2,3,-4,5,-10\}$, $D_2 = \{-1,2,-3,4,-5,10\}$.

$F_3 \cup F_4 = \{2,3,4,5,1,2\}$, $(F_3 \cup F_4)' = \{1,2,3,4,5,9\}$,

Hence $D_3 = \{1,-2,3,-4,5,-9\}$, $D_4 = \{-1,2,-3,4,-5,9\}$.

Also, $D_5 = \{1,-3,4,-5,-4,3,-1,5\}$ and $D_6 = \{2,-3,4,-5,-4,3,-2,5\}$.

Case 2: x odd.

(a) x=1. The existence of a BCD*(tyz,yz,λ) for any λ follows from that of a BCD*(yz,yz,1) by Theorem 3.2. The existence of a BCD*(k,k,1) is equivalent to the existence of a decomposition of K_k^* into hamiltonian circuits. It is known that the decomposition is impossible for $k=4$ or 6 and it has been proved recently [12] that a BCD*(k,k,1) exists for all $k \ge 8$. When $k=4$, a BCD*(4,4,λ) for λ odd does not exist (see Lemma 4.2). However, a BCD*(8,4,1) exists ([2]). Hence, by Theorem 3.2, a BCD*(4t,4,λ) exists for any $t \ge 2$ and any λ. When $k=6$, a BCD*(6,6,1) does not exist ([3]); but there exists a BCD*(6,6,2), (i.e. $x=2$ in Case 1) and a BCD*(6,6,3) ([2]); thus, by Lemma 2.1, a BCD*(6,6,λ) exists for any $\lambda \ge 2$. Furthermore a BCD*(12,6,1) exists ([2]). Hence, by Theorem 3.2, a BCD*(6t,6,λ) exists for any $t \ge 2$ and any λ.

(b) $x \ge 3$. By Theorem 3.2, it suffices to show the existence of a BCD*(tyz,xyz,x) for $x \le t \le 2x-1$. For $t=x$, this follows from case (a). Thus we will suppose that $x < t \le 2x-1$; hence v is even.

As in the construction for the case where x is even, we partition D' into x sets F_i's and $t-x$ sets S_j's. We now have an odd number of F_i's. Consider the first three, F_1, F_2 and F_3. Put $F = F_1 \cup F_2 \cup F_3$; we will construct D_1^*, D_2^* and D_3^* from F, such that each D_i^* satisfies the condition (4'). The rest of the construction remains the same. Now $|F| = f = 3(\frac{k}{2}-1)$ and we have three possibilities:

(a) $f = 2(v-2)/2$, that is, $F = 2I(v-2)/2$ which implies that $3xyz = 2tyz + 2$. Now $t = (3xyz-2)/2yz$ being an integer implies that $yz = 2$ and $t = (3x-1)/2$. Hence $k = 2x$ and $v = 3x-1$.

Put $D_1^* = \{1, -2, 3, -4, \ldots, (2x-3), -(2x-2)\}$,
$D_2^* = \{-1, 2, -3, 4, \ldots, -(x-2)\} \cup \{(2x-2), -(2x-1), \ldots, (3x-3)\}$
and $D_3^* = \{(x-1), -x, \ldots, -(2x-3)\} \cup \{(2x-1), -2x, \ldots, -(3x-3)\}$.
It is easy to check that D_j^*, $j = 1, 2, 3$, satisfies (4').

(b) $f < v-2$, that is, $F \subset 2I_{(v-2)/2}$. Let
$F = \{1,2,\ldots, \frac{v-2}{2}\} \cup \{1,2,\ldots, p\}$, where $1 \le p < (v-2)/2$,
and let $G = \bigcup_{i=1}^{3} D_i^* = \{\pm 1, \pm 2, \ldots, \pm\frac{v-2}{2}\} \cup \{\pm 1, \pm 2, \ldots, \pm p\}$

$$\equiv \{\pm 1, \pm 2, \ldots, \pm p\} \cup \{\pm\tfrac{v}{2}, \pm(\tfrac{v}{2}+1), \ldots, \pm(v-2)\}.$$

Note that $3(k-2) = v-2+2p$ implies that $p = \frac{3k-v-4}{2}$.

Put $q = k-p-3$ and $s = 2k-p-\frac{v}{2}-3$. Hence $s \le p$ and $p+(q+1) = (p-s) + (\frac{v}{2}-1) = s + (\frac{v}{2}-q-2) = k-2$.

If p and $\frac{v}{2}$ are both even, put

$D_1^* = \{1, -2, \ldots, (p-1), -p\} \cup \{\frac{v}{2}, -(\frac{v}{2}+1), \ldots, -(\frac{v}{2}+q)\}$,

$D_2^* = \{-1, 2, \ldots, (s-1), -s\} \cup \{(\frac{v}{2}+q+1), -(\frac{v}{2}+q+2), \ldots, (v-2)\}$,

and $D_3^* = \{(s+1), -(s+2), \ldots, p\} \cup \{-\frac{v}{2}, (\frac{v}{2}+1), \ldots, -(v-2)\}$.

If p is odd and $\frac{v}{2}$ is even, put

$D_1^* = \{1, -2, \ldots, -(p-1), p\} \cup \{-(\frac{v}{2}), (\frac{v}{2}+1), \ldots, -(\frac{v}{2}+q)\}$,

$D_2^* = \{(p-s+1), -(p-s+2), \ldots, -p\} \cup \{(\frac{v}{2}+q+1), -(\frac{v}{2}+q+2), \ldots, -(v-2)\}$

and $D_3^* = \{-1, 2, \ldots, -(p-s)\} \cup \{\frac{v}{2}, -(\frac{v}{2}+1), \ldots, (v-2)\}$.

If p is even and $\frac{v}{2}$ is odd, put

$$D_1^* = \{1, -2, \ldots, (p-1), -p\} \cup \{\tfrac{v}{2}\} \cup \{-(v-q-1), (v-q), \ldots, -(v-2)\},$$

$$D_2^* = \{-1, 2, \ldots, -(s-1), s\} \cup \{-(\tfrac{v}{2}+1), (\tfrac{v}{2}+2), \ldots, (v-q-2)\},$$

and $D_3^* = \{-(s+1), s+2, \ldots, p\} \cup \{-\frac{v}{2}, (\frac{v}{2}+1), \ldots, (v-2)\}$.

Lastly, if p and $\frac{v}{2}$ are both odd, put

$$D_1^* = \{1, -2, \ldots, -(p-1), p\} \cup \{-(\tfrac{v}{2}), (\tfrac{v}{2}+1), \ldots, -(\tfrac{v}{2}+q)\},$$

$$D_2^* = \{-1, 2, \ldots, -s\} \cup \{(\tfrac{v}{2}+q+1), -(\tfrac{v}{2}+q+2), \ldots, (v-2)\},$$

and $D_3^* = \{(s+1), -(s+2), \ldots, -p\} \cup \{\frac{v}{2}, -(\frac{v}{2}+1), \ldots, -(v-2)\}$.

(c) $f > v-2$, that is $3I_{(v-2)/2} \supset F \supset 2I_{(v-2)/2}$.

Put $F+ = \{1, -2, 3, -4, \ldots, \delta(\frac{v}{2}-1)\}$

and $F- = \{-1, 2, -3, 4, \ldots, -\delta(\frac{v}{2}-1)\}$

where

$$\delta = \begin{cases} 1 & \text{if } \frac{v}{2} \text{ is even} \\ -1 & \text{if } \frac{v}{2} \text{ is odd} \end{cases}$$

Now $-a \equiv v-1-a$ for $1 \le a \le v-2$ implies that $F+$ and $F+' = \{-\delta\frac{v}{2}, \delta(\frac{v}{2}+1), \ldots, -(v-2)\}$ are equivalent and $F-$ and $F-' = \{\delta\frac{v}{2}, -\delta(\frac{v}{2}+1), \ldots, (v-2)\}$ are equivalent.

Let p be such that

$2p + 2(v-2) = 3(k-2)$, and $p = (3k - 2v - 2)/2 \ge 1$. Put $P+ = \{1, -2, \ldots, \theta p\}$ and $P- = \{-1, 2, \ldots, -\theta p\}$ where

$$\theta = \begin{cases} 1 & \text{if } p \text{ is odd} \\ -1 & \text{if } p \text{ is even} \end{cases}$$

Also $P+$ is equivalent to $P+' = \{-\theta(v-1-p), \ldots, -(v-2)\}$ and $P-$ is equivalent to $P-' = \{\theta(v-1-p), \ldots, (v-2)\}$.

The construction of D_i^*'s is similar to that in (b).

Here

$$G = \bigcup_{i=1}^{3} D_i^* = F+ \cup F- \cup F+' \cup F-' \cup P+ \cup P- .$$

Let s be such that $(\frac{v}{2}-1)+s+p = k-2$, hence $s = (v-k)/2$.

If p and $\frac{v}{2}$ are both even, then s is odd, put

$$D_1^* = F+ \cup \{-(\tfrac{v}{2}+1), (\tfrac{v}{2}+2), \ldots, -(\tfrac{v}{2}+s)\} \cup P+',$$

$$D_2^* = F- \cup \{(\tfrac{v}{2}+s+1), -(\tfrac{v}{2}+s+2), \ldots, (\tfrac{v}{2}+2s)\} \cup P-',$$

and $D_3^* = F+' \cup \{\frac{v}{2}\} \cup \{-1, 2, \ldots, (k-\frac{v}{2}-2)\}$.

If p is odd and $\frac{v}{2}$ is even, then s is even, put

$$D_1^* = F+ \cup \{-(\tfrac{v}{2}+1), (\tfrac{v}{2}+2), \ldots, -(\tfrac{v}{2}+s-1)\} \cup \{(\tfrac{v}{2}+2s)\} \cup P+',$$

$$D_2^* = F- \cup \{(\tfrac{v}{2}+s), -(\tfrac{v}{2}+s+1), \ldots, -(\tfrac{v}{2}+2s-1)\} \cup P-'$$

and D_3^* as above.

If p is even and $\frac{v}{2}$ is odd, then s is even, put

$$D_1^* = F+ \cup \{(\tfrac{v}{2}+1), -(\tfrac{v}{2}+2), \ldots, (\tfrac{v}{2}+s-1)\} \cup \{-(\tfrac{v}{2}+2s)\} \cup P+',$$

$$D_2^* = F- \cup \{-(\tfrac{v}{2}+s), (\tfrac{v}{2}+s+1), \ldots, (\tfrac{v}{2}+2s-1)\} \cup P-'$$

and $D_3^* = F+' \cup \{-\frac{v}{2}\} \cup \{-1, 2, \ldots, -(k-\frac{v}{2}-2)\}$.

Finally, let p and $\frac{v}{2}$ be both odd, then s is odd, put

$$D_1^* = F+ \cup \{(\tfrac{v}{2}+1), -(\tfrac{v}{2}+2), \ldots, (\tfrac{v}{2}+s)\} \cup P+',$$

$$D_2^* = F- \cup \{-(\tfrac{v}{2}+s+1), (\tfrac{v}{2}+s+2), \ldots, -(\tfrac{v}{2}+2s)\} \cup P-'$$

and D_3^* as in the last case.

It is a routine matter to check that D_1^* and D_2^* satisfy condition (4'). As for D_3^*, we will show the circuit associated with it. Let $\frac{v}{2}$ be even, then

$$D_3^* = \{-\tfrac{v}{2}, (\tfrac{v}{2}+1), -(\tfrac{v}{2}+2), (\tfrac{v}{2}+3), \ldots, -(v-2)\} \cup \{\tfrac{v}{2}\} \cup \{-1, 2, -3, 4, \ldots (k-\tfrac{v}{2}-2)\}.$$

Let $C_3 = \left(\infty, \frac{v}{2}, 0, \frac{v}{2}+1, v-2, \frac{v}{2}+2, \ldots, \frac{3v}{4}-1, \frac{3v}{4}, \frac{v}{4}+1, \frac{v}{4}, \frac{v}{4}+2, \frac{v}{4}-1, \ldots, \frac{v-k+4}{2}, \frac{k}{2}\right)$.

The elements in C_3 are distinct, since the set consists of $0, \frac{v}{2}, \frac{v}{2}+1, \ldots, v-2$ and $\frac{v-k+4}{2}, \frac{v-k+6}{2}, \ldots, \frac{k}{2}$; furthermore, the differences between two consecutive elements of C_3 are exactly $D_3^* \cup \{+\infty, -\infty\}$. Hence C_3 is a base circuit of order $v-1$.

Similarly, if $\frac{v}{2}$ is odd, put

$$C_3 = \left(\infty, 0, \frac{v}{2}, v-2, \frac{v}{2}+1, \ldots, \frac{3v-6}{4}, \frac{3v-2}{4}, \frac{v-2}{4}, \frac{v-6}{4}, \frac{v+2}{4}, \frac{v-10}{4}, \frac{v+6}{4}, \ldots, \frac{k-4}{2}, \frac{v-k}{2}\right).$$

If $k=4$ then $xyz=4$ which implies that $x=1$, a contradiction, hence $k \geq 6$.

The proof of Theorem 3.6 is complete.

We have also the following theorems.

THEOREM 3.7. If $xy \leq 2$, there exists a BCD*(v,xyz,λ) for all v and λ satisfying the necessary conditions (cf Lemma 3.1).

Proof. If $y = 1$, $x = 1$ or 2, then $\ell = 1$ and the existence of such design has been proved (Theorem 3.5). If $x = 1$, $y = 2$, then $\ell = (z+1)/2$; by Theorem 3.2, it suffices to prove the existence of a BCD*$(3z+1,2z,1)$ with z odd, which has been done already (Theorem 1.2.13 in [2]). Hence the proof is complete.

THEOREM 3.8. If $xz \leq 2$, there exists a BCD*(v,xyz,λ) for all v and λ satisfying the necessary conditions (cf Lemma 3.1).

Proof. If $z = 1$, then $\ell = 0$ and the existence of such designs have been proved in Theorem 3.6. Similarly, Theorem 3.6 proves the case $z = 2$ and $\ell = 0$. So we assume that $z = 2$, $x = 1$ and $\ell = 1$, hence y is odd. The existence of such a design was conjectured in I.2.16 of [2] and has been proved partially in [6]. By Theorem 3.2, it suffices to show the existence of a BCD*$(3y,2y,1)$; we will divide the construction into two cases.

(i) $y = 4m+1$, hence $v = 12m+3$ and $b = 3(6m+1)$. Let the elements be $V = \{\infty\} \cup V_1 \cup V_2$ where $V_1 = \{0_1, 1_1, 2_1, \ldots, (6m)_1\}$

and $V_2 = \{0_2, 1_2, 2_2, \ldots, (6m)_2\}$. We will construct three base circuits C_1, C_2 and C_3 such that they generate all the b circuits of the design under the automorphism $A = (\infty)(0_1\ 1_1 \ldots (6m)_1)(0_2\ 1_2 \ldots (6m)_2)$. The elements are taken modulo $6m+1$.

Let $C_1 = (0_1, (6m)_2, 1_1, (6m-1)_2, \ldots, (2m)_1, (4m)_2, (4m+1)_1, (4m-1)_2, (4m+2)_1, (4m-2)_2, \ldots, (6m)_1, (2m)_2)$.

When m is even, put $m = 2n$ and

$$C_2 = (\infty, (3n)_1, (-3n)_1, (3n-1)_1, (-3n+1)_1, \ldots, 1_1, (-1)_1, 0_1, 0_2, 1_2, (-1)_2, \ldots, (3n-1)_2, (-3n+1)_2, (3n)_2, (-3n)_2, (5n+2)_1, (9n)_2, (5n+3)_1, (9n-1)_2, \ldots, (7n)_1, (7n+2)_2, (7n+1)_1)$$

$$C_3 = (\infty, (-3n)_2, (3n)_2, (-3n+1)_2, (3n-1)_2, \ldots, (-1)_2, 1_2, 0_2, 0_1, (-1)_1, 1_1, \ldots, (-3n+1)_1, (3n-1)_1, (-3n)_1, (3n)_1, (7n-1)_2, (3n+1)_1, (7n-2)_2, (3n+2)_1, \ldots, (5n-1)_1, (5n)_2).$$

When m is odd, put $m = 2n+1$,

$$C_2 = (\infty, (3n+2)_1, (-3n-1)_1, (3n+1)_1, \ldots, (-2)_1, 2_1, (-1)_1, 1_1, 0_1, 0_2, (-1)_2, 1_2, (-2)_2, 2_2, \ldots, (-3n-1)_2, (3n+1)_2, (-3n-2)_2, (5n+4)_1, (9n+4)_2, (5n+5)_1, (9n+3)_2, \ldots, (7n)_2, (7n-1)_1)$$

and

$$C_3 = (\infty, (-3n-2)_2, (3n+1)_2, (-3n-1)_2, (3n)_2, \ldots, 2_2, (-2)_2, 1_2, (-1)_2, 0_2, 0_1, 1_1, (-1)_1, 2_1, (-2)_1, \ldots, (3n+1)_1, (-3n-1)_1, (3n+2)_1, (7n+3)_2, (3n+3)_1, (7n+2)_2, (3n+4)_1, \ldots, (5n+2)_1, (5n+3)_2).$$

To verify that C_1, C_2 and C_3 are base circuits, one can define the difference of the pair of elements $\{a_i, b_j\}$ as $b-a$ (mod $6m+1$); then it suffices to verify that every integer in $\{1, 2, \ldots, 6m\}$ occurs exactly once as the difference of each of the following pairs of elements:

$\{a_1, b_1\}$, $\{a_1, b_2\}$, $\{a_2, b_1\}$ and $\{a_2, b_2\}$, in the circuit C_1 or C_2 or C_3.

(ii) $y = 4m+3$, hence $v = 12m+9$ and $b = 3(6m+4)$. Let $V = \{\infty\} \cup V_1$ and V_2 where $V_1 = \{0_1, 1_1, \ldots, (6m+3)_1\}$ and $V_2 = \{0_2, 1_2, \ldots, (6m+3)_2\}$. We will construct three base circuits C_1,

C_2 and C_3 such that they generate all the b circuits under $A = (\infty)(0_1,1_1,\dots,(6m+3)_1)(0_2,1_2,\dots,(6m+3)_2)$. The elements are taken modulo $6m+4$.

$$C_1 = (0_1,(4m+2)_1,1_1,(4m+1)_1,2_1,(4m)_1,\dots,(2m+2)_1,(2m+1)_1,(2m+1)_2,(2m+2)_2,\dots,(4m+1)_2,1_2,(4m+2)_2,0_2) .$$

When m is even, put $m = 2n$ and let

$$C_2 = (\infty,0_1,1_1,(-1)_1,2_1,(-2)_1,\dots,(-n)_1,(n+1)_1,n_2,(n+2)_1,(n-1)_2,(n+3)_1,\dots,(7n+2)_1,(-5n-1)_2,(-9n-2)_2,(-5n-2)_2,(-9n-1)_2,(-5n-3)_2,\dots,(-8n-3)_2,(-6n-1)_2) ,$$

and

$$C_3 = (\infty,0_2,(-1)_2,1_2,(-2)_2,2_2,\dots,n_2,(-n-1)_2,(-n)_1,(-n-2)_2,(-n+1)_1,(-n-3)_2,\dots,(-7n-2)_2,(5n+1)_1,(9n+2)_1,(5n+2)_1,(9n+1)_1,(5n+3)_1,\dots,(8n+3)_1,(6n+1)_1) .$$

When m is odd, let $m = 2n+1$, put

$$C_2 = (\infty, 0_1, 1_1, (-1)_1, 2_1, (-2)_1, \dots, (-n)_1, (n+1)_1, n_2,(n+2)_1,(n-1)_2,(n+3)_1,\dots,(7n+5)_1,(-5n-4)_2,(-9n-7)_2,(-5n-5)_2, (-9n-6)_2, (-5n-6)_2,\dots,(-8n-7)_2, (-6n-5)_2)$$

and

$$C_3 = (\infty,0_2,(-1)_2,1_2,(-2)_2,2_2,\dots,n_2,(-n-1)_2,(-n)_1,(-n-2)_2,(-n+1)_1,(-n-3)_2,\dots,(-7n-5)_2,(5n+4)_1,(9n+7)_1, (5n+5)_1,(9n+6)_1,(5n+6)_1,\dots,(8n+7)_1,(6n+5)_1) .$$

4. BCD*(v,k,λ) for k even, $4 \le k \le 16$.

We will prove, in this section, that the necessary condition for the existence of a BCD*(v,k,λ) where k is even and $4 \le k \le 16$, namely, $\lambda v(v-1) \equiv 0 \pmod{k}$ with $v \ge k$, is also sufficient, except for $v=k=4$, λ odd and $v=k=6$, $\lambda = 1$. Some of the designs have already been proved to exist; for a list of these see [4].

THEOREM 4.1. A necessary and sufficient condition for the existence of a BCD*$(v,4,\lambda)$ is

$v \equiv 0$ or $1 \pmod 4$ and $v > 4$ for $\lambda \ge 1$
$v \equiv 2$ or $3 \pmod 4$ or $v = 4$ for λ even.

The proof is straightforward: Theorem 3.6 implies the existence of a BCD*$(v,4,\lambda)$ where $v \equiv 0 \pmod 4$, $v > 4$ and $x = \lambda = 1$, where $v = 4$ or $v \equiv 2 \pmod 4$ and $x = \lambda = 2$; Theorem 3.5 implies the existence of a BCD*$(v,4,\lambda)$ where $v \equiv 1 \pmod 4$ and $x = \lambda = 1$ and where $v \equiv 3 \pmod 4$ and $x = \lambda = 2$.

LEMMA 4.2. A BCD*$(4,4,\lambda)$ does not exist for λ odd.

Proof. Assume that a BCD*$(4,4,\lambda)$ exists with λ being odd. Without loss of generality, let the elements be $0,1,2,3$ and the arc $(0,1)$ occur in circuits $C_1 = (0,1,2,3)$ and $C_2 = (0,1,3,2)$. Let the multiplicity of C_1, C_2 be m_1 and m_2 respectively, then $m_1+m_2 = \lambda$. Similarly, the arc $(1,2)$ occurs in C_1 and $C_3 = (1,2,0,3)$ which has multiplicity m_3, hence $m_1+m_3 = \lambda$ which implies that $m_2 = m_3$. But the arc $(2,0)$ occurs only in a circuit of type C_2 or C_3, hence $m_2+m_3 = 2m_2 = \lambda$, a contradiction; thus a BCD*$(4,4,\lambda)$ cannot exist for λ odd.

The proofs of the rest of the theorems are similar to that of Theorem 4.1, by applying Theorems 3.5 and 3.6. Exceptions, however, will be stated.

THEOREM 4.3. A necessary and sufficient condition for the existence of a BCD*$(v,6,\lambda)$ is

$v \equiv 0$ or $1 \pmod 3$ and $v > 6$ for $\lambda \ge 1$
$v \equiv 2 \pmod 3$ or $v = 6$ for $\lambda \equiv 0 \pmod 3$.

The only cases which we cannot prove by applying Theorems 3.5 and 3.6 are:

(1) $v \equiv 3 \pmod 6$, $\lambda = 1$. But a BCD*$(9,6,1)$ exists ([2] or [3]). We can apply Theorem 3.2 by letting $v_o = 9$ to prove the existence of a BCD*$(6t+3,6,1)$ for $t \ge 1$.

(2) $v \equiv 4 \pmod 6$, $\lambda = 1$. Now $x = 1$, $y = 2$ and $z = 3$ and the existence of BCD*$(6t+4,6,1)$ is proved in Theorem 3.7.

THEOREM 4.4. A necessary and sufficient condition for the existence of a BCD*$(v,8,\lambda)$ is

$v \equiv 0$ or $1 \pmod 8$ for $\lambda \ge 1$
$v \equiv 0$ or $1 \pmod 4$ for $\lambda \equiv 2 \pmod 4$
any $v \ge 8$ for $\lambda \equiv 0 \pmod 4$.

The proof is obtained by using Theorems 3.5 and 3.6.

THEOREM 4.5. A necessary and sufficient condition for the existence of a BCD*(v,10,λ) is

$v \equiv 0$ or $1 \pmod 5$ for $\lambda \geq 1$

any $v \geq 10$ for $\lambda \equiv 0 \pmod 5$.

The exceptions are: (1) $v = 10t+5$, $\lambda = 1$, in which case $x = 1$, $y = 5$, $z = 2$, $\ell = 1$ and apply Theorem 3.8; (2) $v = 10t+6$, $\lambda = 1$, in which case $x = 1$, $y = 2$, $z = 5$, $\ell = 3$ and apply Theorem 3.7.

THEOREM 4.6. A necessary and sufficient condition for the existence of a BCD*(v,12,λ) is

$v \equiv 0, 1, 4$ or $9 \pmod{12}$ for $\lambda \geq 1$

$v \equiv 0$ or $1 \pmod 4$ for $\lambda \equiv 0 \pmod 3$

$v \equiv 0$ or $1 \pmod 3$ for $\lambda \equiv 0 \pmod 2$

any $v \geq 12$ for $\lambda \equiv 0 \pmod 6$

The exceptional cases are: $v = 12t+4$, $\lambda = x = 1$; $v = 12t+9$, $\lambda = x = 1$; $v = 6t+3$, $\lambda = x = 2$; and $v = 6t+4$, $\lambda = x = 2$, we will prove the existences with the aid of Theorem 3.2 with $v_o = 16$; 21; 15, 21 and 16, 22 respectively. The existence of BCD*(16,12,1) and hence BCD*(16,12,2) and BCD*(21,12,1) and hence BCD*(21,12,2) are given in [2]. The existence of a BCD*(15,12,2) is proved by the following base circuits:

(0,1,3,6,12,2,7,8,10,13,5,9) of order 7,

(∞,0,7,1,10,6,3,2,13,9,4,12) of order 14,

and (∞,0,12,10,3,2,8,13,1,4,5,9) of order 14.

The existence of BCD*(22,12,2) is proved by the following base circuits:

(0,1,20,2,19,3,18,4,17,5,16,∞),

(0,20,1,19,2,18,3,17,4,16,5,∞) and

(0,4,19,5,18,6,17,8,16,9,15,11), all of order 21, and

(0,1,20,2,7,8,6,9,14,15,13,16),

(0,20,1,19,14,13,15,12,7,6,8,5), both of order 7.

THEOREM 4.7. A necessary and sufficient condition for the existence of a BCD*(v,14,λ) is

$v \equiv 0$ or $1 \pmod 7$ for $\lambda \geq 1$

any $v \geq 14$ for $\lambda \equiv 0 \pmod 7$.

The exceptions are $v = 14t+7$ in which case $x = 1$, $y = 7$, $z = 2$ and $\ell = 1$; and $v = 14t+8$ in which case $x = 1$, $y = 2$, $z = 7$ and $\ell = 4$. Theorems 3.8 and 3.7 respectively show their existences.

THEOREM 4.8. A necessary and sufficient condition for the existence of a BCD*(v,16,λ) is

$v \equiv 0$ or $1 \pmod{16}$ for $\lambda \geq 1$

$v \equiv 0$ or $1 \pmod 8$ for $\lambda \equiv 0 \pmod 2$

$v \equiv 0$ or $1 \pmod 4$ for $\lambda \equiv 0 \pmod 4$

any $v \geq 16$ for $\lambda \equiv 0 \pmod 8$

This theorem can be proved by either Theorem 3.5 or Theorem 3.6.

5. The Construction of BCD(v,k,λ) k even.

In this section we consider balanced cycle designs rather than balanced circuit designs; however, as the constructions of these two types of designs are similar, the proofs in this section are shortened.

Lemma 2.5 states that the existence of a BCD*(v,k,λ) implies the existence of a BCD(v,k,2λ); in particular, we have

PROPOSITION 5.1. A BCD(2tyz+1,xyz,x) exists for x even and $t \geq x/2$.

Proof. Let $x = 2x'$. By Theorem 3.5, a BCD*(t(2y)z+1,x'(2y)z,x') exists for $t \geq x'$, hence a BCD(2tyz+1,2x'yz,2x') exists for $t \geq x'$.

PROPOSITION 5.2. A BCD(2tyz,xyz,x) exists for x even and $t \geq x/2$.

Proof. Let $x = 2x'$. Theorem 3.6 implies the existence of a BCD*(t(2y)z,x'(2y)z,x') with $t > x'$ or $t = x' > 1$, hence a BCD(2tyz,xyz,x) exists with $t > x/2$ or $t = x/2 > 1$.

Let $x = 2$ and $t = x' = 1$, hence $v = k = 2yz$. A BCD(2yz,2yz,2) exists with elements $Z_{k-1} \cup \{\infty\}$ and the $2yz-1$ k-cycles generated by the following base cycle:

(∞, 0, 1, k-2, 2, k-3, ..., k/2-1, k/2).

Note. It has been proved that when $3 \le k \le 8$, the necessary condition for the existence of a BCD(v,k,λ) is also sufficient (see, for example [4], [8]). These results, together with Theorem 3.6 provide an alternative proof to Proposition 5.2.

We now consider the case when λ is odd.

PROPOSITION 5.3. A BCD$(2tyz+1,xyz,x)$ exists for x odd, $k = xyz$ even and $(x+1)/2 \le t \le x$.

Proof. A BCD$(2tyz+1,xyz,x)$ must contain $b = t(2tyz+1)$ k-cycles, hence we will construct t base cycles, each of order v.

Let $D = \{1,2,\ldots,2tyz,1,2,\ldots,2tyz,\ldots,1,2,\ldots,2tyz,1,2,\ldots,tyz\}$, that is, $(x-1)/2$ copies of I_{2tyz} followed by one copy of I_{tyz} with the order preserved.

Case 1. Let $k \equiv 0 \pmod 4$. Partition the elements of D so that S_1 contains the first xyz elements, S_2 the second xyz elements and so on, and S_t contains the last xyz elements of D.

Reorder the elements of S_j so that they are in a strictly increasing order, that is,

$$S_j = \{a_{j_1}, a_{j_2}, \ldots, a_{j_k}\} \text{ with } a_{j_i} < a_{j_{i+1}} .$$

As k is even, we have $a_{j_{2i}} - a_{j_{2i-1}} = 1$ for $1 \le j \le t$, $1 \le i \le k/2$.

Consider $D_j = \{a_{j_1}, -a_{j_2}, \ldots, a_{j_{k/2-1}}, -a_{j_{k/2}}, a_{j_{k/2+2}}, -a_{j_{k/2+3}}, \ldots, a_{j_k}, -a_{j_{k/2+1}}\}$ for $1 \le j \le t$. These t sets satisfy condition (1), since

$$\sum_{i=1}^{k} a_{ji} = \sum_{i=1}^{k/4} (-1) + a_{j_{k/2+2}} + \sum_{i=k/4+2}^{k/2} (+1) - a_{j_{k/2+1}} = 0 .$$

We can use these t sets to construct t k-cycles in the same manner as k-circuits were constructed.

Case 2. Let $k \equiv 2 \pmod 4$. The elements of D are partitioned into t sets, $S_1, S_2, \ldots, S_t$, but each S_j satisfies a slightly different condition:

$$S_j = \{a_{j_1}, a_{j_2}, \ldots, a_{j_k}\}$$ where a_{j_i}'s are in a strictly increasing order and $a_{j_{2i}} - a_{j_{2i-1}} = 1$ for any i, $1 \le i \le k/2$ except for one i', $i' \ne (k+2)/4$ with $a_{j_{2i'}} - a_{j_{2i'-1}} = 2$.

The pair $\{a_{j_{2i'-1}}, a_{j_{2i'}}\}$ is called a <u>hook</u>.

To each sequence S_j we associate an ordered set D_j as follows:

$$D_j = \{a_{j_1}, -a_{j_2}, \ldots, a_{j_{k/2-2}}, -a_{j_{k/2-1}}, a_{j_{k/2+1}}, -a_{j_{k/2+2}}, \ldots, -a_{j_{k-1}}, a_{j_k}, -a_{j_{k/2}}\} \text{ if } 1 \le i' \le (k-2)/4, \text{ and}$$

$$D_j = \{a_{j_1}, -a_{j_2}, \ldots, a_{j_{k/2}}, -a_{j_{k/2+1}}, a_{j_{k/2+3}}, -a_{j_{k/2+4}}, \ldots, -a_{j_{k-1}}, a_{j_k}, -a_{j_{k/2+2}}\} \text{ if } (k+6)/4 \le i' \le k/2 .$$

It is easy to verify that the D_j's satisfy condition (1) and the cycles constructed from them are base cycles.

To obtain the sequences S_j, partition the elements of D into $t/2$ collections L_j of length $2k$, if t is even and $(t-1)/2$ collections L_j of length $2k$ and a collection L of length k if t is odd; this is done trivially, that is, let L_1 contain the first $2k$ elements, L_2 the second $2k$ elements and so on.

For each L_j, we can construct two sets S_{2j-1} and S_{2j} in the following manner: for each j, $1 \le j \le [t/2]$, $I_{v-1} \subset L_j \subset 2I_{v-1}$, that is, each element of $\{1,2,\ldots,v-1\}$ occurs once or twice in L_j. Put $L_j = M_j \cup 2N_j$ where M_j contains a sequence of elements which occurs once in L_j and N_j of those which occur twice in increasing order. We have $|M_j| = m$ and $|N_j| = n$ with m and n both even, $m + 2n = 2k$, $m + n = 2tyz$. Hence $m = 2(2t-x)yz \ge 4$. In the case where $m = 4$, $v = 2x + 3$ and $k = 2x$, hence $M_j = \{v-4j, v-4j+1, v-4j+2, v-4j+3\}$ with $v - 4j > k/2$.

We can always choose four consecutive elements from M_j, $M_j' = \{q, q+1, q+2, q+3\}$ where q is odd. Put $M_j = M_j' \cup P_j \cup Q_j$ where $|P_j| = |Q_j| = (m-4)/2$, P_j and Q_j containing pairs of consecutive numbers.

Now let S_{2j} consist of the elements of $N_j \cup P_j \cup \{q,q+2\}$ and S_{2j-1} that of $N_j \cup Q_j \cup \{q+1,q+3\}$, in an increasing order. Hence M_j' provides a hook for S_{2j} and a hook for S_{2j-1}. We have to choose M_j', P_j and Q_j so that the hooks do not occur in the middle of the sequences S_{2j}, S_{2j-1}: if $m \geq 8$, this is always possible; when $m=4$, this is also possible since the elements of M_j are all greater than $k/2$. We can obtain D_{2j} and D_{2j-1} from S_{2j} and S_{2j-1} respectively; the elements of D_{2j} and D_{2j-1} satisfy condition (1), hence base cycles can be constructed.

If t is even, the proof is complete; now consider the case where t is odd. Here we are left with L, a collection of the last k elements of D, that is

$$L = I_{(v-1)/2} \cup \{p, p+1, \ldots, v-1\},$$

where $p = (3t-x)yz+1$. As $tyz \equiv tyz+1 \pmod v$, the sequence S_t associated with L contains a hook $\{tyz-1, tyz+1\}$ and hence the rest follows immediately, except when $tyz-2 = (x-t)yz$: that is, the hook occurs in the middle of S. However, in this case, $(2t-x)yz = 2$ which implies that $2t = x+1$ and $yz = 2$, hence $k = 2x$, $v = 2x+3$ and $x+1 \equiv x+2 \pmod v$ and furthermore

$$D_t = \{1,-2,4,-5,6,\ldots,x-1,-x,x+4,-(x+5),x+6,\ldots,2x+1,\\ -(2x+2),-3,x+2\}$$

satisfies condition (1). Hence the proof is complete.

Example. BCD(13,10,5). We have $x = 5$, $yz = 2$, $t = 3$. Hence $D = \{1,2,\ldots,12,1,2,\ldots,12,1,2,\ldots,6\}$. $L_1 = \{1,2,\ldots,12,1,2,\ldots,8\}$, $L = \{9,10,11,12,1,2,\ldots,6\}$. We get $M_1 = \{9,10,11,12\}$ and $N_1 = \{1,2,\ldots,8\}$ and hence $S_1 = \{1,2,\ldots,8,9,11\}$, $S_2 = \{1,2,\ldots,8,10,12\}$ and $S_3 = \{1,2,\ldots,5,7,9,10,11,12\}$. Put

$$D_1 = \{1,-2,3,-4,5,-6,8,-9,11,-7\},$$

$$D_2 = \{1,-2,3,-4,5,-6,8,-10,12,-7\} \text{ and}$$

$$D_3 = \{1,-2,4,-5, 9,-10,11,-12,-3,7\}$$

<u>Remark</u>. In the case where $t = x = 1$, the result is known already, in fact, a BCD(v,k,1) has been constructed for all k and $v \equiv 1 \pmod{2k}$.

LEMMA 5.4. If a BCD(v,k,x) exists, then a BCD(v+(x+1)yz,k,x) exists as well for $k = xyz$ even and x odd.

Proof. Let $v_1 = v-1$ and $v_2 = (x+1)yz$, hence $BCD(v_1+1,xyz,x)$ exists by hypothesis and $BCD(v_2,xyz,x)$ exists by Proposition 5.3. Also, x is odd implies that v_1 must be even. We can apply Lemma 2.3 if xK_{v_1,v_2} can be shown to be decomposable into cycles of length xyz; but this true since xK_{v_1,v_2} is the edge-disjoint union of $x+1$ graphs isomorphic to $K_{v_1,xyz}$, which can be decomposed into k-cycles by Lemma 2.6.

The following theorem can be proved, with the aid of Lemma 5.4, by induction in exactly the same way as Theorem 3.2 was proved. Also, in view of Lemma 2.1, it suffices to prove the theorem for $\lambda = x$.

THEOREM 5.5. Let xyz be even and x be odd. If a $BCD(\overline{v}+2pyz, xyz,x)$ exists for each p, $0 \leq p \leq (x-1)/2$, then there exists a $BCD(\overline{v}+2qyz,xyz,\lambda)$ for any $q \geq 0$ and $\lambda \equiv 0 \pmod x$.

The following theorem is a corollary of Proposition 5.3 and Theorem 5.5.

THEOREM 5.6. There exists a $BCD(2tyz+1,xyz,\lambda)$ for xyz even, x odd, $\lambda \equiv 0 \pmod x$ and $t \geq (x+1)/2$.

<u>6. BCD(v,k,λ) for k even, 4 ≤ k ≤ 16.</u>

It is proved in this section that the necessary conditions for the existence of a $BCD(v,k,\lambda)$ where k is even and $4 \leq k \leq 16$, namely $\lambda v(v-1) \equiv 0 \pmod{2k}$ with $v \geq k$ and $\lambda(v-1) \equiv 0 \pmod 2$, are also sufficient.

The following two theorems have been proved already (see, for example, [8]).

THEOREM 6.1. A necessary and sufficient condition for the existence of a $BCD(v,4,\lambda)$ is

$v \equiv 1 \pmod 8$	for $\lambda \equiv 1$ or $3 \pmod 4$,
$v \equiv 0$ or $1 \pmod 4$	for $\lambda \equiv 2 \pmod 4$,
$v \geq 4$	for $\lambda \equiv 0 \pmod 4$.

THEOREM 6.2. A necessary and sufficient condition for the existence of a BCD(v,6,λ) is

$v \equiv 1$ or $9 \pmod{12}$ for $\lambda \equiv 1$ or $5 \pmod 6$,
$v \equiv 1 \pmod 4$ for $\lambda \equiv 3 \pmod 6$,
$v \equiv 0$ or $1 \pmod 3$ for $\lambda \equiv 2$ or $4 \pmod 6$
$v \geq 6$ for $\lambda \equiv 0 \pmod 6$.

Lemma 2.5 states that the existence of a BCD*(v,k,λ) implies the existence of a BCD(v,k,2λ); but the existence of a BCD*(v,k,λ) for k even and $4 \leq k \leq 16$ has been decided in section 4, therefore in the following five theorems, we need only consider the cases where λ, and hence v, is odd.

THEOREM 6.3. A necessary and sufficient condition for the existence of a BCD(v,8,λ) is

$v \equiv 1 \pmod{16}$ for $\lambda \equiv 1 \pmod 2$,
$v \equiv 0$ or $1 \pmod 8$ for $\lambda \equiv 2$ or $6 \pmod 8$,
$v \equiv 0$ or $1 \pmod 4$ for $\lambda \equiv 4 \pmod 8$,
$v \geq 8$ for $\lambda \equiv 0 \pmod 8$.

The existence of a BCD(v,8,1) where $v \equiv 1 \pmod{16}$ is implied by Theorem 5.7, in fact, Theorem 5.7 implies that a BCD(v,k,1) exists for k even and $v \equiv 1 \pmod{2k}$.

THEOREM 6.4. A necessary and sufficient condition for the existence of a BCD(v,10,λ) is

$v \equiv 1$ or $5 \pmod{20}$ for $\lambda \equiv 1, 3, 7$ or $9 \pmod{10}$,
$v \equiv 0$ or $1 \pmod 5$ for $\lambda \equiv 2, 4, 6$ or $8 \pmod{10}$,
$v \equiv 1 \pmod 4$ for $\lambda \equiv 5 \pmod{10}$,
$v \geq 10$ for $\lambda \equiv 0 \pmod{10}$.

Theorem 5.7 also implies the existence of a BCD with $\lambda = x = 5$, $v \equiv 1 \pmod 4$. The case $v \equiv 5 \pmod{20}$ follows from Theorem 5.6 with $x = 1$, $\overline{v} + 2pyz = 25$ if a BCD(25,10,1) can be constructed; but a BCD(25,10,1) exists with the following base cycles
(0,1,5,6,10,11,15,16,20,21), of order 5, and
(0,2,24,4,23,5,22,7,21,9) of order 25.

The rest of the proofs are similar.

THEOREM 6.5. A necessary and sufficient condition for the existence of a BCD(v,12,λ) is

$v \equiv 1$ or $9 \pmod{24}$ for $\lambda \equiv 1, 5, 7$ or $11 \pmod{12}$,
$v \equiv 0, 1, 4$ or $9 \pmod{12}$ for $\lambda \equiv 2, 8$ or $10 \pmod{12}$,
$v \equiv 1 \pmod 8$ for $\lambda \equiv 3$ or $9 \pmod{12}$,
$v \equiv 0$ or $1 \pmod 3$ for $\lambda \equiv 4 \pmod{12}$,
$v \equiv 0$ or $1 \pmod 4$ for $\lambda \equiv 6 \pmod{12}$,
$v \geq 12$ for $\lambda \equiv 0 \pmod{12}$.

A BCD(33,12,1) exists with a base cycle of order 11 (0,1,6,13,11,12,17,24,22,23,28,2) and a base cycle of order 33, (0,3,32,5,30,6,29,8,28,9,27,11).

THEOREM 6.6. A necessary and sufficient condition for the existence of a BCD(v,14,λ) is

$v \equiv 1$ or $21 \pmod{28}$ for $\lambda \equiv 1, 3, 5, 9, 11$ or $13 \pmod{14}$,
$v \equiv 0$ or $1 \pmod 7$ for $\lambda \equiv 2, 4, 6, 8, 10$ or $12 \pmod{14}$,
$v \equiv 1 \pmod 4$ for $\lambda \equiv 7 \pmod{14}$,
$v \geq 14$ for $\lambda \equiv 0 \pmod{14}$.

A BCD(21,14,1) exists as follows: let the elements be $\{\infty\} \bigcup_{i=1}^{4} V_i$ where $V_i = \{0_i, 1_i, 2_i, 3_i, 4_i\}$ and the three base cycles, each of order 5 be

$B_1 = (0_1,0_2,0_3,0_4,1_1,2_4,2_2,3_3,3_1,3_4,4_2,3_2,2_1,0_3)$,
$B_2 = (\infty,0_1,1_1,0_2,2_3,1_3,2_4,0_4,3_1,0_3,4_4,1_2,3_4,2_2)$ and
$B_3 = (\infty,0_3,2_3,4_1,2_1,4_2,1_2,3_1,1_4,2_4,4_3,0_2,3_3,0_4)$.

Lastly, we have

THEOREM 6.7. A necessary and sufficient condition for the existence of a BCD(v,16,λ) is

$v \equiv 1 \pmod{32}$ for $\lambda \equiv 1 \pmod 2$,
$v \equiv 0$ or $1 \pmod{16}$ for $\lambda \equiv 2, 6, 10, 12$ or $14 \pmod{16}$,
$v \equiv 0$ or $1 \pmod 8$ for $\lambda \equiv 4 \pmod{16}$,
$v \equiv 0$ or $1 \pmod 4$ for $\lambda \equiv 8 \pmod{16}$,
$v \geq 16$ for $\lambda \equiv 0 \pmod{16}$.

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