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LINE GRAPHS OF HYPERGRAPHS I

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We define the k-line graph of a hypergraph H as the graph whose vertices are the edges of H, two vertices being joined if the edges they represent intersect in at least k elements. In this paper we show that for any integer k and any graph G there exists a partial hypergraph H of some complete h-partite hypergraph $K^h_{h\times N}$ such that G is the k-line graph of H. We also prove that, for any integer p, there exist graphs which are not the $(h-p)$-line graph of some h-uniform hypergraph. As a corollary we answer a problem of C. Cook. Further we show that it is not possible to characterize the $(h-1)$-line graphs by excluding a finite number of forbidden induced subgraphs.

1. Introduction

1.1

A hypergraph H will always be defined by its vertex-set X and its edge-set $\mathscr{E} = \{E_i\}_{1\leq i\leq m}$. The hypergraph H is said to be h-uniform if $|E_i| = h$ for each i $(1 \leq i \leq m)$. In what follows we shall always consider h-uniform hypergraphs.

1.2

Let H be a given hypergraph, then we define the *k-line graph* of H, denoted by $L_k(H)$, as the graph (without loops or multiple edges) whose vertices (e_i) can be put in a one-to-one correspondence with the edges E_i in such a way that two vertices e_i and e_j in $L_k(H)$ are adjacent (joined) if and only if their corresponding edges in H, E_i and E_j, have at least k elements in common.

1.3

Let $K^h_{h\times N}$ be the complete h-partite hypergraph defined as follows: its vertex-set X is the union of h disjoint sets X_i $(1\leq i\leq h)$ with $|X_i| = N$, and its edges are all the subsets E of X where $|E| = h$ and $|E\cap X_i| = 1$ for each i $(1\leq i\leq h)$. Properties of $L_k(K^h_{h\times N})$ are studied in [2].

1.4

We define a *k-plane graph* as the k-line graph of an hypergraph H which is a partial hypergraph of $K^h_{h\times N}$ for some N.

1.5

A k-plane graph can be considered as a graph with distinct ordered h-tuples of integers as its vertex-set, in which two vertices are joined if and only if the corresponding h-tuples agree on k or more coordinates.

Such a graph is defined by Cook [3] as a plane graph in the case $k = 1$, and an arrow graph in the case $k = h - 1$.

1.6

In [3] Cook has shown that any graph G is a plane graph, that $K_4 - x$ is not an arrow graph and has asked whether any graph G is, for some h, an $(h - p)$-plane graph where p is a given integer.

In this paper we show that for any k, any graph G is a k-plane graph (Theorem 2.1), but that, for any p, there exist graphs which are not $(h - p)$-line graphs (Theorem 3.3) and consequently, not $(h - p)$-plane graphs. This answers Cook's question.

Furthermore we prove (Theorem 4.3) that, contrary to the line-graphs of graphs [1, Ch. 18], it is not possible to characterize the $(h - 1)$-plane graphs by excluding a finite number of forbidden induced subgraphs.

1.7

In what follows we shall always denote by $d(e_i, e_j)$ the distance between two vertices e_i and e_j of a graph G and by $D(e_i)$ the subgraph of G all of whose vertices are adjacent to e_i in G. We shall define an edge of an h-uniform hypergraph by the sequence of its vertices $\{x_1, x_2, \dots, x_h\}$. The notation $\{x_1, \dots, \hat{x}_j, \dots, x_h\}$ signifies that we consider the sequence of vertices $(x_i)_{1 \leq i \leq h}$ except x_j. All the definitions not given here can be found in [1].

2. Theorem 2.1

Theorem 2.1. *Let G be a given graph, k a given positive integer, then G is a k-plane graph, that is there exist integers h and N and an h-uniform partial hypergraph H of $K^h_{h \times N}$ such that $G = L_k(H)$.*

Proof. Let G be a given graph. We shall prove by induction on the number n of vertices of G that there exist an integer h and a family $(E_i)_{1 \leq i \leq n}$ of h-tuples of integers not greater than n with the following property: there exists a one-to-one correspondence between the vertices $(e_i)_{1 \leq i \leq n}$ of G and the $(E_i)_{1 \leq i \leq n}$ such that for i, j, $1 \leq i < j \leq n$, E_i and E_j agree on at least k coordinates if and only if $\{e_i, e_j\}$ is an edge of G.

The case $n = 1$ is trivial.

Suppose that we have proved it for a graph with $(n - 1)$ vertices. Then if G is a graph with n vertices $e_1, \dots, e_n$, by inductive hypothesis applied to $G - e_n$ we can find an integer l and $(n - 1)$ l-tuples $(F_i)_{1 \leq i \leq n-1}$ of integers not greater than $n - 1$ such that for each i, j, $1 \leq i < j \leq n - 1$, F_i and F_j agree on at least k coordinates if and only if $d(e_i, e_j) = 1$.

Let $(e_i)_{i \in I}$ be the adjacent vertices of e_n in G, $I \subset \{1, \dots, n - 1\}$, and $|I| = d$. We construct n h-tuples $(E_i)_{1 \leq i \leq n}$ with $h = l + dk$, as follows:

$\forall i$, $1 \leq i \leq n - 1$, E_i is obtained from F_i by adding dk coordinates equal to i.

The l first coordinates of E_n are equal to n and, in the dk last ones, k of them are equal to i for each $i \in I$. It is easy to verify that the family (E_i) has the required property.

3. Theorem 3.3

For Theorem 3.3 we need the following lemma:

Lemma 3.1. *Let H be a given h-uniform hypergraph with edges $(E_i)_{1 \leq i \leq n}$, p a given integer, $p < h$, and $G = L_{h-p}(H)$ with vertices (e_i), $1 \leq i \leq n$. If $d(e_i, e_j) = l$, then $|E_i \cap E_j| \geq h - lp$.*

Proof. (By induction on l). According to the definition of $L_{h-p}(H)$ it is true for $l = 1$. Assume we have proved it for $l - 1$. If $d(e_i, e_j) = l$ then there exists a vertex e of G such that $d(e, e_i) = 1$ and $d(e, e_j) = l - 1$ and consequently there exists an edge E of H such that $|E \cap E_i| \geq h - p$ and, by inductive hypothesis, $|E \cap E_j| \geq h - (l - 1)p$. We can write:

$$h \geq |E \cap (E_i \cup E_j)| = |E \cap E_i| + |E \cap E_j| - |E \cap E_i \cap E_j|$$

so $h \geq h - p + h - (l - 1)p - |E \cap E_i \cap E_j|$, therefore $|E_i \cap E_j| \geq |E \cap E_i \cap E_j| \geq h - lp$.

Remark 3.2. In particular, if $d(e_i, e_j) = 2$, then $h - p > |E_i \cap E_j| \geq h - 2p$.

Theorem 3.3. *Let p be a given integer. Then there exist graphs G which are not $L_{h-p}(H)$ for any integer $h > p$ and any h-uniform hypergraph H.*

Proof. We shall show that for any integer p, there exists an integer N depending only on p such that the complete bipartite graph $K_{2, N+1}$ is not $L_{h-p}(H)$ for any integer $h > p$ and any h-uniform hypergraph H. It is sufficient to prove that if $G = L_{h-p}(H)$ and if two vertices e_1 and e_2 of G satisfy $d(e_1, e_2) = 2$, then $D(e_1) \cap D(e_2)$ contains a set of independent vertices of cardinality not greater than N.

Let E_1, E_2 be the corresponding edges of H. From Lemma 3.1, we have: $h-2p \leq |E_1 \cap E_2| < h-p$.

Put

$$|E_1 \cap E_2| = h - q \quad \text{with } p < q \leq 2p, \tag{3.4}$$

$$E_1 = \{x_1, \ldots, x_q, x_{q+1}, \ldots, x_h\},$$

and

$$E_2 = \{y_1, \ldots, y_q, x_{q+1}, \ldots, x_h\},$$

with $y_i \neq x_j$ for any i, j such that $1 \leq i, j \leq q$.

Let e and e' be two other vertices of G both adjacent to e_1 and e_2, and E, E' their corresponding edges in H. We can write

$$E = \{x_i, i \in I \cup K\} \cup \{y_j, j \in J\} \cup \{a_1, \ldots, a_l\},$$

$$E' = \{x_i, i \in I' \cup K'\} \cup \{y_j, j \in J'\} \cup \{b_1, \ldots, b_m\},$$

with I, I', J, J' subsets of $\{1, \ldots, q\}$, K, K' subsets of $\{q+1, \ldots, h\}$ and $a_1, \ldots, a_l$, $b_1, \ldots, b_m$ vertices which do not belong to $E_1 \cup E_2$. Since $d(e, e_1) = d(e', e_2) = 1$, we have

$$|E \cap E_1| = |I| + |K| \geq h - p \tag{3.5}$$

and

$$|E' \cap E_2| = |J'| + |K'| \geq h - p. \tag{3.6}$$

We shall show that if e and e' are not adjacent, then $(I, J) \neq (I', J')$. Indeed assume that $I = I'$ and $J = J'$, then

$$|E \cap E'| \geq |I| + |J| + |K \cap K'|,$$

$$|E \cap E'| \geq |I| + |J| + |K| + |K'| - |K \cup K'|.$$

From inequalities (3.5) and (3.6) we deduce

$$|E \cap E'| \geq 2(h-p) - |K \cup K'|,$$

$$|E \cap E'| \geq 2(h-p) - (h-q),$$

and by (3.4), $|E \cap E'| \geq h - p + q - p > h - p$. This implies that e and e' are adjacent in G.

So the maximum number of independent vertices we can find in $D(e_1) \cap D(e_2)$ is not greater than N', the number of different (I, J), that is $N' = (\sum_{i=1}^{q} C_q^i)^2$. Since, according to (3.4), $N' \leq (\sum_{i=1}^{2p} C_{2p}^i)^2$, this yields the theorem with $N = (\sum_{i=1}^{2p} C_{2p}^i)^2$.

Corollary 3.7 (Answering Cook's problem [3, p. 116]). *For every p, there exist graphs which are not $(h-p)$-plane graphs.*

4. Theorem 4.3

Definition 4.1. Let us denote by $\mathcal{G}_p^h$ the set of graphs G for which there exist integers h and $p, p < h$, and a h-uniform hypergraph H such that $G = L_{h-p}(H)$. Let $\mathcal{G}_p = \bigcup_h \mathcal{G}_p^h$.

Remarks 4.2. We have proved in Theorem 3.3 that $K_{2,N+1} \notin \mathcal{G}_p$. The value obtained in the theorem is not the best possible. For example for $p = 1$, we have $N = 9$, but $K_{2,3} \notin \mathcal{G}_1$ as it can be easily seen.

Moreover there exist non bipartite graphs which do not belong to $\mathcal{G}_p$. For example, for $p = 1$, it can be proved that there are exactly three graphs other than $K_{2,3}$ with less than five vertices which do not belong to $\mathcal{G}_1$: the two non isomorphic graphs obtained by adding an edge to $K_{2,3}$ and the graph obtained from K_5 by deleting an edge.

$\mathcal{G}_1^2$ is nothing else than the class of line graphs of simple graphs, for which many characterizations have been obtained ([1, Ch. 18]) in particular by excluding a finite number of induced subgraphs. If $G \in \mathcal{G}_p$, then any induced subgraph of G belongs to $\mathcal{G}_p$. So we can ask whether there exists a characterization for $\mathcal{G}_p$ by excluding a finite number of induced subgraphs. The next theorem gives a negative answer to this question: we shall exhibit an infinite family of graphs which do not belong to $\mathcal{G}_1$ and whose induced subgraphs belong to $\mathcal{G}_1$.

Theorem 4.3. *Let us denote by W_n the graph which is a wheel with a central vertex e_0 joined to every other vertex e_i, $1 \leq i \leq n-1$ of a cycle of length $n-1$. Thus, for any k, $k \geq 3$,*

(i) *$W_{2k} \notin \mathcal{G}_1$.*

(ii) *Any proper induced subgraph of $W_{2k} \in \mathcal{G}_1$.*

Proof of (i). We break the proof in two parts.

Case 1. $W_{2k+2} \notin \mathcal{G}_1^h$ for $h < k$ and $k \geq 2$.

Suppose that $W_{2k+2} \in \mathcal{G}_1^h$ with $h < k$. Then $K_{1,k}$ which is an induced subgraph of W_{2k+2} belongs to $\mathcal{G}_1^h$. Let e_i, $0 \leq i \leq k$, be the vertices of $K_{1,k}$, with $d(e_i, e_0) = 1$ for $i \in \{1, \ldots, k\}$, and let H be the hypergraph such that $K_{1,k} = L_{h-1}(H)$. Put $E_0 = \{x_1, \ldots, x_h\}$.

Since $d(e_i, e_0) = 1$ for each $i \in \{1, \ldots, k\}$, we must have $|E_i \cap E_0| = h - 1$, so $E_i = (x_1, \ldots, \hat{x}_{r_i}, \ldots, x_h, y_i)$.

If $0 < i < j < k$, we have $d(e_i, e_j) = 2$ and thus, $|E_i \cap E_j| \leq h - 2$ and $r_i \neq r_j$. Therefore we must find k different numbers belonging to $\{1, \ldots, h\}$, which implies $k \leq h$, contradicting $h < k$.

Case 2. $W_{2k+2} \notin \mathcal{G}_1^h$ for $h \geq k$.

We need the following lemma:

Lemma 4.4. *If $G = L_{h-1}(H)$ and if e_1 and e_2 are two adjacent vertices in G then $D(e_1) \cap D(e_2)$ is the vertex-disjoint union of two cliques.*

Proof. Let E_1 and E_2 be the corresponding edges of H. If $E_1 = \{x_1, \ldots, x_h\}$ then $E_2 = \{x_1, \ldots, \hat{x}_r, \ldots, x_h, y\}$ with $y \notin E_1$ and $1 \leq r \leq h$.

If e is both adjacent to e_1 and e_2, then the corresponding edge E of H satisfies $|E \cap E_1| = |E \cap E_2| = h - 1$ and there are only two kinds of such edges:

(a) $E = (x_1, \ldots, \hat{x}_r, \ldots, x_h, z)$ with $z \neq y$ and $z \notin E_1$,

(b) $E = (x_1, \ldots, \hat{x}_s, \ldots, x_h, y)$ with $1 \leq s \leq h$ and $s \neq r$.

All the edges of each kind have $(h-1)$ vertices in common, so the corresponding vertices e form a clique. Moreover, for any $z, z \neq y$ and $z \notin E_1$, and for any $s, 1 \leq s \leq h$ and $s \neq r$, we have

$$|\{x_1, \ldots, \hat{x}_r, \ldots, x_h, z\} \cap \{x_1, \ldots, \hat{x}_s, \ldots, x_h, y\}| = h - 2,$$

thus the vertices corresponding to two edges not of the same kind a or b, are not joined.

Remark 4.5. By this lemma, if in a graph $G = L_{h-1}(H)$ two vertices e and e' are both adjacent to e_1 and e_2, where $d(e_1, e_2) = 1$, but not mutually adjacent, their corresponding edges in H are each of a different kind a or b and so, if one is known, the kind of the other is well determined.

We come back to the proof of the theorem. Assume that there exists an hypergraph H such that $W_{2k+2} = L_{h-1}(H)$. Let us denote by $C(P_{2r+1})$ the graph obtained by joining a vertex e_0 to each point of a path P_{2r+1} of length $2r$. For any $r, 1 \leq r \leq k-1$, $C(P_{2r+1})$ is a subgraph of W_{2k+2}, and thus, $C(P_{2r+1}) = L_{h-1}(H_r)$ where H_r is a partial hypergraph of H. By induction on r, we shall find the necessary form of the edges of H_r for $r \in \{1, \ldots, k-1\}$.

For $r = 1$, if P_3 has vertices e_{2k+1}, e_1, e_2 and edges $\{e_1, e_2\}$ $\{e_1, e_{2k+1}\}$, put $E_0 = \{x_1, \ldots, x_h\}$; without loss of generality, we can assume $E_1 = \{\hat{x}_1, x_2, \ldots, x_h, y_1\}$ with $y_1 \notin E_0$. According to Lemma 4.4, as e_2 and e_{2k+1} are both adjacent to e_0 and e_1 but not mutually adjacent, we can suppose without loss of generality that

$$E_{2k+1} = (\hat{x}_1, x_2, \ldots, x_h, y_0) \quad \text{with } y_0 \neq y_1, \qquad y_0 \notin E_0$$

and $E_2 = (x_1, \hat{x}_2, \ldots, x_h, y_1)$ and thus H_1 is well determined.

Suppose that $H_{r-1} = (X, \mathscr{E})$ with $X = \{x_1, \ldots, x_h, y_0, \ldots, y_{r-1}\}$ and $\mathscr{E} = \{E_{2k+1}, E_i, 0 \leq i \leq 2r-2\}$, where $E_0 = \{x_1, \ldots, x_h\}$, $E_{2k+1} = \{\hat{x}_1, x_2, \ldots, x_h, y_0\}$ with $y_0 \notin E_0$, and for any $i \in \{1, \ldots, r-1\}$:

$$E_{2i-1} = \{x_1, \ldots, \hat{x}_i, \ldots, x_h, y_i\},$$

$$E_{2i} = \{x_1, \ldots, \hat{x}_{i+1}, \ldots, x_h, y_i\},$$

with $y_i \notin E_0$ and $y_i \neq y_j$ for any $j, j \neq i$.

We construct H_r from H_{r-1} by adding two edges E_{2r-1} and E_{2r} corresponding to the vertices e_{2r-1}, e_{2r} we add to $C(P_{2r-1})$ to obtain $C(P_{2r+1})$. First e_{2r-1} and e_{2r-3} are both adjacent to e_0 and e_{2r-2}. Since we have

$$E_{2r-2} = \{x_1, \ldots, \hat{x}_r, \ldots, x_h, y_{r-1}\}$$

and

$$E_{2r-3} = \{x_1, \ldots, \hat{x}_{r-1}, \ldots, x_h, y_{r-1}\},$$

by Lemma 4.4 and Remark 4.5, we must have $E_{2r-1} = \{x_1, \ldots, \hat{x}_r, \ldots, x_h, y_r\}$ with $y_r \neq y_{r-1}$ and $y_r \notin E_0$.

Moreover as $d(e_{2r-1}, e_i) > 1$ for any $i \in \{1, \ldots, 2r-3, 2k+1\}$, then $y_r \neq y_j$ for any $j \in \{0, \ldots, r-1\}$.

Furthermore e_{2r} and e_{2r-2} are both adjacent to e_0 and e_{2r-1} with $E_{2r-1} = \{x_1, \ldots, \hat{x}_r, \ldots, x_h, y_r\}$ and $E_{2r-2} = \{x_1, \ldots, \hat{x}_r, \ldots, x_h, y_{r-1}\}$. So by Lemma 4.4 and Remark 4.5 we must have $E_{2r} = \{x_1, \ldots, \hat{x}_s, \ldots, x_h, y_r\}$ with $s \in \{1, \ldots, h\}$, $s \neq r$. Moreover, as $d(e_{2r}, e_i) > 1$ for any $i \in \{1, \ldots, r-2\}$ then we have $s > r$. Without loss of generality, we can take $s = r+1$. This is possible for $r+1 \leq k \leq h$, and we have $E_{2r} = \{x_1, \ldots, \hat{x}_{r+1}, \ldots, x_h, y_r\}$. So for $r = k-1$ we have found a unique hypergraph H_{k-1} such that $C(P_{2k-1}) = L_{h-1}(H_{k-1})$.

If we want to construct H_k such that $C(P_{2k+1}) = L_{h-1}(H_k)$ there are two cases:

(a) $k = h$. By the method just shown, we find $E_{2k-1} = \{x_1, \ldots, \hat{x}_k, y_k\}$, but then we cannot construct E_{2k} such that e_{2k} is both adjacent to e_0 and e_{2k-1} but not to e_i for $i \in \{1, \ldots, 2k-2\}$ because we cannot find $s > k$ as done before. So $W_{2h+1} \notin \mathscr{G}_1^h$.

(b) $k > h$. By the same method, we find $E_{2k-1} = \{x_1, \ldots, \hat{x}_k, \ldots, x_h, y_k\}$ and $E_{2k} = \{x_1, \ldots, \hat{x}_{k+1}, \ldots, x_h, y_k\}$ with $y_k \neq y_i$ for each $i \in \{1, \ldots, k-1\}$ and $y_k \notin E_0$. But $|E_{2k} \cap E_{2k+1}| = h-2$, so e_{2k} and e_{2k+1} cannot be adjacent. Thus $W_{2k+1} \notin \mathscr{G}_1^h$.

Proof of (ii). The proof of (i) shows that the subgraph obtained by deleting one vertex (here e_{2k}) belongs to $\mathscr{G}_1$. So any induced subgraph of W_{2k} containing e_0 belongs to $\mathscr{G}_1$. If an induced subgraph does not contain e_0, it is a subgraph of a cycle which belongs to $\mathscr{G}_1$.

Problems. It would be interesting:

(a) to find a simple way to construct all the graphs which do not belong to $\mathscr{G}_1$,

(b) to study the class $\mathscr{G}_p$, for $p > 1$; in particular, is it possible to characterize the class $\mathscr{G}_p$ by excluding forbidden induced subgraphs?

Note added in proof. Further results will appear in M.C. Heydemann and D. Sotteau, Line graphs of hypergraphs II, in: Proc. Coll. Keszthely (North-Holland, Amsterdam, 1977).

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