DECOMPOSITION OF COMPLETE GRAPHS INTO ISOMORPHIC SUBGRAPHS WITH FIVE VERTICES

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Abstract

A graph $H$ is said to be $G$-decomposable if the edges of $H$ can be partitioned into subgraphs isomorphic to $G$; this is denoted by $H \rightarrow G$. When $H$ is the complete graph on $n$ vertices, $K_n$, the decomposition is called a $G$-design. This paper deals with necessary and sufficient conditions for $K_n \rightarrow G$, where $G$ is a graph on five vertices, none of which is isolated.

I. Introduction.

A graph $H$ is said to be $G$-decomposable if the edges of $H$ can be partitioned into subgraphs isomorphic to $G$. This situation is denoted by $H \rightarrow G$; $H$ is also said to admit a $G$-decomposition. A $G$-design is a $G$-decomposition of $K_n$, the complete graph on $n$ vertices. The existence of a $G$-design for various graphs $G$ has been studied in literature (for reference see [1]); in particular, the case where $G$ is a graph with at most four vertices has been solved completely [2]. In this paper, we consider the case where $G$ is a graph with five vertices, none of which is isolated.

It is easy to see the following:

PROPOSITION 1.1: Let $G$ be a graph with $k$ vertices and $e$ edges. If $K_n \rightarrow G$ then

(i) $n \geq k$

(ii) $n(n-1) \equiv 0 \pmod{2e}$

(iii) $n-1 \equiv 0 \pmod{f}$

where $f$ is the greatest common divisor of the degrees of the vertices of $G$.

In [15], it was proved that these necessary conditions are asymptotically sufficient, that is, there exists an integer $n(G)$ such that $K_n \rightarrow G$ for $n \geq n(G)$, and $n$ satisfying the necessary conditions of Proposition 1.1. Our objective is to prove that these necessary conditions are always sufficient except possibly in some cases to be specified below (usually for certain small values of $n$).

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Table 1 exhibits all graphs on five vertices (none of them isolated). Necessary conditions for $K_n \rightarrow G$ are listed as well, and the last column describes the results obtained in each case. In Section II, some general methods for proving $K_n \rightarrow G$ are given while in Section III, each graph $G$ is considered individually.

In the remainder of the paper, $n$ is assumed to be greater than or equal to five.

II. General Construction Methods.

The necessary and sufficient conditions for $K_n \rightarrow G$ have already been obtained for some graphs $G$ with five vertices. The graphs and the respective references are:

- $G_3$, a path of length four and $G_4$ [8],
- $G_5$, a star of four edges [12,16],
- $G_{10}$, a cycle of length five [3,11] and
- $G_{23}$, the complete graph on five vertices [5].

We will not duplicate the constructions here but we will point out that the methods used in this paper are as a rule simpler.

Generally, two methods are used in construction: the method of differences and the composition method. The first method is well-known (see for example [4] or [1]), hence we will discuss just the second method.

Given a graph $G$, the necessary conditions for $K_n \rightarrow G$ follow immediately from Proposition 1.1. Notice that the maximum degree of a vertex in a graph with five vertices is four and that there are no 3-regular graphs on five vertices, hence condition (iii) in the Proposition 1.1 applies only to those graphs whose vertex-degrees are multiples of 2, namely $G_{10}, G_{15}, G_{17}$ and $G_{23}$. We then proceed to investigate whether a necessary condition is also sufficient.

To prove $K_n \rightarrow G$, we will instead show that $K_{n_i} \rightarrow G$ and $K_{n_1, n_2, \ldots, n_h} \rightarrow G$ for some values of $i$, on account of the following two lemmas (for whose proofs see [1]):

**Lemma A.** If $K_{n_i} \rightarrow G$ for $1 \leq i \leq h$ and $K_{n_1, n_2, \ldots, n_h} \rightarrow G$,

then $K_n \rightarrow G$ where $n = \sum_{i=1}^{h} n_i$. 

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LEMMA B. If \( n_{i+1} \rightarrow G \) for \( 1 \leq i \leq h \) and \( n_i, n_{i+1}, \ldots, n_h \rightarrow G \), then
\[
K_n \rightarrow G \quad \text{where} \quad n = \sum_{i=1}^h n_{i+1}.
\]

The graph \( K_{n_1, n_2, \ldots, n_h} \) is the complete multipartite graph on
vertices \( X = \bigcup_{i=1}^h X_i \) where \( |X_i| = n_i \) with edges joining each pair of
vertices from different sets \( X_i, X_j, i \neq j \).

In this paper, the value of \( h \) is 2, 3, 4, or 5. Unless otherwise stated, the vertices of \( K_{n_1} \), that is, the elements of \( X_1 \) will
be the elements of a group \( \Gamma \), which is isomorphic to \( \mathbb{Z}_{n_1} \), the group
of residues modulo \( n_1 \), or \( \mathbb{Z}_x \times \mathbb{Z}_y \), where \( x \cdot y = n_1 \); the elements of
\( K_{n_1+1} \) will be \( \Gamma \cup \{\infty\} \). If \( n = \sum_{i=1}^h n_i \) (or \( n = \sum_{i=1}^h n_{i+1} \)), the
 elements of \( K_n \) will be \( \bigcup_{i=1}^h X_i \) (or \( \bigcup_{i=1}^h X_i \cup \{\infty\} \) respectively).

We now have two different types of constructions to consider:

**TYPE I:** \( K_{n_1} \rightarrow G \).

**TYPE II:** \( K_{n_1, n_2, \ldots, n_h} \rightarrow G \).

Furthermore, since Lemmas A and B can be applied inductively, only some
small values of \( n_i \)'s need be considered.

In Type I construction, we use usually the method of differences
and we will give only the base graphs of the decomposition since the
rest of the graphs can be obtained by applying an automorphism of the
group \( \Gamma \) on the vertices of the base graph, as illustrated in the fol-
lowing example.
EXAMPLE 1. Let $G$ be the graph $G_1 = \begin{array}{ccc} x & y & z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$, which is denoted by $[(x,y,z) (u,v)]$. The proof of $K_6 \to G_1$ is simply:

$V(K_6) = Z_5 \cup \{\omega\}, \{(0,1,3) (\omega,2)\} \mod 5$. The statement indicates the vertices of $K_6$ and a base graph of the decomposition. Hence the graphs of the decomposition are the graphs $[(i,1,1+i,3+i)(\omega,2+i)]$ for $i \in Z_5$.

Similarly for the cases $n = 7, 9$ and $10$:

For $K_7 \to G_1$, we have $V(K_7) = Z_7$, and the base graph is $[(0,1,4) (3,5)] \mod 7$, which means that the graphs of the decomposition are $[(i,1+i,4+i) (3+i,5+i)]$ for $i \in Z_7$.

For $K_9 \to G_1$, $V(K_9) = Z_3 \times Z_3$ and the base graphs are $[((0,0),(0,1),(1,0)) ((1,1),(2,2))] \mod (3,3)$ and $[((0,0),(1,0),(2,0)) (0,1),(2,1))] \mod (-3)$. Hence the graphs of the decomposition are the graphs $[((i,j),(1,i+j),(1+j,i))] ((1+i,1+j),(2+i,2+j))$ for all $i,j \in Z_3$ and the graphs $[((0,i),(1,i),(2,1)) ((0,1+i),(2,1+i))]$ for $i \in Z_3$.

Finally, for $K_{10} \to G_1$, $V(K_{10}) = Z_2 \times Z_5$ and the base graphs are $[((0,0),(0,0),(1,1)) ((1,2),(1,3))] \cup [((1,0),(0,1),(0,3)) ((1,2),(1,4))] \cup [((0,0),(1,0),(0,2)) ((0,1),(1,3))] \mod (-5)$. Hence the graphs of the decomposition are the graphs $[((0,1+i),(0,i),(1,1+i)) ((1,2+i),(1,3+i))]$, $[((i,1),(0,1+i),(0,3+i))) ((1,2+i),(1,4+i))]$ and $[((0,i),(1,i),(0,2+i)) ((0,1+i),(1,3+i))]$ for $i \in Z_5$.

Type II construction is similar and as a rule we indicate only the base graphs of a decomposition; only occasionally do we exhibit all the graphs of a decomposition. Take the case of $K_{n_1,n_2,\ldots,n_h} \to G$ with $n_i = r$ for all $i$; the construction is again illustrated by an example.

EXAMPLE 2. Let $G$ be $G_{13} = \begin{array}{ccccc} u & x & 0 & v \\ y & \end{array}$ and denoted by $[x,y,z,u,v]$.

The proof of $K_{4',4,4} \to G_{13}$ is simply $[0,1,2,3,2,3,3,0] \cup [0,1,3,2,3,2,2,2]$ \mod 4. It is understood that $V(K_{4',4,4}) = \bigcup_{i=1}^{3} X_i$ where $X_i = \{0,1,2,3,4\}$ and that the graphs of the decomposition are $[i_1,(i+3), (2+i), (3+i), 2, 3, 2]$ and $[i_1,i_3,(i+3),1,(2+i),1]$ for $i \in Z_4$.

In the case where $n_1 = r$ for $1 \leq i \leq h - 1$ and $n_h = r' > r$, the vertices of $X_n$ are $g_i$ for $g \in Z_r$ and $\omega, \omega', \omega'', \ldots, \omega(r'-r-1)$.  

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EXAMPLE 3. The base graphs of $K_{4,4,4} \rightarrow G_{13}$ are \([0_1,1_2,0_3,0_2,=] \cup
\{2_2,3_3,3_1,=,1\} \cup \{3_2,1_3,3_1,=,0,1\} \mod 4$. Hence $V(K_{4,4,4}) = \cup_{i=1}^3 X_i$ with $X_i = \{0,1,2,3\}$ for $i = 1$ or $2$ and $X_3 = \{0_3,1_3,2_3,3_3,\}$ for $i = 3$. The graphs of the decomposition are $[i_1,(1+i)2_1,i_2,=], [(2+i)2_2,(3+i)3_3,(3+i)3_1,=,1_1]$ and $[(3+i)2_2,(1+i)3_2,(3+i)3_1,=,1_1]$ for $i \in \mathbb{Z}_4$.

Sometimes a different method is used in proving $K_r,r,r,\ldots,r \rightarrow G$: we let $X_i = \{i-1,1-i+\ldots,1-i+(r-1)1\}$ for $i = 1,2,\ldots,h$. Hence $V(K_r,r,r,\ldots,r) = \mathbb{Z}_r$ and base graphs of the decomposition will be taken modulo $rh$, as seen in:

EXAMPLE 4. The proof of $K_{4,4,4,4} \rightarrow G_{13}$ is: $V(K_{4,4,4,4}) = \mathbb{Z}_{16}$ and the base graph is $[0,10,9,14] \mod 16$, which means that the graphs of the decomposition are $[i_1,10+i,9+i,14+i,7+i]$ for $i \in \mathbb{Z}_{16}$.

To prove that $K_{n_1,n_2,\ldots,n_h} \rightarrow G$, Lemmas C, D, E, and F below are also useful.

**LEMMA C.** If $K_{r_1,r_2} \rightarrow G$ and $K_{r_1,r_2} \rightarrow G$, then $K_{ar_1,br_2+ar_2} \rightarrow G$ for integers $a \geq 1$ and $b$ or $c \geq 1$.

**LEMMA D.** If $K_{r_1,r_2} \rightarrow G$ and $K_{r_1,r_2} \rightarrow G$, then $K_{ar_1,ar_2,br_1+ar_2} \rightarrow G$ for integers $a,b$ with $0 \leq b \leq a$.

**LEMMA E.** If $K_{r,r} \rightarrow G$ and $K_{r,r} \rightarrow G$ then $K_{p,n,p,r,q} \rightarrow G$ for $p \neq 2,6,0 \leq q \leq p$.

**LEMMA F.** If $K_{r,r} \rightarrow G$ and $K_{r,r} \rightarrow G$ then $K_{p,n,p,r,q} \rightarrow G$ for $p \neq 2,3,6,10,14$ and $0 \leq q \leq p$.

Let us remark that in Lemma E, the condition $K_{r,r} \rightarrow G$ is unnecessary if $q = 0$ and the lemma is true for all $p \geq 1$; the condition $K_{r,r} \rightarrow G$ is unnecessary if $q = p$. Similar interpretation can be used in other lemmas, and in the case of Lemma F when $q = 0$, the condition $K_{r,r} \rightarrow G$ is unnecessary and the lemma is true for all $p \neq 2,6$.

The proofs of Lemmas C and D can be found in [1]. Before we prove Lemmas E and F we need two auxiliary results.
PROPOSITION 2.1. If \( p \neq 2,6 \) and \( 0 \leq q \leq p \), then \( K_{p,p,p,q} \rightarrow \{K_4, K_3\} \).

Proof. The existence of a decomposition of \( K_{p,p,p,p} \) into \( K_4 \)'s is equivalent to that of a pair of orthogonal Latin squares of order \( p \); the latter is known to exist for \( p \neq 2 \) or 6 [14]. If we delete the vertices of \( K_{p,p,p,p} \) that are not in \( K_{p,p,p,q} \) from the \( K_4 \)'s which contain them, we get \( K_3 \)'s. Hence it is possible to partition the edges of \( K_{p,p,p,q} \) with \( p \neq 2,6 \) into subgraphs isomorphic to \( K_4 \) or \( K_3 \).

PROPOSITION 2.2. For \( p \neq 2,3,6,10,14 \) and \( 0 \leq q \leq p \), \( K_{p,p,p,p,q} \rightarrow \{K_4, K_3\} \).

The proof is similar to the proof of Proposition 2.1:
\( K_{p,p,p,p,q} \rightarrow K_3 \) is equivalent to the existence of three pairwise orthogonal Latin squares of order \( p \); these are known to exist for \( p \neq 2,3,6,10,14 \) [13].

Now let \( G \) be a graph with vertex set \( \{1,2,\ldots,k\} \) and let \( G \odot S_n \) be the lexicographic product of \( G \) by an independent set of \( n \) elements, that is, a graph with vertex set the disjoint union of \( k \) independent sets \( X_i \), where \( |X_i| = n \) such that two vertices are joined by an edge if and only if they belong to two different sets \( X_i, X_j \) and \( \{i,j\} \) is an edge in \( G \).

It is easy to see that \( K_{p,r,p,r,q,r} = K_p \odot S_r \odot S_r \), which, by Proposition 2.1, is decomposable into subgraphs isomorphic to \( K_4 \odot S_r \) or \( K_3 \odot S_r \) for \( p \neq 2,6 \). Since \( K_4 \odot S_r = K_r \odot S_r, K_3 \odot S_r = K_r \odot S_r, \) both of which by the hypothesis are decomposable into subgraphs isomorphic to \( G \), Lemma E is proved.

The proof of Lemma F is quite similar (except that one uses Proposition 2.2).

III. Constructions for particular graphs.

In this section, we prove the results given in Table I namely, we find necessary and sufficient conditions for \( K_n \rightarrow G \), where \( G \) is any graph on five vertices, none of which is isolated. Most of the constructions will be based on the composition method and the lay-out for each graph \( G \) will be:
(a) the graph $G$, its labels and notation
(b) necessary conditions for $K_n + G$;
(c) constructions showing $K_{n_1} + G$ or $K_{n_1,n_2,\ldots,n_h} + G$ for some small values of $n_i$ in order to start the composition;
(d) compositions, with lemmas and parameters used indicated;
(e) conclusions.

Explanations will be given in the first few cases but after that, only the information is given.

(1) (a) $G_1$: 0—0—0—0, denoted by $[(x,y,z)(u,v)]$
(b) Necessary conditions for $K_n + G_1$: $n \equiv 0$ or $1 \pmod{3}$
(c) $K_{2,3} + G_1$: the subgraphs of the decomposition are

$$([(0_2, l_1, 2_2),(0_1, l_2)] \text{ and } [(0_2, 0_1, 2_2)(1_1, 1_2)]$$

- $K_n + G_1$ for $n = 6, 7, 9$ or $10$: see Example 1 in Section II.
(d) $K_n + G_1$ (with $h = 3$) by Lemma

| $n$ | $n_1$ | $n_2$ | $K_{n_1,n_2} + G$ by Lemma C \\
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$3t, t \geq 4$</td>
<td>A</td>
<td>6</td>
<td>3t-6</td>
</tr>
<tr>
<td>$3t+1, t \geq 4$</td>
<td>B</td>
<td>6</td>
<td>3t-6</td>
</tr>
</tbody>
</table>

(e) Theorem: $K_n + G_1$ if and only if $n \equiv 0$ or $1 \pmod{3}$, $n \geq 5$.

Explanation: $K_{3t} + G_1$ is proved by using Lemma A, which states that if $K_6 + G_1, K_{3t-6} + G_1$, and $K_{6,3t-6} + G_1$, then $K_{3t} + G_1$. The case $K_6 + G_1$ was proved in part (c); $K_{3t-6} + G_1$ is true for $t \geq 4$ by induction, and finally $K_{6,3t-6} + G_1$ is proved by using Lemma C, which in this case states that if $K_{2,3} + G$ then $K_{6,3(t-2)} + G_1$ (since $c = 0$). The case $K_{2,3} + G$ is proved in (c).

The second line in the array can be interpreted similarly.

(2) (a) $G_2$: $x \rightarrow y \rightarrow 0 \rightarrow v$, denoted by $[(x,y,z)(u,v)]$
(b) Necessary conditions for $K_n + G_2$: $n \equiv 0$ or $1 \pmod{8}$
(c) \( K_{2,2,2} \to G_2 \): the base graph is
\[
[(0,1,2,3)(1,2)] \mod (-,3)
\]

- \( K_{2,2,2,2} \to G_2 \): \([(0,1,2,3)(1,0,4)] \cup
\[(0,2,3,4)(0,1,2)] \cup
\[(0,3,4,0)(2,1,4)] \mod (2,-).\]

- \( K_8 \to G_2 \): \( V(K_8) = \mathbb{Z}_7 \cup \{\infty\}, [(0,1,3)(\infty,2)] \mod 7.\)

- \( K_9 \to G_2 \): \( V(K_9) = \mathbb{Z}_9, [(0,1,3)(2,6)] \mod 9.\)

- \( K_{16} \to G_2 \): \( V(K_{16}) = \mathbb{Z}_{15} \cup \{\infty\},
\[(0,1,7)(\infty,2)] \cup [(3,5,8)(2,6)] \mod 15.\)

- \( K_{17} \to G_2 \): \( V(K_{17}) = \mathbb{Z}_{17},
\[(0,1,7)(2,6)] \cup [(3,5,8)(2,10)] \mod 17.\)

- \( K_{40} \to G_2 \): \( V(K_{40}) = \mathbb{Z}_{39} \cup \{\infty\}, [(0,18,19)(1,\infty)] \cup
\[(0,15,17)(1,7)] \cup [(0,13,16)(1,9)] \cup
\[(0,10,14)(1,12)] \cup [(0,7,12)(1,10)] \mod 39.\)

- \( K_{41} \to G_2 \): \( V(K_{41}) = \mathbb{Z}_{41}, \) the first base graph is
\[(0,18,19)(1,21)] \) the other 4 are just
the last four base graphs for the case
\( K_{40} \to G_2, \) except that the elements are
taken modulo 41.
(d) \[ K_n \rightarrow G_2 \] (with \( h=4 \))

by Lemma

<table>
<thead>
<tr>
<th>( n )</th>
<th>( n_1 = n_2 = n_3 )</th>
<th>( n_4 )</th>
<th>( K_{n_1, n_2, n_3, n_4} \rightarrow G_2 ) by Lemma E with ( r=2, p=4t ) and ( q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>24t, ( t \geq 1 )</td>
<td>A</td>
<td>8t</td>
<td>0</td>
</tr>
<tr>
<td>24t+1, ( t \geq 1 )</td>
<td>B</td>
<td>8t</td>
<td>0</td>
</tr>
<tr>
<td>24t+8, ( t \geq 1 )</td>
<td>A</td>
<td>8t</td>
<td>8</td>
</tr>
<tr>
<td>24t+9, ( t \geq 1 )</td>
<td>B</td>
<td>8t</td>
<td>8</td>
</tr>
<tr>
<td>24t+16, ( t \geq 2 )</td>
<td>A</td>
<td>8t</td>
<td>16</td>
</tr>
<tr>
<td>24t+17, ( t \geq 2 )</td>
<td>B</td>
<td>8t</td>
<td>16</td>
</tr>
</tbody>
</table>

(e) Conclusion: \( K_n \rightarrow G_2 \) if and only if \( n \equiv 0 \) or \( 1 \pmod{8} \).

Explanation: The fourth row of the array means that \( K_{24t+9} \rightarrow G_2 \) can be proved to be true by Lemma B if \( K_{8t+1} \rightarrow G_2 \), \( K_9 \rightarrow G_2 \) and \( K_{8t, 8t, 8t, 8} \rightarrow G_2 \).

\( K_{8t+1} \rightarrow G_2 \) can be proved by induction, \( K_9 \rightarrow G_2 \) is in part (c) and \( K_{8t, 8t, 8t, 8} \rightarrow G_2 \) is proved by Lemma E, which states that \( K_{8t, 8t, 8t, 8} \rightarrow G_2 \) if \( K_{2, 2, 2} \rightarrow G_2 \) and \( K_{2, 2, 2, 2} \rightarrow G_2 \), which have been proved in part (c).

The other rows can be interpreted similarly.

(3) \( G_3: \) ~0~0~0~0~0, ~0

\( G_4: \) ~0~0~0~0<0 and \( G_5: \) 0<0

As we mentioned in section II, the necessary and sufficient condition for \( K_n \rightarrow G \), \( G = G_3, G_4 \) or \( G_5 \) has already been found to be \( n \equiv 0 \) or \( 1 \pmod{8} \).

(4) \( G = G_6, G_7, G_8 \) or \( G_9 \)

\[ G_6: \ x \quad 0 \quad u \quad v, \quad G_7: \ x \quad 0 \quad u \quad v, \quad G_8: \ x \quad u \quad v \quad 0 \]

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and \( G_9: \) \[
\begin{array}{c}
\_ \_ \_ \_ \_ \_ \\
0
\end{array}
\]
and are all denoted by \([x, y, z, u, v]\).

These four cases are considered together because the necessary conditions for \( K_n \to G \) for each \( G \) are the same: \( n \equiv 0 \) or \( 1 \) (mod 5); also, they belong to a class of graphs called dragons [11]. It was shown in the same paper [11] that the condition \( n \equiv 0 \) or \( 1 \) (mod 10) is also sufficient for \( K_n \to G \). In a different paper [7], the case \( n \equiv 1 \) or \( 5 \) (mod 10) was considered; it was shown that \( K_n \to G \) exists for \( n \geq 5 \) when \( G = G_6 \) and for \( n > 5 \) when \( G = G_7, G_8 \) or \( G_9 \). The construction used in [7] is based on the method of differences and an additional condition was also satisfied, namely, that each vertex of \( K_n \) occurs in the same number of graphs - such decomposition is usually called a balanced graph design.

We will consider the only unsolved case, \( n \equiv 6 \) (mod 10); we will deal with \( G_6, G_7 \) and \( G_8 \) jointly, and with \( G_9 \) separately. We will use the composition method (which, of course, could be used to deal with the case \( n \equiv 0 \) or \( 1 \) or \( 5 \) (mod 10) as well).

(A) \( G = G_6, G_7 \) or \( G_8 \)

(c) \(- K_{5, 5, 5} \to G: \{ u \ [0_1, 0_{j+1}, 1_{j+2}, 2_{j+3}, a] \} \mod 5 \)

where \( a = 3_{j+2}, 3_{j+1} + 0_{j+2} \)

\( G = G_6, G_7 \) or \( G_8 \) respectively.

\(- K_{5, 5, 5} \to G: \{ u \ [0_1, 1_{j+2}, 0_{j+1}, 2_{j+3}, a]\} \mod 4 \)

\([1_4, 3_2, 0_1, 3_3, b] \cup [3_3, 1_1, 0_2, 2_4, c] \mod 5 \)

where \( a_1 = 1_1, 0_3 \) or \( 2_4 \)
\( a_2 = 1_2, 0_4 \) or \( 2_1 \)
\( a_3 = 4_3, 2_1 \) or \( 0_2 \)
\( a_4 = 3_4, 3_2 \) or \( 0_3 \)
\( b = 2_3, 4_1 \) or \( 4_2 \)
\( c = 3_4, 4_2 \) or \( 0_1 \)

for \( G = G_6, G_7 \) or \( G_8 \) respectively.

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\(- K_6 \rightarrow G: \{[1,5,0,2,4],[4,0,3,1,2],[2,3,5,4,1]\}\)
\(- K_6 \rightarrow G_7: \{[1,5,0,2,3],[4,0,3,1,5],[2,3,5,4,0]\}\)
\(- K_6 \rightarrow G_8: \{[1,5,0,2,3],[4,1,3,0,2],[5,2,4,3,0]\}\)
\(- K_{26} \rightarrow G: \text{V}(K_{26}) = \mathbb{Z}_2 \times \mathbb{Z}_{13}; \)
\[
\{(0,0),(0,1),(1,0),(0,4),a\} \cup \{(0,0),(0,2),(1,3),(0,5),b\} \]
\[
\cup \{(0,0),(0,3),(1,5),(1,4),c\} \cup \{(1,0),(1,1),(0,3),(1,3),d\} \]
\[
\cup \{(1,0),(1,2),(0,6),(1,4),e\} \mod (-,13).\]

where \(a = (0,7),(0,6)\) or \((0,10)\)
\(b = (1,8),(1,6)\) or \((1,11)\)
\(c = (1,11),(1,8)\) or \((0,9)\)
\(d = (1,6),(1,5)\) or \((1,8)\)
\(e = (1,8),(1,6)\) or \((1,10)\)

for \(G = G_6, G_7\) or \(G_8\) respectively.

\(- K_{36} \rightarrow G: \text{V}(K_{36}) = \mathbb{Z}_5 \times \mathbb{Z}_7 \cup \{\infty\},\)
\[
3
\{
\{u \cup [(0,0),(0,j),(2,q_j),(1,j),a]\} \mod (5,7)
\}
\]

where \(q_1 = 3, q_2 = 6\) and \(q_3 = 1\)

and \(a = (1,3+2j),(1,3+j)\) or \((0,4)\) for \(G = G_6, G_7\) or \(G_8\);
the remaining three base graphs are obtained from \(K_6 \rightarrow G\)
with \(\infty\) replacing the element 0 and \((1,0)\) replacing \(i, i = 1,2,\ldots,5.\)

For instance, in the case \(G = G_6\), the three graphs are
\[
\{(1,0),(5,0),\infty,(2,0),(4,0)\} \cup \{(4,0),\infty,(3,0),(1,0),(2,0)\}
\]
\[
\cup \{(2,0),(3,0),(5,0),(4,0),(1,0)\} \mod (-,7).\]

\[(d)\]

<table>
<thead>
<tr>
<th>(n)</th>
<th>(K \rightarrow G) by Lemma</th>
<th>(n_1 = n_2 = n_3)</th>
<th>(n_4)</th>
<th>(K_{n_1,n_2,n_3,n_4} \rightarrow G) by Lemma E with (r=5, p) and (q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>30t+6, (t=2, t\geq 4)</td>
<td>B</td>
<td>10t</td>
<td>5</td>
<td>(2t)</td>
</tr>
<tr>
<td>30t+16, (t\geq 0)</td>
<td>B</td>
<td>10t+5</td>
<td>0</td>
<td>(2t+1)</td>
</tr>
<tr>
<td>30t+26, (t\geq 1)</td>
<td>B</td>
<td>10t+5</td>
<td>10</td>
<td>(2t+1)</td>
</tr>
</tbody>
</table>

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(B) \( G = G_9 \)

(c) - \( K_6 + G_9 \): Assume that \( K_6 + G_9 \), that means we have three graphs in the decomposition, and hence a total of three vertices of degree 3, nine vertices of degree 2 and three vertices of degree 1. However, since a vertex of \( K_6 \) has degree 5, three vertices in \( K_6 \) must each occur as a vertex of degree 3 in one graph and a vertex of degree 2 in another, while the remaining three must each occur as a vertex of degree 2 in two graphs and vertex of degree 1 in the other. One could, with a bit of effort, show that such a decomposition is impossible.

- \( K_{5,5} + G_9 \): \([0,1,0,2,1,1,2] \mod 5\)
- \( K_{16} \to G_9 \): \( V(K_{16}) = Z_5 \times Z_3 \cup \{\cdot\} \)
\[
3 \cup [(0,0),(1,0),(1,1),(1,1),(1,3,0),(1,1)] \cup \\
[(0,0),(4,0),(2,0),(4,1),(1,2)] \cup [(0,0),(2,1),(1,0), \\
(2,2),(3,2)] \cup [(4,0),(3,1),(2,0),(3,2),\cdot] \cup \\
[(1,0),(0,2),(4,1),(4,2),(3,1)] \mod (-3,3).
\]

(d)

<table>
<thead>
<tr>
<th>( K_n \to G_9 ) by Lemma</th>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>( K_{n_1,n_2} \to G_9 ) by Lemma C with</th>
</tr>
</thead>
<tbody>
<tr>
<td>10t+6, ( t \geq 2 )</td>
<td>B</td>
<td>((2t-1)5)</td>
<td>( r_1 = r_2 = 5, a = 2t-1, b = 2, c )</td>
</tr>
</tbody>
</table>

(e) \( K_n \to G_6 \) if and only if \( n \equiv 0 \) or \( 1 \) (mod 5)

\( K_n \to G_7 \) or \( G_8 \) if and only if \( n \equiv 0 \) or \( 1 \) (mod 5), \( n \geq 6 \)

\( K_n \to G_9 \) if and only if \( n \equiv 0 \) or \( 1 \) (mod 5), \( n > 6 \).

(5) \( G_{10} \):

\[
\begin{array}{ccc}
\bullet & - & \bullet \\
\bullet & - & \bullet \\
\end{array}
\]

This case was dealt with already in Section II.

Conclusion: \( K_n \to G_{10} \) if and only if \( n \equiv 1 \) or \( 5 \) (mod 10).
(6) (a) $G_{11}: y \xrightarrow{0} x$, [x,y,z,u,v].

(b) $n \equiv 0, 1, 4 \text{ or } 9 \pmod{12}.$

(c) $K_{4,4,4,4} \to G_{11}: [2_1, 0_3, 1_1, 0_1, 0_2], [3_2, 1_3, 0_1, 1_1, 1_2],$

$[2_2, 2_3, 2_1, 1_1, 1_2], [3_2, 3_3, 1_1, 2_1, 0_2],$

$[2_1, 0_2, 0_3, 1_3, 3_2], [3_1, 1_2, 1_3, 0_3, 2_2],$

$[0_1, 0_2, 2_3, 3_3, 3_2] \text{ and } [1_1, 1_2, 3_3, 2_3, 2_2].$

$- K_{4,4,4,4} \to G_{11}: \text{ V}(K_{4,4,4,4}) = Z_{16}, [0, 7, 5, 6, 2] \pmod{16}$

$- K_{2,2,2,2} \to G_{11}: [0_1, 1_3, 0_2, 1_4, 0_4], [0_2, 1_4, 1_1, 0_3, 1_3],$

$[0_3, 1_1, 1_2, 0_4, 1_4] \text{ and } [0_4, 1_2, 0_1, 1_3, 0_3].$

$- K_{4,4,7} \to G_{11}: [i_1, i_3, (1+i)_2, i_2, \tau^*=],$

$[i_1, (2+i)_2, (3+i)_3, \tau^*, (1+i)_1]$ and

$[i_1, (3+i)_2, \tau^*, (1+i)_3, (2+i)_1]$ for $i \in Z_4$ with

$\tau^* = \tau$ when $i = 0$ or 1

$\tau^* = \tau'$ when $i = 2$ or 3.

$- K_9 \to G_{11}: \text{ V}(K_9) = Z_3 \times Z_3, [(0,0), (0,1), (1,0),$

$(2,0), (0,2)] \cup [(1,0), (2,0), (2,1), (1,1), (1,0), (0,0)] \pmod{(3)}.$

$- K_{12} \to G_{11}: \text{ V}(K_{12}) = Z_{11} \cup \{\tau\}, [1, 0, 5, 3, \tau] \pmod{11}.$

$- K_{13} \to G_{11}: \text{ V}(K_{13}) = Z_{13}, [1, 0, 5, 3, 11] \pmod{11}.$

$- K_{16} \to G_{11}: \text{ V}(K_{16}) = Z_3 \times Z_5 \cup \{\tau\},$

$[(0,0), (0,1), (1,0), (2,0), (1,1)] \cup$

$[(0,0), (1,3), (0,2), \tau, (1,4)] \cup$

$[(1,0), (2,0), (2,1), (1,2), \tau] \cup$

$[(2,0), (2,2), (0,4), (1,3), (2,1)] \pmod{(5)}.$

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\(-K_{21} \to G_{11}: \quad V(K_{21}) = \mathbb{Z}_3 \times \mathbb{Z}_7,
\[(0,1), (0,0), (1,3), (1,1), (1,6)] \cup
\[(1,6), (0,2), (0,0), (1,0), (0,3)] \cup
\[(2,2), (2,0), (1,3), (0,2), (2,1)] \cup
\[(2,1), (2,0), (0,6), (0,4), (2,5)] \cup
\[(1,2), (1,0), (2,3), (2,2), (2,6)] \mod (-7).
\]

\(-K_{24} \to G_{11}: \quad V(K_{24}) = \mathbb{Z}_{23} \cup \{\infty\}, [0,11,4,5,\infty] \cup
[0,10,2,1,5] \mod 23.
\]

\(-K_{52} \to G_{11}: \quad V(K_{52}) = \mathbb{Z}_3 \times \mathbb{Z}_{17} \cup \{\infty\},
\[(0,0), (0,7), (1,7), (1,9), \infty\] \cup
\[(0,1), (0,0), (1,13), (0,5), (0,10)] \cup
\[(0,2), (0,0), (1,6), (0,8), (0,1)] \cup
\[(1,1), (0,0), (1,8), (0,3), (2,0)] \cup
\[(1,5), (1,0), (2,13), (0,6), (2,3)] \cup
\[(1,4), (1,0), (2,6), (0,7), (2,14)] \cup
\[(2,0), (1,0), (1,1), (2,5), (1,7)] \cup
\[(2,3), (1,0), (2,7), (1,10)] \cup
\[(1,3), (1,0), (2,15), (2,14), \infty\] \cup
\[(2,1), (2,0), (0,16), (0,14), (2,4)] \cup
\[(2,2), (2,0), (0,10), (0,9), (2,16)] \cup
\[(2,0), (2,3), (0,4), (0,5), \infty\] \cup
\[(2,0), (2,6), (0,6), (1,13), (2,3)] \mod (-17).
\)

(d) We have two arrays:


<table>
<thead>
<tr>
<th>n</th>
<th>K_n \to G_{11} by Lemma</th>
<th>n_1 = n_2 = n_3</th>
<th>n_4</th>
<th>K_{n_1,n_2,n_3,n_4} \to G_{11} by Lemma E with r p q</th>
</tr>
</thead>
<tbody>
<tr>
<td>33</td>
<td>B</td>
<td>8</td>
<td>8</td>
<td>2 4 4</td>
</tr>
<tr>
<td>36t+13, t=1</td>
<td>B</td>
<td>12</td>
<td>12</td>
<td>4 3 3</td>
</tr>
<tr>
<td>36t+33, t=1</td>
<td>B</td>
<td>20</td>
<td>8</td>
<td>4 5 2</td>
</tr>
</tbody>
</table>

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Note: It is not true that \( K_{2,2,2} \rightarrow G_{11} \) however, (see remark after Lemma F), the condition \( K_{r,r,r} \rightarrow G \) is not necessary for Lemma E when \( p = q \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( K_n \rightarrow G_{11} ) by Lemma</th>
<th>( n_3 )</th>
<th>( a )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>36t, ( t \geq 1 )</td>
<td>A</td>
<td>12t</td>
<td>3t</td>
<td>0</td>
</tr>
<tr>
<td>36t+1, ( t \geq 1 )</td>
<td>B</td>
<td>12t</td>
<td>3t</td>
<td>0</td>
</tr>
<tr>
<td>36t+4, ( t \geq 1 )</td>
<td>B</td>
<td>12t+3</td>
<td>3t</td>
<td>1</td>
</tr>
<tr>
<td>36t+9, ( t \geq 1 )</td>
<td>A</td>
<td>12t</td>
<td>3t+3</td>
<td>3</td>
</tr>
<tr>
<td>36t+12, ( t \geq 1 )</td>
<td>A</td>
<td>12t+4</td>
<td>3t+1</td>
<td>0</td>
</tr>
<tr>
<td>36t+13, ( t \geq 2 )</td>
<td>B</td>
<td>12t+12</td>
<td>3t+4</td>
<td>4</td>
</tr>
<tr>
<td>36t+16, ( t \geq 2 )</td>
<td>B</td>
<td>12t+15</td>
<td>3t</td>
<td>5</td>
</tr>
<tr>
<td>36t+21, ( t \geq 1 )</td>
<td>A</td>
<td>12t+13</td>
<td>3t+3</td>
<td>3</td>
</tr>
<tr>
<td>36t+24, ( t \geq 1 )</td>
<td>A</td>
<td>12t+16</td>
<td>3t+3</td>
<td>4</td>
</tr>
<tr>
<td>36t+25, ( t \geq 0 )</td>
<td>B</td>
<td>12t+8</td>
<td>3t+2</td>
<td>1</td>
</tr>
<tr>
<td>36t+28, ( t \geq 0 )</td>
<td>B</td>
<td>12t+8</td>
<td>3t+2</td>
<td>1</td>
</tr>
<tr>
<td>36t+33, ( t \geq 2 )</td>
<td>A</td>
<td>12t+25</td>
<td>3t+1</td>
<td>7</td>
</tr>
</tbody>
</table>

(e) \( K_n \rightarrow G_{11} \) if and only if \( n \equiv 0, 1, 4 \) or 9 (mod 12).

The next case \( G_{12} \) is similar to the case of \( G_{11} \), hence some proofs are omitted.

(7) (a) \( G_{12} \): \( yO \rightarrow 0v \), \([x,y,z,u,v]\).

(b) \( n \equiv 0, 1, 4 \) or 9 (mod 12).

(c) \( -K_{4,4,4} \rightarrow G_{12} \): If \( K_{2,2,2} \rightarrow G_{12} \), then \( K_{4,4,4} \rightarrow G_{12} \) by Lemma D with \( r_1 = r_2 = r_3 = 2 \), \( a = 2 \) and \( b = 0 \);
\( K_{2,2,2} \rightarrow G_{12} \) is easy to prove, for example,
\([0_3,0_1,0_2,0_3] \) and \([1_3,1_3,1_2,0_2,0_1] \) are two graphs partitioning the edges of \( K_{2,2,2} \).

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\[-K_{4,4,4,4} + G_{12}: \quad V(K_{4,4,4,4}) = Z_{16} \cdot [0,7,5,6,3] \text{ mod } 16,\]
\[-K_{2,2,2,2} + G_{12}: \quad [0,1,1,3,0,2,1,4,0,3], [0,2,1,4,0,3,1,1,0,4],
\quad [1,1,0,3,0,4,1,2,1,3] \quad \text{and} \quad [1,2,0,4,0,1,1,3,1,4].\]
\[-K_{4,4,7} + G_{12}: \quad [0,3,0,1,0,2,1,2,2,3,0,3], [0,1,2,3,3,0,1,3,0,2,2,3,3,0,3,0,3,0,2,2,3,3,0,3,0,3] \mod 4.\]
\[-K_9 + G_{12}: \quad V(K_9) = Z_3 \times Z_3,\]
\quad [(0,1), (0,0), (1,0), (2,0), (1,2)] \cup
\quad [(2,0), (1,0), (1,3), (1,0), (0,2)] \mod (-,3).\]
\[-K_{12} + G_{12}: \quad V(K_{12}) = Z_{11} \cup \{\infty\}, \quad [1,0,5,3,\infty] \text{ mod } 11.\]
\[-K_{13} + G_{12}: \quad V(K_{13}) = Z_{13}, \quad [1,0,5,3,7] \text{ mod } 13.\]
\[-K_{16} + G_{12}: \quad V(K_{16}) = Z_3 \times Z_5 \cup \{\infty\},\]
\quad [(0,1), (0,0), (1,0), (2,0), (1,3)] \cup
\quad [(1,3), (0,0), (0,2), \infty, (1,4)] \cup
\quad [(2,0), (1,0), (2,1), (1,2), \infty] \cup
\quad [(2,0), (2,2), (0,4), (1,3), (0,3)] \mod (-,5).\]
\[-K_{21} + G_{12}: \quad \text{The base graphs are the same as in } K_{21} + G_{11}
\quad \text{with the last elements of each graph replaced by}
\quad (0,4), (1,2), (2,5), (0,2) \text{ and } (2,0), \text{ in the order given.}\]
\[-K_{24} + G_{12}: \quad V(K_{24}) = Z_{23} \cup \{\infty\}, \quad [0,11,4,5,\infty] \cup
\quad [0,10,2,1,3] \text{ mod } 23.\]
\[-K_{52} + G_{12}: \quad \text{The base graphs are the same as those of}
\quad K_{52} + G_{11} \quad \text{with the last elements in each graph replaced}
\quad \text{by } \infty, (1,4), (1,7), (2,10), (1,11), (1,12), (2,7), (2,11), \infty,
\quad (0,13), (0,13), \infty \text{ and } (0,3), \text{ in the order given.}\]

(d) see the case \(G_{11}\).
(a) The conclusion is the same as the case $G_{11}$ as well:

$$K_n \rightarrow G_{12} \text{ if and only if } n \equiv 0, 1, 4 \text{ or } 9 \pmod{12}.$$  

The case $K_{n} \rightarrow G_{13}$ can be solved similarly to the last two cases, i.e. when $G = G_{11}$ or $G_{12}$, hence a part of the proof is omitted; however, due to the fact that we do not know whether $K_{24} \rightarrow G_{13}$ some variations have to be made.

(b) $n \equiv 0, 1, 4 \text{ or } 9 \pmod{12}$.

(c) $K_{4,4,4} \rightarrow G_{13}$: see Example 2 of Section II.

- $K_{4,4,4} \rightarrow G_{13}$: see Example 4 of Section II.

- $K_{2,2,2,2} \rightarrow G_{13}$: $V(K_{2,2,2,2}) = Z_8 \times [0,2,5,3,1], [0,6,1,7,5], [2,4,1,3,7]$ and $[6,4,5,7,3]$.

- $K_{4,4,7} \rightarrow G_{13}$: see Example 3 of Section II.

- $K_{9} \rightarrow G_{13}$: $V(K_{9}) = Z_3 \times Z_3$,

  $$[(1,0),(1,1),(0,0),(0,1),(2,2)] \cup [(2,0),(2,1),(0,0),(1,0),(0,1)] \text{ mod } (-3).$$

- $K_{12} \rightarrow G_{13}$: $V(K_{12}) = Z_{12} \times [3,7,8,2,10], [6,4,3,11,9], [4,8,11,0,5], [10,11,5,6,4], [0,1,8,6,5], [0,2,9,8,7], [0,3,8,10,9], [1,3,5,7,6], [1,9,7,11,10], [2,4,7,6,1]$ and $[2,5,9,11,10]$.

- $K_{13} \rightarrow G_{13}$: $V(K_{13}) = Z_{13} \times [5,0,3,1,6] \pmod{13}$.

- $K_{16} \rightarrow G_{13}$: $V(K_{16}) = Z_3 \times Z_5 \cup \{\}$,

  $$[(0,1),(1,2),(0,3),(2,0),(0,0)] \cup [(0,1),(1,2),(1,0),(0,0),(0,4)] \cup [(2,2),(2,0),(1,0),(1,1),(0,1)] \cup$$

  $$227$$
\[(2,1),(1,2),(2,4),(0,1),(2,0)\] \mod (-5).

- \(K_{21} \rightarrow G_{13}\): \(V(K_{21}) = Z_3 \times Z_7\),
\[
\{(0,0),(0,1),(1,1),(1,4),(0,3)\} \cup \\
\{(0,1),(1,6),(0,0),(1,2),(1,4)\} \cup \\
\{(2,1),(2,4),(1,0),(1,1),(0,4)\} \cup \\
\{(1,1),(2,6),(1,0),(2,2),(2,4)\} \cup \\
\{(0,6),(2,1),(0,2),(2,0),(2,2)\} \mod (-7).
\]

- \(K_{24} \rightarrow G_{13}\): we have no proof for this case, but we conjecture that it is true.

- \(K_{52} \rightarrow G_{13}\): \(V(K_{52}) = Z_3 \times Z_7 \cup \{\ast\}\),
\[
\{\ast,(1,1,6),(0,8),(2,8),(0,0)\} \cup \\
\{(0,0),(0,1),(1,4),(1,1),(0,8)\} \cup \\
\{(0,0),(0,2),(1,8),(1,2),(0,6)\} \cup \\
\{(0,1),(1,11),(0,0),(1,5),(0,4)\} \cup \\
\{(0,1),(1,15),(0,0),(1,13),(0,6)\} \cup \\
\{(1,0),(1,1),(2,4),(2,1),(1,5)\} \cup \\
\{(1,0),(1,2),(2,8),(2,2),(1,9)\} \cup \\
\{(1,1),(2,5),(1,0),(2,13),(2,12)\} \cup \\
\{(1,0),(2,8),(1,1),(2,15),(2,16)\} \cup \\
\{(0,15),(2,10),(1,0),(2,9),(2,11)\} \cup \\
\{(2,0),(2,2),(1,2),(0,2),(0,3)\} \cup \\
\{(0,14),(2,1),(0,8),(2,0),(2,5)\} \cup \\
\{(0,16),(2,1),(0,12),(2,0),(2,6)\} \mod (-17).
\]

In addition, we have

- \(K_{12,12,12,12,12,12,12,12,12,12,12,12} \rightarrow G_{13}\): \(V(K_{12,12,12,12,12,12,12,12,12,12,12,12}) = Z_{72}\),
\[
[0,3,64,31,1] \cup [0,9,67,35,4] \cup [0,20,1,22,7] \cup \\
[0,23,57,10,8] \cup [0,55,71,44,29] \mod 72.
\]
(d) The two arrays are the same as those in the case of
\( G = G_{11} \) with the following exception: since we do not
know whether \( K_{24} \rightarrow G_{13} \) is true, the statements in the
longer array for the cases \( n = 36t \) and \( n = 36t+9 \)
are true only when \( t \geq 1 \) and \( t \neq 2 \). The cases thus
not included in the array, namely \( n = 36t \) and \( n = 36t+9 \)
for \( t = 2 \) can be proved as follows:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( K_n \rightarrow G_{13} ) by Lemma</th>
<th>( n_i, i=1,2,\ldots,h )</th>
<th>( K_{n_1}, n_2, \ldots, n_h \rightarrow G_{13} ) by Lemma</th>
<th>( E ) with ( r=4, p=q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>36t, ( t=2 )</td>
<td>A</td>
<td>( n_1 = 12, i=1,2,\ldots,6 )</td>
<td>- ( (\text{proved in part (c)} )</td>
<td></td>
</tr>
<tr>
<td>36t+9, ( t=2 )</td>
<td>B</td>
<td>( n_1 = 20, i=1,2,3,4 )</td>
<td></td>
<td>5</td>
</tr>
</tbody>
</table>

(e) \( K_n \rightarrow G_{13} \) if and only if \( n \equiv 0,1,4 \) or \( 9 \pmod{12} \),
except possibly when \( n = 24 \).

(9) (a) \( G_{14} : \)

| [x,y,z,u,v]. \( (G_{14} \approx K_2,3) \)

(b) \( n \equiv 0,1,4 \) or \( 9 \pmod{12} \).

(c) \( K_9 \rightarrow G_{14} \): The degrees of vertices of \( G_{14} \) are 2 and
3, since a vertex of \( K_9 \) has degree 8, it must
appear either in two subgraphs with degree 3 and in
one subgraph with degree 2 (in which case, the vertex
is said to be type 1), or in four subgraphs with degree
2 (type 2). Suppose that there are \( a \) vertices in \( K_9 \)
which are type 1 and \( b \) vertices which are type 2, then
obviously \( a + b = 9 \). As there are 6 subgraphs
isomorphic to \( G_{14} \) in the decompositions, by counting
in two ways the number of vertices with degree 3 (or 2),
we have

\[
2a = 12 \\
3a + 4b = 15
\]

Solving the equations, we have \( a = 6, b = 3 \). Now,
in a \( G_{14} \), the vertices of degree 2 are adjacent

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only to vertices of degree 3, the three vertices of type 2 can never be joined to each other, a contradiction to $K_9 \rightarrow G_{14}$.

$- K_{12} \rightarrow G_{14}$: the proof is similar, though longer than the one above, and is hence omitted.

$- K_{13} \rightarrow G_{14}$: $V(K_{13}) = Z_{13}, [0,1,2,4,6] \mod 13$.

$- K_{16} \rightarrow G_{14}$: $V(K_{16}) = Z_3 \times Z_5 \cup \{\infty\}$,

$= \{(0,0),(1,2),(0,0),(2,0),(2,1)\} \cup$

$\{(0,0),(0,4),(0,0),(1,0),(2,0)\} \cup$

$\{(0,0),(2,2),(0,2),(2,3),(2,4)\} \cup$

$\{(0,0),(1,0),(1,3),(1,4),(2,2)\} \mod (-,5)$.

$- K_{21} \rightarrow G_{14}$: $V(K_{21}) = Z_3 \times Z_7$,

$= \{(0,0),(1,1),(0,1),(0,3),(1,3)\} \cup$

$\{(0,0),(2,2),(0,2),(1,2),(1,4)\} \cup$

$\{(1,0),(1,2),(0,1),(1,3),(2,6)\} \cup$

$\{(1,0),(2,0),(2,1),(2,2),(2,3)\} \cup$

$\{(2,0),(2,1),(0,2),(0,4),(0,6)\} \mod (-,7)$.

$- K_{24} \rightarrow G_{14}$: $V(K_{24}) = \{\infty,0,1,\ldots,6\} \cup \bigcup_{i=1}^{2} X_i$ where

$X_i = \{0,4,1,\ldots,7\}$. Since $K_{16} \rightarrow G_{14}$, we can construct 20 subgraphs from the vertices of $X_1 \cup X_2$.

The remaining 26 subgraphs are:

$[0,1,3,4,1,4,2], [0,1,0,2,1,2,3]$

$[1,2,4,5,1,5,2], [1,1,1,2,0,2,3]$

$[2,3,5,4,1,4,2], [2,1,2,0,1,3]$

$[3,4,6,5,1,5,2], [3,1,3,2,0,1,2]$

$[4,5,0,1,7,1,2], [6,1,6,2,0,1,2]$

$[5,6,1,4,1,4,2], [6,1,6,2,3,4,5]$

$[6,0,2,5,1,5,2], [7,1,7,2,0,1,2]$

$[7,1,7,2,3,6,\infty]$
and \([\omega_i, i+1, i_1, i_2] \) for \(i = 0, 1, \ldots, 6, \)

\([j_1, j_2, 4, 5, 6] \) for \(j = 0, 1, 2, 3. \)

(d) \[
\begin{array}{c|c|c|c|c|c}
 n & K_n \rightarrow G_{14} & n_1 & n_2 & K_{n_1, n_2} \rightarrow G_{14} \text{ by Lemma } C \\
 & \text{by Lemma} & & & \\
 12t, \ t \geq 3 & B & 12t(t-2)+8 & 15 & a & b \\
 12t+1, \ t \geq 2 & B & 12(t-1) & 12 & 6(t-1)+4 & 5 \\
 12t+4, \ t \geq 2 & B & 12(t-1) & 15 & 6(t-1) & 5 \\
 12t+9, \ t \geq 2 & B & 12(t-1)+8 & 12 & 6(t-1)+4 & 4 \\
\end{array}
\]

(e) \(K_n \rightarrow G_{14} \) iff \(n \equiv 0, 1, 4, 9 \pmod{12}, \ n \geq 13. \)

(10) (a) \(G_{15}: \)

(b) \(n \equiv 1 \) or \(9 \pmod{12}. \)

(c) The case \(n \equiv 1 \pmod{12} \) was solved in [7].

Let \(n = 12t+9. \)

It is well-known that a resolvable decomposition into triangles exists [12], that is \(K_n \rightarrow \) and furthermore, the triangles of the decomposition can be partitioned into \(6t+4\) resolution classes, each containing \(4t+3\) disjoint triangles (and thus covering all the vertices of \(K_n\)).

Partition the resolution classes into pairs (there is an even number of them) and consider a particular pair of resolution classes. By the theorem on simultaneous representatives [4, pg.51] applied to the two classes, we get common representatives of the triangles in them.

Joining the two triangles which contain the common representative \(x\) at \(x\) yields a \(G_{15}'\). Hence we have \(K_{12t+9} \rightarrow G_{15}'. \)

(d) \(K_n \rightarrow G_{15} \) if and only if \(n \equiv 1 \) or \(9 \pmod{12}. \)
(11) (a) \( G_{16} : \begin{array}{c}
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\text{[x,y,z,u,v].}
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\end{array}}

(b) \( n \equiv 0 \text{ or } 1 \pmod{7} \).

(c) - \( K_7 + G_{16} \): use degree argument, similar to the case \( K_9 + G_{14} \).

- \( K_8 + G_{16} \): use degree argument.

- \( K_7,7,7,7 \leftrightarrow G_{16} : [0,1,1,2,1,3,0,4,0,2] \cup [0,1,2,3,6,4,6,3] \cup [0,1,3,2,5,3,4,1,4] \cup [4,2,0,1,0,3,4,3,3] \cup [4,3,0,1,6,2,2,4,6,4] \pmod{7} \).

- \( K_7,7,7,7,7 \leftrightarrow G_{16} : V(K_7,7,7,7,7) = Z_{35} \),
\([0,1,4,13,7] \cup [0,2,8,19,14] \pmod{35} \).

- \( K_{14} + G_{16} : V(K_{14}) = Z_{13} \cup \{\infty\}, [0,1,4,6,\infty] \pmod{13} \).

- \( K_{15} + G_{16} : V(K_{15}) = Z_{15} \cup [0,1,4,6,7] \pmod{15} \).

- \( K_{21} + G_{16} : V(K_{21}) = Z_4 \times Z_5 \cup \{\infty\},
\[ (0,0),(0,1),(1,3),(2,0),(1,1) \cup
(1,0),(0,0),(2,1),(2,3),(3,2) \cup
(0,0),(0,2),(3,2),(3,3),(2,2) \cup
(3,1),(3,3),(2,0),(2,1),(1,0) \cup
(1,0),(1,1),(2,0),(3,4),(1,2) \cup
\{\infty,(0,0),(1,4),(3,4),(2,0) \pmod{(-5)} \).\)

- \( K_{22} + G_{16} : V(K_{22}) = Z_2 \times Z_{11},
\[ (0,0),(0,4),(1,0),(1,1),(0,5) \cup
(1,5),(0,0),(0,2),(0,3),(1,1) \cup
(0,0),(1,4),(1,6),(1,9),(1,10) \pmod{(-11)} \).

- \( K_{28} + G_{16} : V(K_{28}) = Z_{27} \cup \{\infty\}, [0,3,4,14,\infty] \cup
[0,6,8,15,5] \pmod{27} \).
$\mathbb{K}_{29} \to G_{16}$: \( V(\mathbb{K}_{29}) = \mathbb{Z}_{29} \lor [0,1,3,11,6] \lor [0,4,9,16,14] \mod 29.$

$\mathbb{K}_{35} \to G_{16}$: \( V(\mathbb{K}_{35}) = \mathbb{Z}_2 \times \mathbb{Z}_1 \lor \{=\},
\{(0,0),(0,6),(0,7),(1,8),(0,4)\} \lor \{(0,0),(0,3),(0,5),(1,9),(1,13)\} \lor 
\{(0,0),(1,5),(1,11),(1,14),\{\}\} \lor 
\{(1,0),(0,0),(1,10),(1,15),(1,1)\} \lor 
\{(1,0),(1,4),(0,5),(0,14),\{\}\} \mod (-,17).$

$\mathbb{K}_{36} \to G_{16}$: \( V(\mathbb{K}_{36}) = \mathbb{Z}_4 \times \mathbb{Z}_9,
\{(0,0),(0,2),(1,8),(2,4),(0,1)\} \lor 
\{(0,0),(0,4),(1,7),(2,0),(0,3)\} \lor 
\{(1,0),(1,1),(2,0),(3,6),(1,2)\} \lor 
\{(1,0),(1,3),(2,4),(3,3),(1,4)\} \lor 
\{(0,3),(2,0),(2,4),(3,5),(1,4)\} \lor 
\{(0,5),(1,5),(3,0),(3,4),(1,7)\} \lor 
\{(2,0),(2,1),(2,3),(3,3),(0,1)\} \lor 
\{(0,0),(3,0),(3,1),(3,3),(3,6)\} \lor 
\{(1,5),(0,0),(2,3),(3,7),(2,2)\} \lor 
\{(1,4),(0,0),(2,7),(3,5),(3,2)\} \mod (-,9).$

$\mathbb{K}_{42} \to G_{16}$: \( V(\mathbb{K}_{42}) = \mathbb{Z}_{41} \lor \{=\}, \{0,3,4,20,15\} \lor 
\{0,8,10,19,14\} \lor \{0,5,12,18,\} \mod 41.$

$\mathbb{K}_{43} \to G_{16}$: \( V(\mathbb{K}_{43}) = \mathbb{Z}_{43}, \{0,8,9,19,6\} \lor 
\{0,5,7,20,12\} \lor \{0,3,17,21,16\} \mod 43.$

$\mathbb{K}_{49} \to G_{16}$: \( V(\mathbb{K}_{49}) = \mathbb{Z}_7 \times \mathbb{Z}_7,
\{(i,0),(i,1),(i+1,5),(i+3,3),(i+2,j)\}
with \(j = \begin{cases} 4 & \text{for } i = 0,1,2,3,4, \\
0 & \text{for } i = 5,6. 
\end{cases}
\lor \{(i,0),(i+1,1),(i+1,3),(i+3,4),(i+3,j)\}}

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\[ \text{with } j = \begin{cases} 6 & \text{for } i = 0, 1, 2, 3, 4, 6. \\ 0 & \text{for } i = 5. \end{cases} \]
\[ \{ (i, 0), (i+1, 6), (i+3, 1), (i+3, 5), (k, j) \} \]
\[ \text{with } k = 6, j = 0 \text{ for } i = 0 \\
\quad k = i+1, j = 2 \text{ for } i = 1, 2, 3, 4 \\
\quad k = 2, j = 0 \text{ for } i = 5, 6 \]
\[ \{ (1, 0), (2, 0), (3, 0), (0, 0), (4, 0) \} \]
\[ \{ (4, 0), (5, 0), (6, 0), (3, 0), (2, 0) \} \]
\[ \{ (0, 4), (1, 6), (5, 0), (6, 2), (4, 4) \} \mod (-7). \]

- \( K_{50} \to G_{16} : V(K_{50}) = Z_2 \times Z_{25} \)
  \[ \{ (0, 0), (0, 1), (1, 1), (1, 3), (0, 8) \} \cup \\
  \{ (0, 0), (0, 2), (0, 5), (1, 10), (0, 12) \} \cup \\
  \{ (0, 0), (0, 4), (0, 11), (1, 15), (1, 9) \} \cup \\
  \{ (0, 0), (0, 6), (0, 15), (1, 22), (1, 12) \} \cup \\
  \{ (0, 0), (1, 19), (1, 20), (1, 24), (1, 18) \} \cup \\
  \{ (1, 0), (1, 3), (1, 9), (0, 11), (1, 11) \} \cup \\
  \{ (1, 0), (1, 7), (1, 15), (0, 19), (1, 12) \} \mod (-25). \]

- \( K_{56} \to G_{16} : V(K_{56}) = Z_{55} \cup \{ \infty \}, [0, 5, 6, 26, 3] \cup \\
  [0, 11, 13, 25, 17] \cup [0, 10, 28, 32, 19] \cup \\
  [0, 8, 15, 24, \infty] \mod 55. \]

- \( K_{57} \to G_{16} : V(K_{57}) = Z_{57}, [0, 9, 10, 26, 5] \cup \\
  [0, 11, 13, 25, 8] \cup [0, 3, 23, 27, 18] \cup \\
  [0, 6, 21, 28, 19] \mod 57. \]

- \( K_{63} \to G_{16} : V(K_{63}) = Z_2 \times Z_{31} \cup \{ \infty \}, \\
  \{ (0, 0), (0, 1), (0, 13), (1, 0), (1, 11) \} \cup \\
  \{ (0, 0), (0, 2), (0, 6), (1, 8), (1, 12) \} \cup \\
  \{ (0, 0), (0, 3), (0, 10), (1, 13), (1, 16) \} \cup \\
  \{ (0, 0), (0, 5), (0, 16), (1, 20), (1, 19) \} \cup \\
  \{ (0, 0), (0, 8), (0, 17), (1, 22), \infty \} \cup \\

\text{234} \]
[(1,0),(1,1),(1,4),(1,10),(0,8)] \cup
[(1,0),(1,2),(1,13),(0,4),(0,6)] \cup
[(1,0),(1,7),(1,15),(0,14),(0,3)] \cup
[(1,0),(1,5),(1,17),(1,10),\infty] \mod (-,31).

- \mathbb{K}_{64} \rightarrow \mathbb{G}_{16} : \mathbb{V}(\mathbb{K}_{64}) = \mathbb{Z}_7 \times \mathbb{Z}_9 \cup \{\infty\},

\left[(1,0),(1,1),(i+1,1),(i+3,1),(i+2,6) \right] i = 0,1,\ldots,6
\cup \left[(1,0),(1,4),(i+1,7),(i+3,2),(i+2,j) \right]
\text{with } \begin{cases} j = 1 & \text{for } i = 0,1,2,3,4,5 \\ j = 3 & \text{for } i = 6 \end{cases}
\cup \left[(1,0),(i+1,6),(i+1,8),(i+3,4),(k,j) \right]
\text{with } k = 2, j = 3, \text{ for } i = 0
k = i+1, j = 5 \text{ for } i = 1,2
k = i+1, j = 2 \text{ for } i = 3,4,5,6
\cup \left[(1,0),(i+1,4),(i+3,3),(i+3,6),(k,j) \right]
\text{with } k = 3, j = 8 \text{ for } i = 0
k = 5, j = 4 \text{ for } i = 1
k = 6, j = 4 \text{ for } i = 2
k = 6, j = 5 \text{ for } i = 3
k = 0, j = 5 \text{ for } i = 4
k = 0, j = 3 \text{ for } i = 5
k = 2, j = 8 \text{ for } i = 6
\cup \left[(4,1),(0,0),(1,2),(3,5),\infty\right]
\cup \left[(4,5),(1,0),(2,2),(5,1),(6,8) \right]
\cup \left[(5,5),(2,0),(3,2),\infty,(6,1) \right]
\cup \left[(6,4),(0,0),(1,5),\infty,(3,5) \right] \mod (-,9).

- \mathbb{K}_{70} \rightarrow \mathbb{G}_{16} : \mathbb{V}(\mathbb{K}_{70}) = \mathbb{Z}_{69} \cup \{\infty\}, [0,11,13,25,1]
\cup [0,3,29,34,20] \cup [0,10,28,32,30]
\cup [0,17,23,50,21] \cup [0,8,15,24,\infty] \mod 69.

- \mathbb{K}_{71} \rightarrow \mathbb{G}_{16} : \mathbb{V}(\mathbb{K}_{71}) = \mathbb{Z}_{71}, [0,11,13,25,1]
\cup [0,3,29,34,20] \cup [0,10,28,32,30]
\cup [0,17,23,50,19] \cup [0,8,15,24,35] \mod 71.
'K_{77} \rightarrow G_{16}: V(K_{77}) = Z_{4} \times Z_{19} \cup \{\infty\},
[(0,0),(1,0),(2,0),(3,0),\infty]
\cup [(0,\alpha),(1,2\alpha),(2,3\alpha),(3,4\alpha),(0,2\alpha)]
with \alpha = 1, 2, 3, 4, 5, 6, 7, 8, 9
\cup [(1,2\alpha),(0,\alpha),(2,3\alpha),(3,4\alpha),(1,3\alpha-3)]
with \alpha = 10, 11, 12
\cup [(2,3\alpha),(0,\alpha),(1,2\alpha),(3,4\alpha),(2,4\alpha-6)]
with \alpha = 13, 14, 15
\cup [(3,4\alpha),(0,\alpha),(1,2\alpha),(2,3\alpha),(3,5\alpha-9)]
with \alpha = 16, 17, 18
\cup [(i,0),(i,1),(i,4),(1,6),\infty]
for i = 1, 2, 3 \mod (-,19).

K_{78} \rightarrow G_{16}: V(K_{78}) = Z_{2} \times Z_{39},
[(0,0),(0,16),(0,18),(0,19),(1,4)]
\cup [(0,0),(0,8),(0,15),(1,35),(1,6)]
\cup [(0,0),(0,4),(0,13),(1,26),(1,7)]
\cup [(0,0),(0,6),(0,11),(1,3),(1,11)]
\cup [(0,0),(0,10),(0,22),(1,1),(1,14)]
\cup [(1,0),(1,16),(1,18),(1,19),(0,24)]
\cup [(1,0),(1,8),(1,15),(0,37),(0,23)]
\cup [(1,0),(1,4),(1,14),(0,34),(0,15)]
\cup [(1,0),(1,6),(1,11),(0,16),(0,11)]
\cup [(1,0),(1,9),(1,22),(0,1),(0,7)]
\cup [(0,0),(0,14),(1,0),(1,12),(1,33)] \mod (-,39).

K_{91} \rightarrow G_{16}: V(K_{91}) = Z_{2} \times Z_{45} \cup \{\infty\},
[(0,0),(0,1),(0,4),(0,20),\infty]
\cup [(0,0),(0,2),(0,9),(1,10),(1,6)]
\cup [(0,0),(0,5),(0,15),(1,40),(1,11)]
\[ u \ [ (0,0), (0,8), (0,22), (1,44), (1,17) ] \]
\[ u \ [ (0,0), (0,11), (0,24), (1,27), (1,19) ] \]
\[ u \ [ (1,0), (1,1), (1,4), (1,20), \cdots ] \]
\[ u \ [ (1,0), (1,2), (1,8), (0,4), (0,21) ] \]
\[ u \ [ (1,0), (1,5), (1,15), (0,27), (0,15) ] \]
\[ u \ [ (0,0), (0,6), (0,18), (1,20), (1,13) ] \]
\[ u \ [ (0,0), (0,17), (1,0), (1,9), (1,42) ] \]
\[ u \ [ (1,0), (1,7), (1,24), (0,40), (0,14) ] \]
\[ u \ [ (1,0), (1,11), (1,23), (0,30), (0,13) ] \]
\[ u \ [ (1,0), (1,13), (1,27), (0,6), (0,11) ] \mod (-45). \]

- \( K_{92} + G_{16} : \ V(K_{92}) = \mathbb{Z}_4 \times \mathbb{Z}_{23}, \)
  \[ [ (0, \alpha), (1,2 \alpha), (2,3 \alpha), (3,4 \alpha), (0,2 \alpha+1) ] \]
  with \( \alpha = 0, 1, \ldots, 10 \)
  \[ u \ [ (1,2 \alpha), (0, \alpha), (2,3 \alpha), (3,4 \alpha), (1,3 \alpha-4) ] \]
  with \( \alpha = 11, 12, 13, 14 \)
  \[ u \ [ (2,3 \alpha), (0, \alpha), (1,2 \alpha), (3,4 \alpha), (2,4 \alpha-8) ] \]
  with \( \alpha = 15, 16, 17, 18 \)
  \[ u \ [ (3,4 \alpha), (0, \alpha), (1,2 \alpha), (2,3 \alpha), (3,5 \alpha-12) ] \]
  with \( \alpha = 19, 20, 21, 22 \)
  \[ u \ [ (1,0), (1,1), (1,4), (1,6), (1,11) ] \]
  for \( i = 1, 2, 3 \mod (-23). \)

- \( K_{92} + G_{16} : \ V(K_{92}) = \mathbb{Z}_4 \times \mathbb{Z}_{37}, \)
  \[ [ (0, \alpha), (1,2 \alpha), (2,3 \alpha), (3,4 \alpha), (0,2 \alpha+1) ] \]
  with \( \alpha = 0, 1, 2, \ldots, 17 \)
  \[ u \ [ (1,2 \alpha), (0, \alpha), (2,3 \alpha), (3,4 \alpha), (1,3 \alpha-10) ] \]
  with \( \alpha = 18, 19, \ldots, 28 \)
  \[ u \ [ (2,3 \alpha), (0, \alpha), (1,2 \alpha), (3,4 \alpha), (2,4 \alpha-14) ] \]
  with \( \alpha = 29, 30, 31, 32 \)

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\[ u \ [(3,4a), (0,a), (1,2a), (2,3a), (3,5a-18)] \]

with \( \alpha = 33,34,35,36 \)

\[ u \ [(1,0), (1,1), (1,4), (1,6), (1,7)] \]

\[ u \ [(1,0), (1,1), (1,7), (1,11), (1,8)] \]

\[ u \ [(1,0), (1,2), (1,5), (1,14), (1,13)] \]

\( i = 2,3 \mod (-,37). \)

(d)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( K_{n\rightarrow G_{16}} ) by Lemma</th>
<th>( n_1 = n_2 = n_3 = n_4 )</th>
<th>( n_5 )</th>
<th>( K_{n_1,n_2,n_3,n_4,n_5\rightarrow G_{16}} ) by Lemma F with ( r = 7 )</th>
<th>( p^m )</th>
<th>( q^m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>28t, ( t \geq 3 ), ( t #6,17,21,29 )</td>
<td>A</td>
<td>7t</td>
<td>0</td>
<td>t</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>28t+1, ( t \geq 3 ), ( t #6,17,29 )</td>
<td>B</td>
<td>7t</td>
<td>0</td>
<td>t</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>28t+7, ( t \geq 6 ), ( t #7,11,15,18,22,30 )</td>
<td>A</td>
<td>7t-7</td>
<td>35</td>
<td>t-1</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>28t+8, ( t \geq 6 ), ( t #7,11,15,18,30 )</td>
<td>B</td>
<td>7t-7</td>
<td>35</td>
<td>t-1</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>28t+14, ( t \geq 4 ), ( t #6,10,14,17,21,29 )</td>
<td>A</td>
<td>7t</td>
<td>14</td>
<td>t</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>28t+15, ( t \geq 4 ), ( t #6,10,14,17,29 )</td>
<td>B</td>
<td>7t</td>
<td>14</td>
<td>t</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>28t+21, ( t \geq 4 ), ( t #6,10,14,17,21,29 )</td>
<td>A</td>
<td>7t</td>
<td>21</td>
<td>t</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>28t+22, ( t \geq 4 ), ( t #6,10,14,17,29 )</td>
<td>B</td>
<td>7t</td>
<td>21</td>
<td>t</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>28t, ( t = 6,17,21,29 )</td>
<td>A</td>
<td>7t-7</td>
<td>28</td>
<td>t-1</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>28t+1, ( t = 6,17,29 )</td>
<td>B</td>
<td>7t-7</td>
<td>28</td>
<td>t-1</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>28t+7, ( t = 11,15,18,22,30 )</td>
<td>A</td>
<td>7t-14</td>
<td>63</td>
<td>t-2</td>
<td>9</td>
<td></td>
</tr>
</tbody>
</table>

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<table>
<thead>
<tr>
<th>t=11,15,18,30</th>
<th>B</th>
<th>7t-14</th>
<th>63</th>
<th>t-2</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>t=10,14,17,21,29</td>
<td>A</td>
<td>7t-7</td>
<td>42</td>
<td>t-1</td>
<td>6</td>
</tr>
<tr>
<td>t=10,14,17,29</td>
<td>B</td>
<td>7t-7</td>
<td>42</td>
<td>t-1</td>
<td>6</td>
</tr>
<tr>
<td>t=10,14,17,21,29</td>
<td>A</td>
<td>7t-7</td>
<td>49</td>
<td>t-1</td>
<td>7</td>
</tr>
<tr>
<td>t=10,14,17,29</td>
<td>B</td>
<td>7t-7</td>
<td>49</td>
<td>t-1</td>
<td>7</td>
</tr>
</tbody>
</table>

The array is not complete and the missing cases are considered below.

(i) Whether $K_n \to G_{16}$ for $n = 28t+7$ and $t = 4, 5,$ or 7, and $n = 28t+8$ for $t = 4$ or 7 is still undecided.

(ii) $n = 28t+14$ or $28t+15$ for $t = 3$: if we have $K_2,2,2,2,2,2,2 \to K_4$, then it can easily be seen that $K_{14,14,14,14,14,14} \to K_{7,7,7,7}$ and, by Lemma A and B, respectively, we get $K_{98} \to G_{16}$ and $K_{99} \to G_{16}$. The decomposition of $K_2,2,2,2,2,2 \to G_{16}$ is

$$\{ \cup [0^j_0, 0^j_1, 0^j_2, 1^j_0, 1^j_1, 1^j_2] \mod 2.$$ $j=1$

(iii) $n = 28t+14$ or $28t+15$ for $t = 6$: similarly, if $K_{182} \to G_{16}$ and $K_{183} \to G_{16}$, then trivially $K_{14,14,14,14,14,14,14,14,14} \to K_{7,7,7,7} \to G_{16}$ and thus $K_{182} \to G_{16}$ and $K_{183} \to G_{16}$. The decomposition of $K_2,2,2,2,2,2 \to G_{16}$ is given by

$$\{ \cup [0^j_0, 0^j_2, 0^j_3, 0^j_5, 1^j_0, 1^j_1, 1^j_2, 1^j_3] \cup [0^j_0, 0^j_2, 0^j_3, 0^j_5, 1^j_0, 1^j_1, 1^j_2, 1^j_3] \mod 2.$$ $j=1$

(iv) $n = 28t+21$ or $28t+22$ for $t = 3$: if we have $K_3,3,3,3,3 \to K_4$, then $K_{21,21,21,21,21,21} \to K_{7,7,7,7} \to G_{16}$ and by Lemma A and B, respectively, we have $K_{105} \to G_{16}$ and $K_{106} \to G_{16}$. The case $K_3,3,3,3,3 \to K_4$ is proved by letting $V(K_3,3,3,3,3) = Z_{15}$ and the base graph $[0,1,3,7]$ mod 15.
(v) \( n = 28t+21 \) or \( 28t+22 \) for \( t = 6 \): if we have
\[ K_3,3,3,3,3,3,3,3 \rightarrow K_4, \text{ then } K_{21,21,21,21,21,21,21,21} \rightarrow K_{7,7,7,7} \rightarrow G_{16} \] and we have both \( K_{189} \rightarrow G_{16} \) and \( K_{190} \rightarrow G_{16} = K_{3,3,3,...,3} \rightarrow K_4 \) is proved by letting \( V(K_{3,3,...,3}) = Z_3 \times Z_3 \times Z_3 \) and the base graphs \([(0,2,0),(0,0,1),(1,1,2),(1,1,0)] \cup [(0,0,0),(1,2,2),(0,0,2),(1,0,0)] \) mod \((3,3,3)\).

(e) \( K_n \rightarrow G_{16} \) if and only if \( n \equiv 0 \) or \( 1 \pmod{7} \), \( n \neq 7,8 \)
except possibly when \( n = 119,120,147,203,204 \).

(12) (a) \( G_{17} \) \( \rightarrow (x,y,z,u,v) \) \( (G_{17} \cong K_{1,1,3}) \).

(b) \( n \equiv 1 \) or \( 7 \pmod{14} \).

(c) \( K_{7,7,7} \rightarrow G_{17} = [0_2,0_3,1_1,2_1,3_1] \)
\( \cup [0_3,1_1,2_2,3_2] \cup [0_1,0_2,1_3,2_3,3_3] \mod 7. \)

\( K_{7,7,21} \rightarrow G_{17} \): since \( G_{17} = K_{1,1,3} \), the decomposition

\[ r_1 = r_2 = 1, \quad r_3 = 3, \quad a = 7 \quad \text{and} \quad b = 0. \]

\( K_7 \rightarrow G_{17} \): \( V(K_7) = Z_2 \times Z_3 \cup \{\} \),
\([0,0),(1,0),(0,1),(1,1),\{\} \mod (-,3)\). \( K_{15} \rightarrow G_{17} \): \( V(K_{15}) = Z_{15}, [0,1,3,5,7] \mod 15. \)

\( K_{29} \rightarrow G_{17} \): \( V(K_{29}) = Z_{29}, [0,1,4,6,8] \)
\( \cup [0,2,11,12,15] \mod 29. \)

\( K_{49} \rightarrow G_{17} \): since \( K_7 \rightarrow K_3 \) (equivalent to a Steiner triple system of order 7) then \( K_{7,7,7,7,7,7,7} \rightarrow K_{7,7,7} \rightarrow G_{17} \) and, since it has been proved that \( K_7 \rightarrow G_{17} \), we have \( K_{49} \rightarrow G_{17} \) by Lemma A.
(d) \[
\begin{array}{|c|c|c|c|c|}
\hline
n & K \rightarrow G_{17} & n_1 = n_2 & n_3 & n_1, n_2, \rightarrow G_{17} \\
\hline
42t+1, t \geq 1 & B & 14t & 14t & 2t \quad 0 \\
42t+7, t \geq 2 & A & 14t-7 & 14t+21 & 2t-1 \quad 2 \\
42t+15, t \geq 1 & B & 14t & 14t+14 & 2t \quad 1 \\
42t+21, t \geq 0 & A & 14t+7 & 14t+7 & 2t+1 \quad 0 \\
42t+29, t \geq 1 & B & 14t & 14t+28 & 2t \quad 2 \\
42t+35, t \geq 0 & A & 14t+7 & 14t+21 & 2t+1 \quad 1 \\
\hline
\end{array}
\]

(e) \( K_n \rightarrow G_{17} \) if and only if \( n \equiv 1 \) or \( 7 \pmod{14} \).

Remark: The case \( K_n \rightarrow G_{17} \) for \( n \equiv 1 \pmod{14} \) has already been solved, using a different method, in [6], furthermore, the decomposition satisfies an extra condition and is called a balanced G-design for \( G = G_{17} \).

(13) (a) \( G_{18} : \)

(b) \( n \equiv 0 \) or \( 1 \pmod{7} \).

(c) \(- K_8 + G_{18}: \) We use degree argument again. The degree of a vertex in \( K_8 \) is 7 and since the degrees of vertices in \( G_{18} \) are 2 or 3, any vertex of \( K_8 \) must appear in one graph of the decomposition with degree 3 and in two others with degree 2. However, this contradicts the fact that in the four graphs of the decomposition, there are altogether \( 4 \times 4 \) vertices of degree 3 and \( 4 \times 1 \) vertices of degree 2.

\(- K_{14} + G_{18}: \) use similar argument as above.

\(- K_{7,7,7} + G_{18}: \) \([0,0,0,0,1,3,2,2]\) \n\[ [0,4,1,2,2,5] \cup [5,2,1,0,6,5] \mod 7.\]
$K_{7,7,14} \rightarrow G_{18}$: A slightly different notation is used here, namely, though the elements of the first two sets of elements $X_1, X_2$ are still $g_i, g \in Z_7, i = 1, 2$ the third set is now assumed to be a union of two sets $X_3$ and $X'_3$ with elements $g_3$ and $g'_3$ respectively, $g \in Z_7$. The base graphs are

\[ [0_1,0_2,0_3,1_3,2_2] \cup [1_1,2_2,4_3,0_3',1_2] \]
\[ \cup [1_1,1_2,1_3',4_3',0_2'] \cup [5_2,2_3,0_1',6_1,5_3] \]
\[ \cup [3_2,5_3',0_1',6_1,2_3'] \mod 7. \]

$K_{7,7,7,7} \rightarrow G_{18}$:  
\[ [0_1',0_3',0_2',1_2',1_4'] \]
\[ \cup [0_1',2_2,3_3',4_3',4_4'] \cup [1_3,3_4,0_1',6_1,5_2] \]
\[ \cup [0_3',2_1,1_4',4_4',5_2'] \cup [3_2',6_4,0_1',1_3,4_2] \]
\[ \cup [3_4',4_3',5_1',2_2',5_4'] \mod 7. \]

$K_7 \rightarrow G_{18}$:  
$V(K_7) = Z_2 \times Z_3 \cup \{\}:
\{(0,0),(1,0),(0,1),(1,1),\} \mod (-,3)$. 

$K_{15} \rightarrow G_{18}$:  $V(K_{15}) = Z_{15}': [0,1,3,6,10] \mod 15$. 

$K_{22} \rightarrow G_{18}$:  $V(K_{22}) = Z_2 \times Z_{11}$:
\[ [(0,0),(1,6),(1,3),(1,10),(1,5)] \]
\[ \cup [(0,0),(0,1),(0,3),(1,1),(1,0)] \]
\[ \cup [(1,2),(0,4),(0,0),(0,9),(1,5)] \mod (-,11). \]

$K_{29} \rightarrow G_{18}$:  $V(K_{29}) = Z_{29}, [0,1,10,13,24]$
\[ \cup [0,2,8,5,12] \mod 29. \]

$K_{35} \rightarrow G_{18}$:  $V(K_{35}) = Z_2 \times Z_{17} \cup \{\}$,
\[ [(0,9),(1,9),(0,0),(1,2),\} \]
\[ \cup [(1,2),(1,8),(0,0),(0,1),(0,7)] \]
\[ \cup [(1,4),(1,12),(0,0),(0,1),(1,14)] \]
\[ \cup [(0,2),(0,5),(0,1),(0,0),(1,16)] \]
\[ \cup [(1,5),(1,2),(1,1),(1,0),(0,12)] \mod (-,17). \]
<table>
<thead>
<tr>
<th>$n$</th>
<th>$n_1=n_2$</th>
<th>$n_3$</th>
<th>$K_{n_1,n_2,n_3} \rightarrow_{G_{18}}$ by Lemma</th>
<th>$K_n \rightarrow_{G_{18}}$ by Lemma with $r_1=r_2=r_3=7$; $r'_3=14$, a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>42t, $t \geq 2$, $t \neq 3,4,7$</td>
<td>A</td>
<td>14t</td>
<td>14t</td>
<td>2t</td>
<td>0</td>
</tr>
<tr>
<td>42t+1, $t \geq 1$</td>
<td>B</td>
<td>14t</td>
<td>14t</td>
<td>2t</td>
<td>0</td>
</tr>
<tr>
<td>42t+7, $t \geq 2$, $t \neq 3,4,7$</td>
<td>A</td>
<td>14t</td>
<td>14t+7</td>
<td>2t</td>
<td>1</td>
</tr>
<tr>
<td>42t+8, $t \geq 1$, $t \neq 2,6,8$</td>
<td>B</td>
<td>14t</td>
<td>14t+7</td>
<td>2t</td>
<td>1</td>
</tr>
<tr>
<td>42t+14, $t \geq 5$, $t \neq 6,7$</td>
<td>A</td>
<td>14t</td>
<td>14t+14</td>
<td>2t</td>
<td>2</td>
</tr>
<tr>
<td>42t+15, $t \geq 1$</td>
<td>B</td>
<td>14t</td>
<td>14t+14</td>
<td>2t</td>
<td>2</td>
</tr>
<tr>
<td>42t+21, $t \geq 0$</td>
<td>A</td>
<td>14t+7</td>
<td>14t+7</td>
<td>2t+1</td>
<td>0</td>
</tr>
<tr>
<td>42t+22, $t \geq 1$, $t \neq 2,6,8$</td>
<td>B</td>
<td>14t+7</td>
<td>14t+7</td>
<td>2t+1</td>
<td>0</td>
</tr>
<tr>
<td>42t+28, $t \geq 1$, $t \neq 2,3,6$</td>
<td>A</td>
<td>14t+7</td>
<td>14t+14</td>
<td>2t+1</td>
<td>1</td>
</tr>
<tr>
<td>42t+29, $t \geq 1$, $t \neq 2,6,8$</td>
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<td>14t+7</td>
<td>14t+14</td>
<td>2t+1</td>
<td>1</td>
</tr>
<tr>
<td>42t+35, $t \geq 1$</td>
<td>A</td>
<td>14t+7</td>
<td>14t+21</td>
<td>2t+1</td>
<td>2</td>
</tr>
<tr>
<td>42t+36, $t \geq 3$, $t \neq 5,6,7,8$</td>
<td>B</td>
<td>14t+7</td>
<td>14t+21</td>
<td>2t+1</td>
<td>2</td>
</tr>
</tbody>
</table>

(PART B)

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n_1=n_2$</th>
<th>$n_3$</th>
<th>$K_{n_1,n_2,n_3} \rightarrow_{G_{18}}$ by Lemma</th>
<th>$K_n \rightarrow_{G_{18}}$ by Lemma with $r_1=r_2=r_3=7$; $r'_3=14$, a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>42t, $t=4,7$</td>
<td>A</td>
<td>14t-7</td>
<td>14t+14</td>
<td>2t-1</td>
<td>3</td>
</tr>
<tr>
<td>42t+7, $t=3,4,7$</td>
<td>A</td>
<td>14t-7</td>
<td>14t+21</td>
<td>2t-1</td>
<td>4</td>
</tr>
<tr>
<td>42t+8, $t=6,8$</td>
<td>B</td>
<td>14t-7</td>
<td>14t+21</td>
<td>2t-1</td>
<td>4</td>
</tr>
<tr>
<td>42t+14, $t=3,4,6,7$</td>
<td>A</td>
<td>14t-7</td>
<td>14t+28</td>
<td>2t-1</td>
<td>5</td>
</tr>
<tr>
<td>42t+22, $t=2,6,8$</td>
<td>B</td>
<td>14t</td>
<td>14t+21</td>
<td>2t</td>
<td>3</td>
</tr>
</tbody>
</table>
Finally, for \( n = 49 \), since \( K_{49} \rightarrow \{K_7, K_7, K_7, K_7, 7, 7, 7, 7, 7, 7, 7, 7\} \),
\( K_7 \rightarrow G_{18}, K_7, 7, 7 \rightarrow G_{18} \) and \( K_7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7 \rightarrow K_7, 7, 7 \) (due to \( K_7 \rightarrow K_3 \) we have \( K_{49} \rightarrow G_{18} \).

\[(e)\  K_n \rightarrow G_{18} \text{ if and only if } n \equiv 0 \text{ or } 1 \pmod{7}, \ n \geq 7, \ n \neq 8,14 \text{ except possibly when } n = 36,42,56,92,98,120.\]

Remark: if we had been able to show that \( K_n \rightarrow G_{18} \) for \( n = 36,42 \) and 56, then the existence of \( K_n \rightarrow G_{18} \) for \( n = 92,98 \) or 120 would be implied and Parts (B) and (C) of the array in (d) could have been incorporated into Part (A).

(14) (a) \( G_{19} : 0 \rightarrow x \rightarrow y \rightarrow 0, \ [x,y,z,u,v]. \)

(b) \( n \equiv 0 \text{ or } 1 \pmod{7}. \)

(c) \( - K_8 \rightarrow G_{19} : \text{ again by degree argument.} \)

\(- K_7, 7, 7 \rightarrow G_{19} : \text{ } V(K_7, 7, 7) = Z_{21}, [5,7,0,1,15] \pmod{21}. \)
- $K_{7,7,7,7} \to G_{19}$: $[5_2,0_1,1_3,2_1,0_3]$
  $\cup [2_3,4_1,0_4,0_1,3_4] \cup [1_2,1_4,6_1,3_4,0_1]$
  $\cup [4_2,0_1,3_3,3_4,5_4] \cup [0_2,1_3,3_4,0_3,5_4]$
  $\cup [3_2,2_4,3_1,2_2,6_4] \mod 7$.

- $K_{14,14,14,14} \to G_{19}$: $V(K_{14,14,14,14}) = \mathbb{Z}_{56}$
  $[0,13,27,10,18] \cup [0,15,26,19,9]$
  $\cup [0,22,25,23,21] \mod 56$.

- $K_7 \to G_{19}$: $V(K_7) = \mathbb{Z}_2 \times \mathbb{Z}_3 \cup \{=\}$,
  $[(0,0),(1,0),(1,2),(0,1),\ldots] \mod (-,3)$.

- $K_{14} \to G_{19}$: $V(K_{14}) = \mathbb{Z}_{14}$, $[8,9,0,7,10]$
  $[0,3,2,1,12], [0,6,5,4,13], [11,2,10,0,4]$
  $[8,4,3,1,6], [5,13,8,2,11], [11,1,12,8,6]$
  $[5,9,12,7,1], [3,7,10,5,13], [9,3,11,7,6]$
  $[1,4,10,13,7], [6,12,2,7,10] \text{ and } [13,4,9,2,12]$.

- $K_{15} \to G_{19}$: $V(K_{15}) = \mathbb{Z}_{15}$, $[0,5,7,4,6] \mod 15$.

- $K_{22} \to G_{19}$: $V(K_{22}) = \mathbb{Z}_2 \times \mathbb{Z}_{11}$
  $[(0,0),(0,1),(1,10),(1,0),(0,3)]$
  $\cup [(1,7),(0,4),(0,0),(1,4),(1,9)]$
  $\cup [(0,0),(1,2),(1,6),(0,5),(1,8)] \mod (-,11)$.

- $K_{29} \to G_{19}$: $V(K_{29}) = \mathbb{Z}_{29}$, $[1,0,5,8,10]$
  $\cup [2,0,8,19,15] \mod 29$.

- $K_{36} \to G_{19}$: $V(K_{36}) = \mathbb{Z}_4 \times \mathbb{Z}_9$
  $[(0,0),(0,2),(1,5),(1,8),(1,0)]$
  $\cup [(1,2),(0,0),(0,1),(2,8),(3,5)] \mod (4,9)$ and
  $[(0,0),(0,4),(2,5),(2,0),(2,8)]$
  $\cup [(1,0),(1,4),(3,5),(3,0),(3,8)] \mod (-,9)$.

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\(- K_{50} \rightarrow G_{19}: \ V(K_{50}) = Z_2 \times Z_{25}, \)
\([0,0], (1,13), (1,12), (0,12), (1,2)\]
\(\cup \ [(1,16), (1,14), (0,0), (0,11), (1,24)]\)
\(\cup \ [(1,17), (1,20), (0,0), (0,16), (1,8)]\)
\(\cup \ [(0,0), (1,22), (1,18), (0,8), (0,15)]\)
\(\cup \ [(0,0), (0,1), (1,24), (1,19), (0,5)]\)
\(\cup \ [(0,0), (1,21), (1,15), (0,6), (0,18)]\)
\(\cup \ [(0,0), (1,4), (1,11), (0,3), (0,23)] \ mod \ (-, 25).\)

\(- K_{71} \rightarrow G_{19}: \ V(K_{71}) = [0,17,35,10,2]\)
\(\cup \ [3,7,0,29,30] \cup \ [0,14,30,31,38]\)
\(\cup \ [21,9,0,34,41] \cup \ [11,5,0,19,33] \ mod \ 71.\)

\(- K_{92} \rightarrow G_{19}: \ V(K_{92}) = \cup_{i=1}^{13} V_i \cup \{\} \ with \ |V_i| = 7\)
for \(i = 1, 2, \ldots, 13.\)

\(K_{92}\) is the edge-disjoint union of

(i) \(K_{29}\) on the elements of \(\cup_{i=1}^{4} V_i \cup \{\}\)

(ii) three \(K_{22}\)s on the elements of \(V_{5+i} \cup V_{6+i} \cup V_{7+i} \cup \{\}\) for \(i = 0, 3,\) or \(6.\)

(iii) the graph \(K_{3,3,3,4} \otimes S_7.\)

If we can prove that \(K_{3,3,3,4} + \{K_{3},K_{4}\}\) then
\(K_{3,3,3,4} \otimes S_7 + \{K_{7,7,7},K_{7,7,7}\}\) and since \(K_{7,7,7} + G_{19}, K_{7,7,7} + G_{19},\) we have \(K_{92} \rightarrow G_{19}.\) Now \(K_{3,3,3,4} + \{K_{3},K_{4}\}\) with base graphs \([0,4,2,1,2] \cup [0,4,2,2,3] \cup [0,4,2,3,1] \cup [0,4,0,0,2,0] \cup [0,1,1,2,2] \ mod \ 3.\)
<table>
<thead>
<tr>
<th>n</th>
<th>(K_n + G_{19}) by Lemma</th>
<th>(n_1 = n_2 = n_3)</th>
<th>(n_4)</th>
<th>(K_{n_1, n_2, n_3, n_4} + G_{19}) by Lemma E, with r, p, q</th>
</tr>
</thead>
<tbody>
<tr>
<td>2lt, t ≥ 1</td>
<td>A</td>
<td>7t</td>
<td>0</td>
<td>7 t 0</td>
</tr>
<tr>
<td>2lt+1, t ≥ 2</td>
<td>B</td>
<td>7t</td>
<td>0</td>
<td>7 t 0</td>
</tr>
<tr>
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<td>7t</td>
<td>7</td>
<td>7 t 1</td>
</tr>
<tr>
<td>t ≠ 2, 6</td>
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</tr>
<tr>
<td>2lt+8, t ≥ 5,</td>
<td>B</td>
<td>7(t-1)</td>
<td>28</td>
<td>7 t-1 4</td>
</tr>
<tr>
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<tr>
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<td>14</td>
<td>7 t 2</td>
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<tr>
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<td>7 t 2</td>
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<td>t ≠ 2, 6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2lt+8, t = 7</td>
<td>B</td>
<td>14×3</td>
<td>14×2</td>
<td>14 3 2</td>
</tr>
<tr>
<td>2lt+14, t = 2</td>
<td>A</td>
<td>14</td>
<td>14</td>
<td>14 1 1</td>
</tr>
<tr>
<td>2lt+14, t = 6</td>
<td>A</td>
<td>14×3</td>
<td>14</td>
<td>14 3 1</td>
</tr>
<tr>
<td>2lt+15, t = 2</td>
<td>B</td>
<td>14</td>
<td>14</td>
<td>14 1 1</td>
</tr>
<tr>
<td>2lt+15, t = 6</td>
<td>B</td>
<td>14×3</td>
<td>14</td>
<td>14 3 1</td>
</tr>
</tbody>
</table>

Some cases are missing from the array and are considered here.

\(K_{2lt+7} + G_{19}\) for \(t = 2\) and \(6\). The proofs of both cases are similar, hence only one is mentioned here: as \(K_7 + K_3\), we have \(K_{7,7} + (K_7), K_{7,7}, K_{7,7}\), both of which are decomposable into \(G_{19}\), hence \(K_{49} + G_{19}\); for \(K_{133} + G_{19}\), the only difference in the proof is the fact that \(K_{19} + K_3\).

\(K_{14,14,14} + G_{19}\) hence \(K_{14,14,14} + G_{19}\).

\(K_{35} + G_{19}\) if \(K_{7,7,7,7} + G_{19}\), but the latter can be proved:

let \(V(K_{7,7,7,7}) = Z_{35}\) then [0, 1, 4, 13, 8] \(\cup [2, 0, 19, 8, 14]\) \(\mod 35\) are the base graphs.
(e) $K_n \rightarrow G_{19}$ if and only if $n \equiv 0 \text{ or } 1 \pmod{7}$, $n \neq 8$.

(15) (a) $G_{20}:
\begin{align*}
&\text{un}\ x, y, z, u, v, \\
&\text{ov}, \ [x,y,z,u,v].
\end{align*}$

(b) $n \equiv 0 \text{ or } 1 \pmod{16}$.

(c) $K_{4,4,4,4,4} \rightarrow G_{20}:
\begin{align*}
&V(K_{4,4,4,4,4}) = Z_{20}, \ [0,1,7,3,9] \pmod{20}.
&K_{16,16,16,16} \rightarrow G_{20}:
&V(K_{16,16,16,16}) = Z_{64},
&\ [0,1,15,6,34] \cup [13,2,23,0,40]
&\cup [3,45,28,10,0] \pmod{64}.
&K_{17} \rightarrow G_{20}:
&V(K_{17}) = Z_{17}, \ [0,5,7,1,8] \pmod{17}.
&K_{33} \rightarrow G_{20}:
&V(K_{33}) = Z_{33}, \ [0,16,14,1,22]
&\cup [0,3,8,12,10] \pmod{33}.
&K_{49} \rightarrow G_{20}:
&V(K_{49}) = Z_{49}, \ [0,1,20,24,12]
&\cup [0,8,17,2,22] \cup [0,3,16,21,10] \pmod{49}.
&K_{97} \rightarrow G_{20}:
&V(K_{97}) = Z_{97}, \ [1,21,36,0,39]
&\cup [0,2,42,72,33] \cup [0,4,47,84,28]
&\cup [71,94,0,8,80] \cup [16,45,91,0,26]
&\cup [0,90,85,32,49] \pmod{97}.
&K_{113} \rightarrow G_{20}:
&V(K_{113}) = Z_{113}, \ [0,1,36,43,56]
&\cup [0,3,16,108,14] \cup [0,9,48,98,49]
&\cup [27,31,68,0,65] \cup [81,91,93,0,109]
&\cup [47,53,0,30,99] \cup [29,54,0,80,73] \pmod{113}.
&K_{177} \rightarrow G_{20}:
&V(K_{177}) = Z_{177}, \ [0,5,3,26,55]
&\cup [0,6,10,52,67] \cup [12,20,104,0,63]
&\cup [0,31,24,40,58] \cup [48,62,80,0,87]
&\cup [0,124,160,96,102] \cup [0,15,71,143,69]
&\cup [109,142,0,30,9] \cup [41,60,107,0,119]
\[ u \equiv [0,120,82,37,29] \cup [74,164,0,63,75] \mod 177. \]

(e) \( K_n \rightarrow G_{20} \) for \( n = 17,33,49,97,113 \) and 177.

Remark: We can prove \( K_n \rightarrow G_{20} \) for \( n = 16t \) or \( 16t+1 \) if \( K_n \rightarrow G_{20} \) for \( t = 1,2,3,6,7,8,9,11,26 \) or 27. We have been able to solve only six of the twenty cases.

(16) (a) \( G_{21}: \quad [x,y,z,u,v]. \ (G_{21} \cong K_{1,2,2}) \)

(b) \( n \equiv 0 \) or 1 (mod 16).

(c) \(- K_{16} \rightarrow G_{21} \): by degree argument.

- \( K_{8,8,8} \rightarrow G_{21}: \quad V(K_{8,8,8}) = Z_{24}, \ [0,1,5,22,14] \mod 24. \)

- \( K_{16,16,16} \rightarrow G_{21}: \quad K_{16,16,16} \rightarrow K_{8,8,8} \rightarrow G_{21}. \)

- \( K_{17} \rightarrow G_{21}: \quad V(K_{17}) = Z_{17}, \ [0,1,4,9,7] \mod 17. \)

- \( K_{33} \rightarrow G_{21}: \quad V(K_{33}) = Z_{33}, \ [0,2,19,32,24] \)
  \[ u \equiv [0,5,23,29,26] \mod 33. \]

- \( K_{65} \rightarrow G_{21}: \quad V(K_{65}) = Z_{65}, \ [0,8,35,64,52] \)
  \[ u \equiv [0,7,24,63,16] \cup [0,6,40,62,51] \]
  \[ u \equiv [0,5,28,61,15] \mod 65. \]

(d) Proof of \( K_{16+1} \rightarrow G_{21}, \ t \geq 1. \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( K_n \rightarrow G_{21} ) by Lemma</th>
<th>( n_1 = n_2 )</th>
<th>( n_3 )</th>
<th>( K_{n_1, n_2, n_3} \rightarrow G_{21} ) by Lemma D with ( r_1 = r_2 = r_3 = 16, r_4 = 8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>48t+1, ( t \geq 1 )</td>
<td>B</td>
<td>16t</td>
<td>16t</td>
<td>( a ) ( t ) 0</td>
</tr>
<tr>
<td>48t+17, ( t \geq 3 )</td>
<td>B</td>
<td>16t+16</td>
<td>16t-16</td>
<td>( t+1 ) 4</td>
</tr>
<tr>
<td>48t+33, ( t \geq 1 )</td>
<td>B</td>
<td>16t+16</td>
<td>16t</td>
<td>( t+1 ) 2</td>
</tr>
</tbody>
</table>
We need the following decompositions for the induction used in the array.

- \( K_{16,16,8} \rightarrow G_{21} \): by Lemma D with \( r_1 = r_2 = 2, \)
  \( r_3 = r'_3 = 1, \ a = 8 \) and \( b = 0, \)
  \( V(K_{113}) = \bigcup_{i=1}^{7} \mathbb{Z}_{16} \cup \{ \infty \} \)

- \( K_{113} \rightarrow (K_{17}, K_{16,16,16,16,16,16}) \). It has been shown that \( K_{17} \rightarrow G_{21} \) and also \( K_{16,16,16} \rightarrow G_{21} \),
  the last decomposition with \( K_7 \rightarrow K_3 \) implies that \( K_{16,16,16,16,16,16} \rightarrow G_{21} \).

As for the other infinite family, \( n \equiv 0 \pmod{16} \),
if we can prove that \( K_n \rightarrow G_{21} \) for \( n = 16t \), \( t = 2,3,4,5 \)
or 7, then we would have \( K_n \rightarrow G_{21} \) for the whole family by applying Lemma A and using the same method
as displayed in the array. However, of the five values, a decomposition has been obtained only for one, namely
when \( t = 4 \).

- \( K_{64} \rightarrow G_{21} \): \( V(K_{64}) = \bigcup_{i=1}^{3} V_i \cup \{ \infty \}, \)
  \( V_i = \{ 0_1, 1_1, \ldots, 20_i \}; \ [10_1, 0_1, 2_1, 7_1, 15_2] \)
  \( \cup [20_2, 1_1, 0_1, 4_1, 19_3] \cup [9_1, 0_1, 9_2, 2_1, 6_2] \)
  \( \cup [19_3, 0_1, 6_1, 8_2, 14_2] \cup [12_2, \infty, 0_1, 3_2, 14_3] \)
  \( \cup [8_2, 0_2, 1_2, 3_2, 10_3] \cup [0_2, 3_2, 4_3, 9_2, 18_3] \)
  \( \cup [13_2, 0_1, 0_3, 13_3, 17_2] \cup [12_3, 0_1, 6_3, 7_3, 16_3] \)
  \( \cup [7_3, 0_1, 5_3, 3_1, 14_3] \cup [12_2, 0_1, 3_3, 2_1, 10_3] \)
  \( \cup [0_1, 3_2, 9_3, 20_3, 17_3] \pmod{21}. \)

(e) \( K_n \rightarrow G_{21} \) if \( n \equiv 1 \pmod{16} \) and \( n = 64, K_{16} \rightarrow G_{21} \).

(17) (a) \( G_{22} : \ [x, y, z, u, v]. \ (G_{22} \cong K_{1,1,1,2}). \)

(b) \( n \equiv 0 \) or \( 1 \pmod{9} \).

(c) \( K_n \rightarrow G_{22} \) for \( n = 9,10,18 \): by degree argument.

- \( K_{19} \rightarrow G_{22} \): \( V(K_{19}) = \mathbb{Z}_{19}, [0,1,3,9,15] \pmod{19}. \)

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- $K_6,6,6,6 + G_{22}$: $V(K_6,6,6,6) = Z_{24}$, $[0,3,18,1,13] \mod 24$.

(e) $K_n + G_{22}$ for $n = 9, 10, 18$; however, $K_{19} + G_{22}$.

(18) (a) $G_{23}$:

$G_{23}$ is the complete graph with five vertices and, as mentioned in Section II, $K_n + G_{23}$ has already been shown to hold [5]; that is,

$K_n + G_{23}$ if and only if $n \equiv 1$ or $5 \pmod{20}$.

(Equivalently, a balanced incomplete block design, BIBD $(n, 5, 1)$ exists if and only if $n \equiv 1$ or $5 \pmod{20}$.)

REFERENCES


Table 1: on $K_n + G_i$, where $G_i$ is a graph with five vertices, none of which are isolated and $n \geq 5$.

<table>
<thead>
<tr>
<th>Graph</th>
</tr>
</thead>
</table>
| $G_1$: \[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\] |
| $G_2$: \[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\] |
| $G_3$: \[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\] |
| $G_4$: \[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\] |
| $G_5$: \[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\] |
| $G_6$: \[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\] |
| $G_7$: \[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\] |
| $G_8$: \[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\] |
| $G_9$: \[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\] |
| $G_{10}$: \[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\] |
| $G_{11}$: \[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\] |
| $G_{12}$: \[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\] |
| $G_{13}$: \[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\] |
| $G_{14}$: \[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\] |

<table>
<thead>
<tr>
<th>Necessary Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \equiv 0 \text{ or } 1 \pmod{3}$</td>
</tr>
<tr>
<td>$n \equiv 0 \text{ or } 1 \pmod{8}$</td>
</tr>
<tr>
<td>$n \equiv 0 \text{ or } 1 \pmod{5}$</td>
</tr>
<tr>
<td>$n \equiv 0 \text{ or } 1 \pmod{5}$</td>
</tr>
<tr>
<td>$n \equiv 1 \text{ or } 5 \pmod{10}$</td>
</tr>
<tr>
<td>$n \equiv 0 \text{, } 1, 4 \text{ or } 9 \pmod{12}$</td>
</tr>
<tr>
<td>$n \equiv 0, 1, 4 \text{ or } 9 \pmod{12}$, except possibly when $n = 24$</td>
</tr>
<tr>
<td>$n \equiv 0, 1, 4 \text{ or } 9 \pmod{12}$, $n \geq 13$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Result on Sufficient Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>same*</td>
</tr>
<tr>
<td>same</td>
</tr>
<tr>
<td>same</td>
</tr>
<tr>
<td>same</td>
</tr>
<tr>
<td>same</td>
</tr>
<tr>
<td>same</td>
</tr>
<tr>
<td>$n \equiv 0 \text{ or } 1 \pmod{5}$, $n \geq 6^*$</td>
</tr>
<tr>
<td>$n \equiv 0 \text{ or } 1 \pmod{5}$, $n \geq 6^*$</td>
</tr>
<tr>
<td>$n \equiv 0 \text{ or } 1 \pmod{5}$, $n &gt; 6^*$</td>
</tr>
<tr>
<td>same</td>
</tr>
<tr>
<td>same</td>
</tr>
<tr>
<td>same</td>
</tr>
</tbody>
</table>

n = 24

*possibly
<table>
<thead>
<tr>
<th>Graph</th>
<th>Necessary Condition</th>
<th>Result on Sufficient Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{15}$</td>
<td>$n \equiv 1 \text{ or } 9 \pmod{12}$</td>
<td>\text{same}</td>
</tr>
<tr>
<td>$G_{16}$</td>
<td>$n \equiv 0 \text{ or } 1 \pmod{7}$</td>
<td>$n \equiv 0 \text{ or } 1 \pmod{7}$, $n \neq 7, 8$ except possibly when $n = 119, 120, 147, 203, 204^x$</td>
</tr>
<tr>
<td>$G_{17}$</td>
<td>$n \equiv 1 \text{ or } 7 \pmod{14}$</td>
<td>\text{same}</td>
</tr>
<tr>
<td>$G_{18}$</td>
<td>$n \equiv 0 \text{ or } 1 \pmod{7}$</td>
<td>$n \equiv 0 \text{ or } 1 \pmod{7}$, $n \neq 8, 14$ except possibly when $n = 36, 92, 56, 92, 98, 120^x$</td>
</tr>
<tr>
<td>$G_{19}$</td>
<td>$n \equiv 0 \text{ or } 1 \pmod{7}$</td>
<td>$n \equiv 0 \text{ or } 1 \pmod{7}$, $n \neq 8^x$</td>
</tr>
<tr>
<td>$G_{20}$</td>
<td>$n \equiv 0 \text{ or } 1 \pmod{16}$</td>
<td>$n = 17, 33, 49, 97, 113$ and $117^x$</td>
</tr>
<tr>
<td>$G_{21}$</td>
<td>$n \equiv 0 \text{ or } 1 \pmod{16}$</td>
<td>$n \equiv 1 \pmod{16}$, $n \neq 16$, $n = 64^x$</td>
</tr>
<tr>
<td>$G_{22}$</td>
<td>$n \equiv 0 \text{ or } 1 \pmod{9}$</td>
<td>$n \neq 9, 10, 18$, $n = 19^x$</td>
</tr>
<tr>
<td>$G_{23}$</td>
<td>$n \equiv 1 \text{ or } 5 \pmod{20}$</td>
<td>\text{same}</td>
</tr>
</tbody>
</table>

*same*: The necessary conditions are also sufficient.

+ : The necessary conditions are not always sufficient.

x : Complete solution has not been obtained.