DECOMPOSITION OF COMPLETE GRAPHS INTO ISOMORPHIC SUBGRAPHS WITH FIVE VERTICES

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Abstract

A graph H is said to be G-decomposable if the edges of H can be partitioned into subgraphs isomorphic to G; this is denoted by $H \to G$. When H is the complete graph on n vertices, K_n, the decomposition is called a G-design. This paper deals with necessary and sufficient conditions for $K_n \to G$, where G is a graph on five vertices, none of which is isolated.

I. *Introduction.*

A graph H is said to be *G-decomposable* if the edges of H can be partitioned into subgraphs isomorphic to G. This situation is denoted by $H \to G$; H is also said to admit a *G-decomposition*. A *G-design* is a G-decomposition of K_n, the complete graph on n vertices. The existence of a G-design for various graphs G has been studied in literature (for reference see [1]); in particular, the case where G is a graph with at most four vertices has been solved completely [2]. In this paper, we consider the case where G is a graph with five vertices, none of which is isolated.

It is easy to see the following:

PROPOSITION 1.1: *Let G be a graph with k vertices and e edges. If $K_n \to G$ then*

(i) $n \geq k$

(ii) $n(n-1) \equiv 0 \pmod{2e}$

(iii) $n-1 \equiv 0 \pmod{f}$

where f is the greatest common divisor of the degrees of the vertices of G.

In [15], it was proved that these necessary conditions are asymptotically sufficient, that is, there exists an integer $n(G)$ such that $K_n \to G$ for $n \geq n(G)$, and n satisfying the necessary conditions of Proposition 1.1. Our objective is to prove that these necessary conditions are always sufficient except possibly in some cases to be specified below (usually for certain small vaules of n).

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Table 1 exhibits all graphs on five vertices (none of them isolated). Necessary conditions for $K_n \to G$ are listed as well, and the last column describes the results obtained in each case. In Section II, some general methods for proving $K_n \to G$ are given while in Section III, each graph G is considered individually.

In the remainder of the paper, n is assumed to be greater than or equal to five.

II. *General Construction Methods.*

The necessary and sufficient conditions for $K_n \to G$ have already been obtained for some graphs G with five vertices. The graphs and the respective references are:

G_3, a path of length four and G_4 [8],
G_5, a star of four edges [12,16],
G_{10}, a cycle of length five [3,11] and
G_{23}, the complete graph on five vertices [5].

We will not duplicate the constructions here but we will point out that the methods used in this paper are as a rule simpler.

Generally, two methods are used in construction: the method of differences and the composition method. The first method is well-known (see for example [4] or [1]), hence we will discuss just the second method.

Given a graph G, the necessary conditions for $K_n \to G$ follow immediately from Proposition 1.1. Notice that the maximum degree of a vertex in a graph with five vertices is four and that there are no 3-regular graphs on five vertices, hence condition (iii) in the Proposition 1.1 applies only to those graphs whose vertex-degrees are multiples of 2, namely G_{10}, G_{15}, G_{17} and G_{23}. We then proceed to investigate whether a necessary condition is also sufficient.

To prove $K_n \to G$, we will instead show that $K_{n_i} \to G$ and $K_{n_1,n_2,\ldots,n_h} \to G$ for some values of i, on account of the following two lemmas (for whose proofs see [1]):

LEMMA A. *If* $K_{n_i} \to G$ *for* $1 \le i \le h$ *and* $K_{n_1,n_2,\ldots,n_h} \to G$, *then* $K_n \to G$ *where* $n = \sum_{i=1}^{h} n_i$.

LEMMA B. *If* $K_{n_i+1} \to G$ *for* $1 \le i \le h$ *and* $K_{n_1,n_2,\ldots,n_h} \to G$, *then* $K_n \to G$ *where* $n = \sum_{i=1}^{h} n_i+1$.

The graph $K_{n_1,n_2,\ldots,n_h}$ is the complete multipartite graph on vertices $X = \bigcup_{i=1}^{h} X_i$ where $|X_i| = n_i$ with edges joining each pair of vertices from different sets X_i, X_j, $i \neq j$.

In this paper, the value of h is 2, 3, 4, or 5. Unless otherwise stated, the vertices of K_{n_i}, that is, the elements of X_i will be the elements of a group Γ, which is isomorphic to Z_{n_i}, the group of residues modulo n_i, or $Z_x \times Z_y$ where $x.y = n_i$; the elements of K_{n_i+1} will be $\Gamma \cup \{\infty\}$. If $n = \sum_{i=1}^{h} n_i$ (or $n = \sum_{i=1}^{h} n_i+1$), the elements of K_n will be $\bigcup_{i=1}^{h} X_i$ (or $\bigcup_{i=1}^{h} X_i \cup \{\infty\}$ respectively).

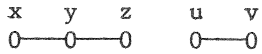
We now have two different types of constructions to consider:

TYPE I: $K_{n_i} \to G$.

TYPE II: $K_{n_1,n_2,\ldots,n_h} \to G$.

Furthermore, since Lemmas A and B can be applied inductively, only some small values of n_i's need be considered.

In Type I construction, we use usually the method of differences and we will give only the *base graphs* of the decomposition since the rest of the graphs can be obtained by applying an automorphism of the group Γ on the vertices of the base graph, as illustrated in the following example.

EXAMPLE 1. Let G be the graph $G_1 =$ 

, which is denoted by $[(x,y,z)\ (u,v)]$. The proof of $K_6 \to G_1$ is simply:

$V(K_6) = Z_5 \cup \{\infty\}$, $[(0,1,3)\ (\infty,2)]$ mod 5. The statement indicates the vertices of K_6 and a base graph of the decomposition. Hence the graphs of the decomposition are the graphs $[(i,1+1,3+i)(\infty,2+i)]$ for $i \in Z_5$.

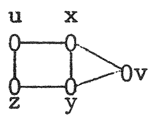
Similarly for the cases $n = 7$, 9 and 10:

For $K_7 \to G_1$, we have $V(K_7) = Z_7$, and the base graph is $[(0,1,4)\ (3,5)]$ mod 7, which means that the graphs of the decomposition are $[(i,1+i,4+i)\ (3+i,5+i)]$ for $i \in Z_7$.

For $K_9 \to G_1$, $V(K_9) = Z_3 \times Z_3$ and the base graphs are $[((0,0),(0,1),(1,0))\ ((1,1),(2,2))]$ mod (3,3) and $[((0,0),(1,0),(2,0))\ ((0,1),(2,1))]$ mod (-,3). Hence the graphs of the decomposition are the graphs $[((i,j),(i,1+j),(1+i,j))\ ((1+i,1+j),(2+i,2+j))]$ for all $i,j \in Z_3$ and the graphs $[((0,i),(1,i),(2,i))\ ((0,i+1),(2,i+1))]$ for $i \in Z_3$.

Finally, for $K_{10} \to G_1$, $V(K_{10}) = Z_2 \times Z_5$ and the base graphs are $[((0,1),(0,0),(1,1))\ ((1,2),(1,3))] \cup [((1,0),(0,1),(0,3))\ ((1,2),(1,4))] \cup [((0,0),(1,0),(0,2))\ ((0,1),(1,3))]$ mod (-,5). Hence the graphs of the decomposition are the graphs $[((0,1+i),(0,i),(1,1+i))\ ((1,2+i),(1,3+i))]$, $[((1,i),(0,1+i),(0,3+i))\ ((1,2+i),(1,4+i))]$ and $[((0,i),(1,i),(0,2+i))\ ((0,1+i),(1,3+i))]$ for $i \in Z_5$.

Type II construction is similar and as a rule we indicate only the base graphs of a decomposition; only occasionally do we exhibit all the graphs of a decomposition. Take the case of $K_{n_1,n_2,\ldots,n_h} \to G$ with $n_i = r$ for all i; the construction is again illustrated by an example.

EXAMPLE 2. Let G be $G_{13} =$ 

and denoted by $[x,y,z,u,v]$.

The proof of $K_{4,4,4} \to G_{13}$ is simply $[0_1,2_3,2_2,3_3,0_2] \cup [0_1,0_3,3_1,2_2,1_2]$ mod 4. It is understood that $V(K_{4,4,4}) = \bigcup_{i=1} X_i$ where $X_i = \{0_i,1_i,2_i,3_i\}$ and that the graphs of the decomposition are $[i_1,(2+i)_3,(2+i)_2,(3+i)_3,i_2]$ and $[i_1,i_3,(i+3)_1,(2+i)_2,(1+i)_2]$ for $i \in Z_4$.

In the case where $n_i = r$ for $1 \le i \le h-1$ and $n_h = r' > r$, the vertices of X_h are g_h for $g \in Z_r$ and $\infty, \infty', \infty'', \ldots, \infty^{(r'-r-1)}$.

EXAMPLE 3. The base graphs of $K_{4,4,7} \to G_{13}$ are $[0_1,1_2,0_3,0_2,\infty] \cup [2_2,3_3,3_1,\infty',0_1] \cup [3_2,1_3,3_1,\infty'',0_1]$ mod 4. Hence $V(K_{4,4,7}) = \bigcup_{i=1}^{3} X_i$ with $X_i = \{0_i,1_i,2_i,3_i\}$ for $i = 1$ or 2 and $X_3 = \{0_3,1_3,2_3,3_3,\infty,\infty',\infty''\}$; the graphs of the decomposition are $[i_1,(1+i)_2,i_3,i_2,\infty]$, $[(2+i)_2,(3+i)_3,(3+i)_1,\infty',i_1]$ and $[(3+i)_2,(1+i)_3,(3+i)_1,\infty'',i_1]$ for $i \in Z_4$.

Sometimes a different method is used in proving $K_{r,r,r,\ldots,r} \to G$: we let $X_i = \{i-1,i-1+h,\ldots,i-1+(r-1)h\}$ for $i = 1,2,\ldots,h$. Hence $V(K_{r,r,\ldots,r}) = Z_{rh}$ and base graphs of the decomposition will be taken modulo rh, as seen in:

EXAMPLE 4. The proof of $K_{4,4,4,4} \to G_{13}$ is : $V(K_{4,4,4,4}) = Z_{16}$ and the base graph is $[0,10,9,14,7]$ mod 16, which means that the graphs of the decomposition are $[i,10+i,9+i,14+i,7+i]$ for $i \in Z_{16}$.

To prove that $K_{n_1,n_2,\ldots,n_h} \to G$, Lemmas C,D,E, and F below are also useful.

LEMMA C. *If* $K_{r_1,r_2} \to G$ *and* $K_{r_1,r_2'} \to G$, *then* $K_{ar_1,br_2+cr_2'} \to G$ *for integers* $a \geq 1$ *and* b *or* $c \geq 1$.

LEMMA D. *If* $K_{r_1,r_2,r_3} \to G$ *and* $K_{r_1,r_2,r_3'} \to G$, *then* $K_{ar_1,ar_2,(a-b)r_3+br_3'} \to G$ *for integers* a,b *with* $0 \leq b \leq a$.

LEMMA E. *If* $K_{r,r,r} \to G$ *and* $K_{r,r,r,r} \to G$ *then* $K_{pr,pr,pr,qr} \to G$ *for* $p \neq 2,6$, $0 \leq q \leq p$.

LEMMA F. *If* $K_{r,r,r,r} \to G$ *and* $K_{r,r,r,r,r} \to G$ *then* $K_{pr,pr,pr,pr,qr} \to G$ *for* $p \neq 2,3,6,10,14$ *and* $0 \leq q \leq p$.

Let us remark that in Lemma E, the condition $K_{r,r,r,r} \to G$ is unnecessary if $q = 0$ and the lemma is true for all $p \geq 1$; the condition $K_{r,r,r} \to G$ is unnecessary if $q = p$. Similar interpretation can be used in other lemmas, and in the case of Lemma F when $q = 0$, the condition $K_{r,r,r,r,r} \to G$ is unnecessary and the lemma is true for all $p \neq 2,6$.

The proofs of Lemmas C and D can be found in [1]. Before we prove Lemmas E and F we need two auxiliary results.

PROPOSITION 2.1. *If* $p \neq 2,6$ *and* $0 \leq q \leq p$ *then* $K_{p,p,p,q} \to \{K_4, K_3\}$.

Proof. The existence of a decomposition of $K_{p,p,p,p}$ into K_4 is equivalent to that of a pair of orthogonal Latin squares of order p; the latter is known to exist for $p \neq 2$ or 6 [14]. If we delete the vertices of $K_{p,p,p,p}$ that are not in $K_{p,p,p,q}$ from the K_4's which contain them, we get K_3's. Hence it is possible to partition the edges of $K_{p,p,p,q}$ with $p \neq 2,6$ into subgraphs isomorphic to K_4 or K_3.

PROPOSITION 2.2. *For* $p \neq 2,3,6,10,14$ *and* $0 \leq q \leq p$, $K_{p,p,p,p,q} \to \{K_5, K_4\}$.

The proof is similar to the proof of Proposition 2.1: $K_{p,p,p,p,q} \to K_5$ is equivalent to the existence of three pairwise orthogonal Latin squares of order p; these are known to exist for $p \neq 2,3,6,10,14$ [13].

Now let G be a graph with vertex set $\{1,2,\ldots,k\}$ and let $G \otimes S_n$ be the lexicographic product of G by an independent set of n elements, that is, a graph with vertex set the disjoint union of k independent sets X_i, where $|X_i| = n$ such that two vertices are joined by an edge if and only if they belong to two different sets X_i, X_j and $\{i,j\}$ is an edge in G.

It is easy to see that $K_{pr,pr,pr,qr} = K_{p,p,p,q} \otimes S_r$, which, by Proposition 2.1, is decomposable into subgraphs isomorphic to $K_4 \otimes S_r$ or $K_3 \otimes S_r$ for $p \neq 2,6$. Since $K_4 \otimes S_r = K_{r,r,r,r}$ and $K_3 \otimes S_r = K_{r,r,r}$, both of which by the hypothesis are decomposable into subgraphs isomorphic to G, Lemma E is proved.

The proof of Lemma F is quite similar (except that one uses Proposition 2.2).

III. Constructions for particular graphs.

In this section, we prove the results given in Table 1 namely, we find necessary and sufficient conditions for $K_n \to G$, where G is any graph on five vertices, none of which is isolated. Most of the constructions will be based on the composition method and the lay-out for each graph G will be:

(a) the graph G, its labels and notation

(b) necessary conditions for $K_n \to G$;

(c) constructions showing $K_{n_i} \to G$ or $K_{n_1,n_2,\dots,n_h} \to G$ for some small values of n_i in order to start the composition;

(d) compositions, with lemmas and parameters used indicated;

(e) conclusions.

Explanations will be given in the first few cases but after that, only the information is given.

(1) (a) G_1:
```
x   y   z     u   v
0———0———0     0———0
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, denoted by $[(x,y,z)(u,v)]$

(b) Necessary conditions for $K_n \to G_1$: $n \equiv 0$ or $1 \pmod 3$

(c) - $K_{2,3} \to G_1$: the subgraphs of the decomposition are $[(0_2,1_1,2_2)\ (0_1,1_2)]$ and $[(0_2,0_1,2_2)(1_1,1_2)]$

- $K_n \to G_1$ for $n = 6,7,9$ or 10: see Example 1 in Section II.

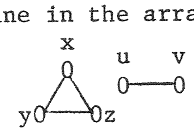
(d)

n	$K_n \to G_1$(with h=3) by Lemma	n_1	n_2	$K_{n_1,n_2} \to G$ by Lemma C with $r_1=2, r_2=3, r_2'=0, a=3$ $c=0$ and b:
$3t, t \geq 4$	A	6	$3t-6$	$t-2$
$3t+1$, $t \geq 4$	B	6	$3t-6$	$t-2$

(e) Theorem: $K_n \to G_1$ if and only if $n \equiv 0$ or $1 \pmod 3$, $n \geq 5$.

Explanation: $K_{3t} \to G_1$ is proved by using Lemma A, which states that if $K_6 \to G_1$, $K_{3t-6} \to G_1$ and $K_{6,3t-6} \to G_1$, then $K_{3t} \to G_1$. The case $K_6 \to G_1$ was proved in part (c); $K_{3t-6} \to G_1$ is true for $t \geq 4$ by induction, and finally $K_{6,3t-6} \to G_1$ is proved by using Lemma C, which in this case states that if $K_{2,3} \to G$ then $K_{6,3(t-2)} \to G_1$ (since $c = 0$). The case $K_{2,3} \to G$ is proved in (c).

The second line in the array can be interpreted similarly.

(2) (a) G_2: 

, denoted by $[(x,y,z)\ (u,v)]$

(b) Necessary conditions for $K_n \to G_2$: $n \equiv 0$ or $1 \pmod 8$

(c) $-\ K_{2,2,2} \to G_2$: the base graph is

$[(0_1,0_2,1_3)(1_1,1_2)]$ mod $(-,3)$

$-\ K_{2,2,2,2} \to G_2$: $[(0_1,0_2,1_3)(1_1,0_4)] \cup$

$[(0_2,0_3,1_4)(0_1,1_2)] \cup$

$[(0_3,0_4,0_1)(1_2,1_4)]$ mod $(2,-)$.

$-\ K_8 \to G_2$: $V(K_8) = Z_7 \cup \{\infty\}$, $[(0,1,3)(\infty,2)]$ mod 7.

$-\ K_9 \to G_2$: $V(K_9) = Z_9$, $[(0,1,3)(2,6)]$ mod 9.

$-\ K_{16} \to G_2$: $V(K_{16}) = Z_{15} \cup \{\infty\}$,

$[(0,1,7)(\infty,2)] \cup [(3,5,8)(2,6)]$ mod 15.

$-\ K_{17} \to G_2$: $V(K_{17}) = Z_{17}$,

$[(0,1,7)(2,6)] \cup [(3,5,8)(2,10)]$ mod 17.

$-\ K_{40} \to G_2$: $V(K_{40}) = Z_{39} \cup \{\infty\}$, $[(0,18,19)(1,\infty)] \cup$

$[(0,15,17)(1,7)] \cup [(0,13,16)(1,9)] \cup$

$[(0,10,14)(1,12)] \cup [(0,7,12)(1,10)]$ mod 39.

$-\ K_{41} \to G_2$: $V(K_{41}) = Z_{41}$, the first base graph is $[(0,18,19)(1,21)]$ the other 4 are just the last four base graphs for the case $K_{40} \to G_2$, except that the elements are taken modulo 41.

(d)

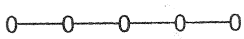
n	$K_n \to G_2$ (with $h=4$) by Lemma	$n_1=n_2=n_3$	n_4	$K_{n_1,n_2,n_3,n_4} \to G_2$ by Lemma E with $r=2, p=4t$ and q
$24t,\ t\ge 1$	A	$8t$	0	0
$24t+1,\ t\ge 1$	B	$8t$	0	0
$24t+8,\ t\ge 1$	A	$8t$	8	4
$24t+9,\ t\ge 1$	B	$8t$	8	4
$24t+16,\ t\ge 2$	A	$8t$	16	8
$24t+17,\ t\ge 2$	B	$8t$	16	8

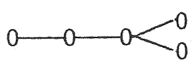
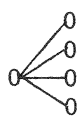
(e) *Conclusion:* $K_n \to G_2$ if and only if $n \equiv 0$ or $1 \pmod 8$.

Explanation: The fourth row of the array means that $K_{24t+9} \to G_2$ can be proved to be true by Lemma B if $K_{8t+1} \to G_2$, $K_9 \to G_2$ and $K_{8t,8t,8t,8} \to G_2$.

$K_{8t+1} \to G_2$ can be proved by induction, $K_9 \to G_2$ is in part (c) and $K_{8t,8t,8t,8} \to G_2$ is proved by Lemma E, which states that $K_{8t,8t,8t,8} \to G_2$ if $K_{2,2,2} \to G_2$ and $K_{2,2,2,2} \to G_2$, which have been proved in part (c).

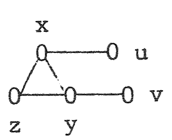
The other rows can be interpreted similarly.

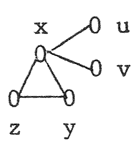
(3) G_3:  ,

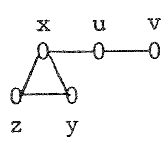
G_4:  and G_5: 

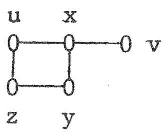
As we mentioned in section II, the necessary and sufficient condition for $K_n \to G$, $G = G_3, G_4$ or G_5 has already been found to be $n \equiv 0$ or $1 \pmod 8$.

(4) $G = G_6, G_7, G_8$ or G_9

G_6: 

, G_7: 

, G_8: 

and G_9: 

and are all denoted by $[x,y,z,u,v]$.

These four cases are considered together because the necessary conditions for $K_n \to G$ for each G are the same: $n \equiv 0$ or 1 (mod 5); also, they belong to a class of graphs called dragons [11]. It was shown in the same paper [11] that the condition $n \equiv 0$ or 1 (mod 10) is also sufficient for $K_n \to G$. In a different paper [7] the case $n \equiv 1$ or 5 (mod 10) was considered; it was shown that $K_n \to G$ exists for $n \geq 5$ when $G = G_6$ and for $n > 5$ when $G = G_7, G_8$ or G_9. The construction used in [7] is based on the method of differences and an additional condition was also satisfied, namely, that each vertex of K_n occurs in the same number of graphs - such decomposition is usually called a balanced graph design.

We will consider the only unsolved case, $n \equiv 6$ (mod 10); we will deal with G_6, G_7 and G_8 jointly, and with G_9 separately. We will use the composition method (which, of course, could be used to deal with the case $n \equiv 0$ or 1 or 5 (mod 10) as well).

(A) $G = G_6, G_7$ or G_8

(c) - $K_{5,5,5} \to G$: $\{ \bigcup_{j=1}^{3} [0_j, 0_{j+1}, 1_{j+2}, 2_{j+1}, a] \}$ mod 5

where $a = 3_{j+2}, 3_{j+1}$ or 0_{j+2},

$G = G_6, G_7$ or G_8 respectively.

- $K_{5,5,5,5} \to G$: $\{ \bigcup_{j=1}^{4} [0_j, 1_{j+2}, 0_{j+1}, 2_{j+1}, a_j] \} \cup$

$[1_4, 3_2, 0_1, 3_3, b] \cup [3_3, 1_1, 0_2, 2_4, c]$ mod 5

where $a_1 = 1_1, 0_3$ or 2_4
$a_2 = 1_2, 0_4$ or 2_1
$a_3 = 4_3, 2_1$ or 0_2
$a_4 = 3_4, 3_2$ or 0_3
$b = 2_3, 4_1$ or 4_2
$c = 3_4, 4_2$ or 0_1
for $G = G_6, G_7$ or G_8 respectively.

- $K_6 \to G_6$: [1,5,0,2,4],[4,0,3,1,2],[2,3,5,4,1]

- $K_6 \to G_7$: [1,5,0,2,3],[4,0,3,1,5],[2,3,5,4,0]

- $K_6 \to G_8$: [1,5,0,2,3],[4,1,3,0,2],[5,2,4,3,0]

- $K_{26} \to G$: $V(K_{26}) = Z_2 \times Z_{13}$,

$[(0,0),(0,1),(1,0),(0,4),a] \cup [(0,0),(0,2),(1,3),(0,5),b]$

$\cup [(0,0),(0,3),(1,5),(1,4),c] \cup [(1,0),(1,1),(0,3),(1,3),d]$

$\cup [(1,0),(1,2),(0,6),(1,4),e] \bmod (-,13)$.

where $a = (0,7),(0,6)$ or $(0,10)$
$b = (1,8),(1,6)$ or $(1,11)$
$c = (1,11),(1,8)$ or $(0,9)$
$d = (1,6),(1,5)$ or $(1,8)$
$e = (1,8),(1,6)$ or $(1,10)$
for $G = G_6, G_7$ or G_8 respectively.

- $K_{36} \to G$: $V(K_{36}) = Z_5 \times Z_7 \cup \{\infty\}$,

$$\{\bigcup_{j=1}^{3} [(0,0),(0,j),(2,q_j),(1,j),a]\} \bmod (5,7)$$

where $q_1 = 3$, $q_2 = 6$ and $q_3 = 1$
and $a = (1,3+2j),(1,3+j)$ or $(0,4)$ for
$G = G_6$, G_7 or G_8;
the remaining three base graphs are obtained from $K_6 \to G$ with ∞ replacing the element 0 and $(i,0)$ replacing i, $i = 1,2,\ldots,5$.

For instance, in the case $G = G_6$, the three graphs are
$[(1,0),(5,0),\infty,(2,0),(4,0)] \cup [(4,0),\infty,(3,0),(1,0),(2,0)]$
$\cup [(2,0),(3,0),(5,0),(4,0),(1,0)] \bmod (-,7)$.

(d)

n	$K_n \to G$ by Lemma	$n_1=n_2=n_3$	n_4	$K_{n_1,n_2,n_3,n_4} \to G$ by Lemma E with r=5, p and q	
30t+6, t=2, t≥4	B	10t	5	2t	1
30t+16, t≥0	B	10t+5	0	2t+1	0
30t+26, t≥1	B	10t+5	10	2t+1	2

(B) $G = G_9$

(c) $- K_6 \not\to G_9$: Assume that $K_6 \to G_9$, that means we have 3 graphs in the decomposition, and hence a total of three vertices of degree 3, nine vertices of degree 2 and three vertices of degree 1. However, since a vertex of K_6 has degree 5, three vertices in K_6 must each occur as a vertex of degree 3 in one graph and a vertex of degree 2 in another, while the remaining three must each occur as a vertex of degree 2 in two graphs and vertex of degree 1 in the other. One could, with a bit of effort, show that such a decomposition is impossible.

$- K_{5,5} \to G_9$: $[0_1, 0_2, 2_1, 1_2, 2_2]$ mod 5

$- K_{16} \to G_9$: $V(K_{16}) = Z_5 \times Z_3 \cup \{\infty\}$,

$\{ \bigcup_{i=0}^{3} [(i,0),(i,1),(i+1,1),(i+3,0), \infty]\} \cup$
$[(0,0),(4,0),(2,0),(4,1),(1,2)] \cup [(0,0),(2,1),(1,0),$
$(2,2),(3,2)] \cup [(4,0),(3,1),(2,0),(3,2),\infty] \cup$
$[(1,0),(0,2),(4,1),(4,2),(3,1)]$ mod $(-,3)$.

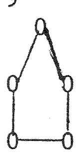
(d)

	$K_n \to G_9$ by Lemma	n_1	n_2	$K_{n_1,n_2} \to G_9$ by Lemma C with $r_1=r_2=5$, $a=2t-1$, $b=2$,c
$10t+6$, $t \geq 2$	B	$(2t-1)5$	10	0

(e) $K_n \to G_6$ if and only if $n \equiv 0$ or $1 \pmod 5$

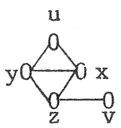
$K_n \to G_7$ or G_8 if and only if $n \equiv 0$ or $1 \pmod 5$, $n \geq 6$

$K_n \to G_9$ if and only if $n \equiv 0$ or $1 \pmod 5$, $n > 6$.

(5) G_{10}: 

This case was dealt with already in Section II.

Conclusion: $K_n \to G_{10}$ if and only if $n \equiv 1$ or $5 \pmod{10}$.

(6) (a) G_{11}: 

, $[x,y,z,u,v]$.

(b) $n \equiv 0,1,4$ or $9 \pmod{12}$.

(c) - $K_{4,4,4} \to G_{11}$: $[2_2,0_3,1_1,0_1,0_2]$, $[3_2,1_3,0_1,1_1,1_2]$, $[2_2,2_3,2_1,3_1,1_2]$, $[3_2,3_3,3_1,2_1,0_2]$, $[2_1,0_2,0_3,1_3,3_2]$ $[3_1,1_2,1_3,0_3,2_2]$, $[0_1,0_2,2_3,3_3,3_2]$ and $[1_1,1_2,3_3,2_3,2_2]$.

- $K_{4,4,4,4} \to G_{11}$: $V(K_{4,4,4,4}) = Z_{16}$, $[0,7,5,6,2] \bmod 16$

- $K_{2,2,2,2} \to G_{11}$: $[0_1,1_3,0_2,1_4,0_4]$, $[0_2,1_4,1_1,0_3,1_3]$, $[0_3,1_1,1_2,0_4,1_4]$ and $[0_4,1_2,0_1,1_3,0_3]$.

- $K_{4,4,7} \to G_{11}$: $[i_1,i_3,(1+i)_2,i_2,\infty^*]$, $[i_1,(2+i)_2,(3+i)_3,\infty'',(1+i)_1]$ and $[i_1,(3+i)_2,\infty^*,(1+i)_3,(2+i)_1]$ for $i \in Z_4$ with

$\infty^* = \infty$ when $i = 0$ or 1

∞' when $i = 2$ or 3.

- $K_9 \to G_{11}$: $V(K_9) = Z_3 \times Z_3$, $[(0,0),(0,1),(1,0),(2,0),(0,2)] \cup [(1,0),(2,0),(2,1),(1,1),(0,0)] \bmod (-,3)$.

- $K_{12} \to G_{11}$: $V(K_{12}) = Z_{11} \cup \{\infty\}$, $[1,0,5,3,\infty] \bmod 11$.

- $K_{13} \to G_{11}$: $V(K_{13}) = Z_{13}$, $[1,0,5,3,11] \bmod 13$.

- $K_{16} \to G_{11}$: $V(K_{16}) = Z_3 \times Z_5 \cup \{\infty\}$,

$[(0,0),(0,1),(1,0),(2,0),(1,1)] \cup$

$[(0,0),(1,3),(0,2),\infty,(1,4)] \cup$

$[(1,0),(2,0),(2,1),(1,2),\infty] \cup$

$[(2,0),(2,2),(0,4),(1,3),(2,1)] \bmod (-,5)$.

- $K_{21} \to G_{11}$: $V(K_{21}) = Z_3 \times Z_7$,

$[(0,1),(0,0),(1,3),(1,1),(1,6)] \cup$

$[(1,6),(0,2),(0,0),(1,0),(0,3)] \cup$

$[(2,2),(2,0),(1,3),(0,2),(2,1)] \cup$

$[(2,1),(2,0)\ (0,6),(0,4),(2,5)] \cup$

$[(1,2),(1,0),(2,3),(2,2),(2,6)] \bmod (-,7)$.

- $K_{24} \to G_{11}$: $V(K_{24}) = Z_{23} \cup \{\infty\}$, $[0,11,4,5,\infty] \cup$

$[0,10,2,1,5] \bmod 23$.

- $K_{52} \to G_{11}$: $V(K_{52}) = Z_3 \times Z_{17} \cup \{\infty\}$,

$[(0,0),(0,7),(1,7),(1,9),\infty] \cup$

$[(0,1),(0,0),(1,13),(0,5),(0,10)] \cup$

$[(0,2),(0,0),(1,6),(0,8),(0,1)] \cup$

$[(1,1),(0,0),(1,8),(0,3),(2,0)] \cup$

$[(1,5),(1,0),(2,13),(0,6),(2,3)] \cup$

$[(1,4),(1,0),(2,6),(0,7),(2,14)] \cup$

$[(2,0),(1,0),(1,1),(2,5),(1,7)] \cup$

$[(2,3),(1,0),(1,2),(2,7),(1,10)] \cup$

$[(1,3),(1,0),(2,15),(2,14),\infty] \cup$

$[(2,1),(2,0),(0,16),(0,14),(2,4)] \cup$

$[(2,2),(2,0),(0,10),(0,9),(2,16)] \cup$

$[(2,0),(2,3),(0,4),(0,5),\infty] \cup$

$[(2,0),(2,6),(0,6),(1,13),(2,3)] \bmod (-,17)$.

(d) We have two arrays:

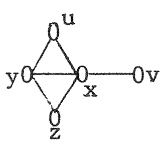
n	$K_n \to G_{11}$ by Lemma	$n_1=n_2=n_3$	n_4	$K_{n_1,n_2,n_3,n_4} \to G_{11}$ by Lemma E with r	p	q
33	B	8	8	2	4	4
36t+13, t=1	B	12	12	4	3	3
36t+33, t=1	B	20	8	4	5	2

Note: It is not true that $K_{2,2,2} \to G_{11}$, however, (see remark after Lemma F), the condition $K_{r,r,r} \to G$ is not necessary for Lemma E when $p = q$.

	$K_n \to G_{11}$ by Lemma			$K_{n_1,n_2,n_3} \to G_{11}$ by Lemma D with $r_1=r_2=r_3=4$, $r_3'=7$,	
n		$n_1=n_2$	n_3	a	b
36t, t≥1	A	12t	12t	3t	0
36t+1, t≥1	B	12t	12t	3t	0
36t+4, t≥1	B	12t	12t+3	3t	1
36t+9, t≥1	A	12t	12t+9	3t	3
36t+12, t≥1	A	12t+4	12t+4	3t+1	0
36t+13, t≥2	B	12t	12t+12	3t	4
36t+16, t≥2	B	12t	12t+15	3t	5
36t+21, t≥1	A	12t+4	12t+13	3t+1	3
36t+24, t≥1	A	12t+4	12t+16	3t+1	4
36t+25, t≥0	B	12t+8	12t+8	3t+2	0
36t+28, t≥0	B	12t+8	12t+11	3t+2	1
36t+33, t≥2	A	12t+4	12t+25	3t+1	7

(e) $K_n \to G_{11}$ if and only if $n \equiv 0,1,4$ or $9 \pmod{12}$.

The next case G_{12} is similar to the case of G_{11}, hence some proofs are omitted.

(7) (a) G_{12}: 

, [x,y,z,u,v].

(b) $n \equiv 0,1,4$ or $9 \pmod{12}$.

(c) - $K_{4,4,4} \to G_{12}$: If $K_{2,2,2} \to G_{12}$, then $K_{4,4,4} \to G_{12}$ by Lemma D with $r_1 = r_2 = r_3 = 2$, $a = 2$ and $b = 0$; $K_{2,2,2} \to G_{12}$ is easy to prove, for example, $[0_3,0_1,0_2,1_2,1_1]$ and $[1_3,1_1,1_2,0_2,0_1]$ are two graphs partitioning the edges of $K_{2,2,2}$.

- $K_{4,4,4,4} \to G_{12}$: $V(K_{4,4,4,4}) = Z_{16}$, $[0,7,5,6,3]$ mod 16,

- $K_{2,2,2,2} \to G_{12}$: $[0_1,1_3,0_2,1_4,0_3]$, $[0_2,1_4,0_3,1_1,0_4]$, $[1_1,0_3,0_4,1_2,1_3]$ and $[1_2,0_4,0_1,1_3,1_4]$.

- $K_{4,4,7} \to G_{12}$: $[0_3,0_1,0_2,1_2,2_2] \cup [0_1,2_2,3_3,\infty,1_3] \cup$ $[0_1,3_2,\infty',\infty'',2_3]$ mod 4.

- $K_9 \to G_{12}$: $V(K_9) = Z_3 \times Z_3$,
$[(0,1),(0,0),(1,0),(2,0),(1,2)] \cup$
$[(2,0),(1,0),(2,1),(1,1),(0,2)]$ mod $(-,3)$.

- $K_{12} \to G_{12}$: $V(K_{12}) = Z_{11} \cup \{\infty\}$, $[1,0,5,3,\infty]$ mod 11.

- $K_{13} \to G_{12}$: $V(K_{13}) = Z_{13}$, $[1,0,5,3,7]$ mod 13.

- $K_{16} \to G_{12}$: $V(K_{16}) = Z_3 \times Z_5 \cup \{\infty\}$,
$[(0,1),(0,0),(1,0),(2,0),(1,3)] \cup$
$[(1,3),(0,0),(0,2),\infty,(1,4)] \cup$
$[(2,0),(1,0),(2,1),(1,2),\infty] \cup$
$[(2,0),(2,2),(0,4),(1,3),(0,3)]$ mod $(-,5)$.

- $K_{21} \to G_{12}$: The base graphs are the same as in $K_{21} \to G_{11}$ with the last elements of each graph replaced by $(0,4),(1,2),(2,5),(0,2)$ and $(2,0)$, in the order given.

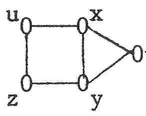
- $K_{24} \to G_{12}$: $V(K_{24}) = Z_{23} \cup \{\infty\}$, $[0,11,4,5,\infty] \cup$ $[0,10,2,1,3]$ mod 23.

- $K_{52} \to G_{12}$: The base graphs are the same as those of $K_{52} \to G_{11}$ with the last elements in each graph replaced by $\infty,(1,4),(1,7),(2,10),(1,11),(1,12),(2,7),(2,11),\infty,$ $(0,13),(0,13),\infty$ and $(0,3)$, in the order given.

(d) see the case G_{11}.

(e) The conclusion is the same as the case G_{11} as well: $K_n \to G_{12}$ if and only if $n \equiv 0,1,4$ or $9 \pmod{12}$.

The case $K_n \to G_{13}$ can be solved similarly to the last two cases, i.e. when $G = G_{11}$ or G_{12}, hence a part of the proof is omitted; however, due to the fact that we do not know whether $K_{24} \to G_{13}$ some variations have to be made.

(8) (a) G_{13}: 

, $[x,y,z,u,v]$.

(b) $n \equiv 0,1,4$ or $9 \pmod{12}$.

(c) - $K_{4,4,4} \to G_{13}$: see Example 2 of Section II.

- $K_{4,4,4,4} \to G_{13}$: see Example 4 of Section II.

- $K_{2,2,2,2} \to G_{13}$: $V(K_{2,2,2,2}) = Z_8$, $[0,2,5,3,1]$, $[0,6,1,7,5]$, $[2,4,1,3,7]$ and $[6,4,5,7,3]$.

- $K_{4,4,7} \to G_{13}$: see Example 3 of Section II.

- $K_9 \to G_{13}$: $V(K_9) = Z_3 \times Z_3$, $[(1,0),(1,1),(0,0),(0,1),(2,2)] \cup$ $[(2,0),(2,1),(0,0),(1,0),(0,1)] \bmod (-,3)$.

- $K_{12} \to G_{13}$: $V(K_{12} = Z_{12}$, $[3,7,8,2,10]$, $[6,4,3,11,9]$, $[4,8,11,0,5]$, $[10,11,5,6,4]$, $[0,1,8,6,5]$, $[0,2,9,8,7]$, $[0,3,8,10,9]$, $[1,3,5,7,6]$, $[1,9,7,11,10]$, $[2,4,7,6,1]$ and $[2,5,9,11,10]$.

- $K_{13} \to G_{13}$: $V(K_{13}) = Z_{13}$, $[5,0,3,1,6] \bmod 13$.

- $K_{16} \to G_{13}$: $V(K_{16}) = Z_3 \times Z_5 \cup \{\infty\}$,
$[\infty,(1,2),(0,3),(2,0),(0,0)] \cup$
$[(0,1),(1,2),(1,0),(0,0),(0,4)] \cup$
$[(2,2),(2,0),(1,0),(1,1),(0,1)] \cup$

$[(2,1),(1,2),(2,4),(0,1),(2,0)] \bmod (-,5)$.

- $K_{21} \to G_{13}$: $V(K_{21}) = Z_3 \times Z_7$,

  $[(0,0),(0,1),(1,1),(1,4),(0,3)] \cup$

  $[(0,1),(1,6),(0,0),(1,2),(1,4)] \cup$

  $[(2,1),(2,4),(1,0),(1,1),(0,4)] \cup$

  $[(1,1),(2,6),(1,0),(2,2),(2,4)] \cup$

  $[(0,6),(2,1),(0,2),(2,0),(2,2)] \bmod (-,7)$.

- $K_{24} \to G_{13}$: we have no proof for this case, but we conjecture that it is true.

- $K_{52} \to G_{13}$: $V(K_{52}) = Z_3 \times Z_{17} \cup \{\infty\}$,

  $[\infty,(1,16),(0,8),(2,8),(0,0)] \cup$

  $[(0,0),(0,1),(1,4),(1,1),(0,8)] \cup$

  $[(0,0),(0,2),(1,8),(1,2),(0,6)] \cup$

  $[(0,1),(1,11),(0,0),(1,5),(0,4)] \cup$

  $[(0,1),(1,15),(0,0),(1,13),(0,6)] \cup$

  $[(1,0),(1,1),(2,4),(2,1),(1,5)] \cup$

  $[(1,0),(1,2),(2,8),(2,2),(1,9)] \cup$

  $[(1,1),(2,5),(1,0),(2,13),(2,12)] \cup$

  $[(1,0),(2,8),(1,1),(2,15),(2,16)] \cup$

  $[(0,15),(2,10),(1,0),(2,9),(2,11)] \cup$

  $[(2,0),(2,2),(1,2),(0,2),(0,3)] \cup$

  $[(0,14),(2,1),(0,8),(2,0),(2,5)] \cup$

  $[(0,16),(2,1),(0,12),(2,0),(2,6)] \bmod (-,17)$.

  In addition, we have

- $K_{12,12,12,12,12,12} \to G_{13}$: $V(K_{12,12,12,12,12,12}) = Z_{72}$,

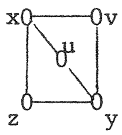
  $[0,3,64,31,1] \cup [0,9,67,35,4] \cup [0,20,1,22,7] \cup$

  $[0,23,57,10,8] \cup [0,55,71,44,29] \bmod 72$.

(d) The two arrays are the same as those in the case of $G = G_{11}$ with the following exception: since we do not know whether $K_{24} \to G_{13}$ is true, the statements in the longer array for the cases $n = 36t$ and $n = 36t+9$ are true only when $t \geq 1$ and $t \neq 2$. The cases thus not included in the array, namely $n = 36t$ and $n = 36t+9$ for $t = 2$ can be proved as follows:

n	$K_n \to G_{13}$ by Lemma	n_i, $i=1,2,\ldots,h$	$K_{n_1,n_2,\ldots,n_h} \to G_{13}$ by Lemma E with $r=4, p=q$
$36t$, $t=2$	A	$n_i=12$, $i=1,2,\ldots,6$	- (proved in part (c)
$36t+9$, $t=2$	B	$n_i=20$, $i=1,2,3,4$	5

(e) $K_n \to G_{13}$ if and only if $n \equiv 0,1,4$ or $9 \pmod{12}$, except possibly when $n = 24$.

(9) (a) G_{14}: 

, $[x,y,z,u,v]$. ($G_{14} \simeq K_{2,3}$)

(b) $n \equiv 0,1,4$ or $9 \pmod{12}$.

(c) - $K_9 \not\to G_{14}$: The degrees of vertices of G_{14} are 2 and 3, since a vertex of K_9 has degree 8, it must appear either in two subgraphs with degree 3 and in one subgraph with degree 2 (in which case, the vertex is said to be type 1), or in four subgraphs with degree 2 (type 2). Suppose that there are a vertices in K_9 which are type 1 and b vertices which are type 2, then obviously $a + b = 9$. As there are 6 subgraphs isomorphic to G_{14} in the decompositions, by counting in two ways the number of vertices with degree 3 (or 2), we have

$$2a = 12$$
$$a + 4b = 18.$$

Solving the equations, we have $a = 6$, $b = 3$. Now, in a G_{14}, the vertices of degree 2 are adjacent only to vertices of degree 3, the three vertices of type 2 can never be joined to each other, a contradiction to $K_9 \to G_{14}$.

- $K_{12} \not\to G_{14}$: the proof is similar, though longer than the one above, and is hence omitted.

- $K_{13} \to G_{14}$: $V(K_{13}) = Z_{13}$, $[0,1,2,4,6] \bmod 13$.

- $K_{16} \to G_{14}$: $V(K_{16}) = Z_3 \times Z_5 \cup \{\infty\}$,

  $[(1,0),(1,2),(0,0),(2,0),(2,1)] \cup$

  $[\infty,(0,4),(0,0),(1,0),(2,0)] \cup$

  $[(0,0),(2,2),(0,2),(2,3),(2,4)] \cup$

  $[(0,0),(1,0),(1,3),(1,4),(2,2)] \bmod (-,5)$.

- $K_{21} \to G_{14}$: $V(K_{21}) = Z_3 \times Z_7$,

  $[(0,0),(1,1),(0,1),(0,3),(1,3)] \cup$

  $[(0,0),(2,2),(0,2),(1,2),(1,4)] \cup$

  $[(1,0),(1,2),(0,1),(1,3),(2,6)] \cup$

  $[(1,0),(2,0),(2,1),(2,2),(2,3)] \cup$

  $[(2,0),(2,1),(0,2),(0,4),(0,6)] \bmod (-,7)$.

- $K_{24} \to G_{14}$: $V(K_{24}) = \{\infty,0,1,\ldots,6\} \cup \bigcup_{i=1}^{2} X_i$ where $X_i = \{0_i,1_i,\ldots,7_i\}$. Since $K_{16} \to G_{14}$, we can construct 20 subgraphs from the vertices of $X_1 \cup X_2$. The remaining 26 subgraphs are:

  $[0,1,3,4_1,4_2]$, $[0_1,0_2,1,2,3]$

  $[1,2,4,5_1,5_2]$, $[1_1,1_2,0,2,3]$

  $[2,3,5,4_1,4_2]$, $[2_1,2_2,0,1,3]$

  $[3,4,6,5_1,5_2]$, $[3_1,3_2,0,1,2]$

  $[4,5,0,7_1,7_2]$, $[6_1,6_2,0,1,2]$

  $[5,6,1,4_1,4_2]$, $[6_1,6_2,3,4,5]$

  $[6,0,2,5_1,5_2]$, $[7_1,7_2,0,1,2]$

  $[7_1,7_2,3,6,\infty]$

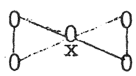
and $[\infty,i,i+1,i_1,i_2]$ for $i = 0,1,\ldots,6$,

$[j_1,j_2,4,5,6]$ for $j = 0,1,2,3$.

(d)

	$K_n \to G_{14}$ by Lemma			$K_{n_1,n_2} \to G_{14}$ by Lemma C with $r_1=2, r_2=3, r_2'=0, c=0$,	
n		n_1	n_2	a	b
$12t$, $t\geq 3$	B	$12t(t-2)+8$	15	$6(t-2)+4$	5
$12t+1$, $t\geq 2$	B	$12(t-1)$	12	$6(t-1)$	4
$12t+4$, $t\geq 2$	B	$12(t-1)$	15	$6(t-1)$	5
$12t+9$, $t\geq 2$	B	$12(t-1)+8$	12	$6(t-1)+4$	4

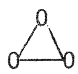
(e) $K_n \to G_{14}$ iff $n \equiv 0,1,4,9 \pmod{12}$, $n \geq 13$.

(10) (a) G_{15}: 

(b) $n \equiv 1$ or $9 \pmod{12}$.

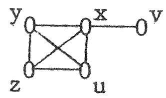
(c) The case $n \equiv 1 \pmod{12}$ was solved in [7].

Let $n = 12t+9$.

It is well-known that a resolvable decomposition into triangles exists [12], that is $K_n \to$  and furthermore, the triangles of the decomposition can be partitioned into $6t + 4$ resolution classes, each containing $4t + 3$ disjoint triangles (and thus covering all the vertices of K_n).

Partition the resolution classes into pairs (there is an even number of them) and consider a particular pair of resolution classes. By the theorem on simultaneous representatives [4,pg.51] applied to the two classes, we get common representatives of the triangles in them. Joining the two triangles which contain the common representative x at x yields a G_{15}. Hence we have $K_{12t+9} \to G_{15}$.

(d) $K_n \to G_{15}$ if and only if $n \equiv 1$ or $9 \pmod{12}$.

(11) (a) G_{16}: 

, $[x,y,z,u,v]$.

(b) $n \equiv 0$ or $1 \pmod 7$.

(c) - $K_7 \not\to G_{16}$: use degree argument, similar to the case $K_9 \not\to G_{14}$.

- $K_8 \not\to G_{16}$: use degree argument.

- $K_{7,7,7,7} \to G_{16}$: $[0_1,1_2,1_3,0_4,0_2] \cup [0_1,2_2,3_3,6_4,6_3] \cup [0_1,3_2,5_3,5_4,1_4] \cup [4_2,0_1,0_3,4_4,3_3] \cup [5_2,0_1,2_3,3_4,6_4] \cup [4_3,0_1,6_2,2_4,6_4] \bmod 7$.

- $K_{7,7,7,7,7} \to G_{16}$: $V(K_{7,7,7,7,7}) = Z_{35}$, $[0,1,4,13,7] \cup [0,2,8,19,14] \bmod 35$.

- $K_{14} \to G_{16}$: $V(K_{14}) = Z_{13} \cup \{\infty\}$, $[0,1,4,6,\infty] \bmod 13$.

- $K_{15} \to G_{16}$: $V(K_{15}) = Z_{15}$, $[0,1,4,6,7] \bmod 15$.

- $K_{21} \to G_{16}$: $V(K_{21}) = Z_4 \times Z_5 \cup \{\infty\}$,
$[(0,0),(0,1),(1,3),(2,0),(1,1)] \cup$
$[(1,0),(0,0),(2,1),(2,3),(3,2)] \cup$
$[(0,0),(0,2),(3,2),(3,3),(2,2)] \cup$
$[(3,1),(3,3),(2,0),(2,1),(1,0)] \cup$
$[(1,0),(1,1),(2,0),(3,4),(1,2)] \cup$
$[\infty,(0,0),(1,4),(3,4),(2,0)] \bmod (-,5)$.

- $K_{22} \to G_{16}$: $V(K_{22}) = Z_2 \times Z_{11}$,
$[(0,0),(0,4),(1,0),(1,1),(0,5)] \cup$
$[(1,5),(0,0),(0,2),(0,3),(1,1)] \cup$
$[(0,0),(1,4),(1,6),(1,9),(1,10)] \bmod (-,11)$.

- $K_{28} \to G_{16}$: $V(K_{28}) = Z_{27} \cup \{\infty\}$, $[0,3,4,14,\infty] \cup [0,6,8,15,5] \bmod 27$.

- $K_{29} \to G_{16}$: $V(K_{29}) = Z_{29}$, $[0,1,3,11,6] \cup [0,4,9,16,14] \bmod 29$.

- $K_{35} \to G_{16}$: $V(K_{35}) = Z_2 \times Z_{17} \cup \{\infty\}$,

  $[(0,0),(0,6),(0,7),(1,8),(0,4)] \cup$

  $[(0,0),(0,3),(0,5),(1,9),(1,13)] \cup$

  $[(0,0),(1,5),(1,11),(1,14),\infty] \cup$

  $[(1,0),(0,0),(1,10),(1,15),(1,1)] \cup$

  $[(1,0),(1,4),(0,5),(0,14),\infty] \bmod (-,17)$.

- $K_{36} \to G_{16}$: $V(K_{36}) = Z_4 \times Z_9$,

  $[(0,0),(0,2),(1,8),(2,4),(0,1)] \cup$

  $[(0,0),(0,4),(1,7),(2,0),(0,3)] \cup$

  $[(1,0),(1,1),(2,0),(3,6),(1,2)] \cup$

  $[(1,0),(1,3),(2,4),(3,3),(1,4)] \cup$

  $[(0,3),(2,0),(2,4),(3,5),(1,4)] \cup$

  $[(0,5),(1,5),(3,0),(3,4),(1,7)] \cup$

  $[(2,0),(2,1),(2,3),(3,3),(0,1)] \cup$

  $[(0,0),(3,0),(3,1),(3,3),(3,6)] \cup$

  $[(1,5),(0,0),(2,3),(3,7),(2,2)] \cup$

  $[(1,4),(0,0),(2,7),(3,5),(3,2)] \bmod (-,9)$.

- $K_{42} \to G_{16}$: $V(K_{42}) = Z_{41} \cup \{\infty\}$, $[0,3,4,20,15] \cup [0,8,10,19,14] \cup [0,5,12,18,\infty] \bmod 41$.

- $K_{43} \to G_{16}$: $V(K_{43}) = Z_{43}$, $[0,8,9,19,6] \cup [0,5,7,20,12] \cup [0,3,17,21,16] \bmod 43$.

- $K_{49} \to G_{16}$: $V(K_{49}) = Z_7 \times Z_7$,

  $[(i,0),(i,1),(i+1,5),(i+3,3),(i+2,j)]$

  with $j = \begin{cases} 4 & \text{for } i = 0,1,2,3,4. \\ 0 & \text{for } i = 5,6. \end{cases}$

  $\cup\ [(i,0),(i+1,1),(i+1,3),(i+3,4),(i+3,j)]$

with $j = \begin{cases} 6 & \text{for } i = 0,1,2,3,4,6. \\ 0 & \text{for } i = 5, \end{cases}$

$\cup$ [(i,0),(i+1,6),(i+3,1),(i+3,5),(k,j)]

with k = 6, j = 0 for i = 0

k = i+1, j = 2 for i = 1,2,3,4

k = 2, j = 0 for i = 5,6

$\cup$ [(1,0),(2,0),(3,0),(0,0),(4,0)]

$\cup$ [(4,0),(5,0),(6,0),(3,0),(2,0)]

$\cup$ [(0,4),(1,6),(5,0),(6,2),(4,4)] mod (-,7).

- $K_{50} \to G_{16}$: $V(K_{50}) = Z_2 \times Z_{25}$,

[(0,0),(0,1),(1,1),(1,3),(0,8)] $\cup$

[(0,0),(0,2),(0,5),(1,10),(0,12)] $\cup$

[(0,0),(0,4),(0,11),(1,15),(1,9)] $\cup$

[(0,0),(0,6),(0,15),(1,22),(1,12)] $\cup$

[(0,0),(1,19),(1,20),(1,24),(1,18)] $\cup$

[(1,0),(1,3),(1,9),(0,11),(1,11)] $\cup$

[(1,0),(1,7),(1,15),(0,19),(1,12)] mod (-,25).

- $K_{56} \to G_{16}$: $V(K_{56}) = Z_{55} \cup \{\infty\}$, [0,5,6,26,3] $\cup$

[0,11,13,25,17] $\cup$ [0,10,28,32,19] $\cup$

[0,8,15,24,∞] mod 55.

- $K_{57} \to G_{16}$: $V(K_{57}) = Z_{57}$, [0,9,10,26,5] $\cup$

[0,11,13,25,8] $\cup$ [0,3,23,27,18] $\cup$

[0,6,21,28,19] mod 57.

- $K_{63} \to G_{16}$: $V(K_{63}) = Z_2 \times Z_{31} \cup \{\infty\}$,

[(0,0),(0,1),(0,13),(1,0),(1,11)] $\cup$

[(0,0),(0,2),(0,6),(1,8),(1,12)] $\cup$

[(0,0),(0,3),(0,10),(1,13),(1,16)] $\cup$

[(0,0),(0,5),(0,16),(1,20),(1,19)] $\cup$

[(0,0),(0,8),(0,17),(1,22),∞] $\cup$

$[(1,0),(1,1),(1,4),(1,10),(0,8)] \cup$

$[(1,0),(1,2),(1,13),(0,4),(0,6)] \cup$

$[(1,0),(1,7),(1,15),(0,14),(0,3)] \cup$

$[(1,0),(1,5),(1,17),(1,10),\infty] \bmod (-,31).$

- $K_{64} \to G_{16}$: $V(K_{64}) = Z_7 \times Z_9 \cup \{\infty\}$,

$[(i,0),(i,1),(i+1,1),(i+3,1),(i+2,6)]$ $i = 0,1,\dots,6$

$\cup [(i,0),(i,4),(i+1,7),(i+3,2),(i+2,j)]$

with $\begin{cases} j = 1 & \text{for } i = 0,1,2,3,4,5 \\ j = 3 & \text{for } i = 6 \end{cases}$

$\cup [(i,0),(i+1,6),(i+1,8),(i+3,4),(k,j)]$

with $k = 2$, $j = 3$, for $i = 0$

$k = i+1$, $j = 5$ for $i = 1,2$

$k = i+1$, $j = 2$ for $i = 3,4,5,6$

$\cup [(i,0),(i+1,4),(i+3,3),(i+3,6),(k,j)]$

with $k = 3$, $j = 8$ for $i = 0$

$k = 5$, $j = 4$ for $i = 1$

$k = 6$, $j = 4$ for $i = 2$

$k = 6$, $j = 5$ for $i = 3$

$k = 0$, $j = 5$ for $i = 4$

$k = 0$, $j = 3$ for $i = 5$

$k = 2$, $j = 8$ for $i = 6$

$\cup [(4,1),(0,0),(1,2),(3,5),\infty]$

$\cup [(4,5),(1,0),(2,2),(5,1),(6,8)]$

$\cup [(5,5),(2,0),(3,2),\infty,(6,1)]$

$\cup [(6,4),(0,0),(1,5),\infty,(3,5)] \bmod (-,9).$

- $K_{70} \to G_{16}$: $V(K_{70}) = Z_{69} \cup \{\infty\}$, $[0,11,13,25,1]$

$\cup [0,3,29,34,20] \cup [0,10,28,32,30]$

$\cup [0,17,23,50,21] \cup [0,8,15,24,\infty] \bmod 69.$

- $K_{71} \to G_{16}$: $V(K_{71}) = Z_{71}$, $[0,11,13,25,1]$

$\cup [0,3,29,34,20] \cup]0,10,28,32,30]$

$\cup [0,17,23,50,19] \cup [0,8,15,24,35] \bmod 71.$

- $K_{77} \to G_{16}$: $V(K_{77}) = Z_4 \times Z_{19} \cup \{\infty\}$,

  $[(0,0),(1,0),(2,0),(3,0),\infty]$

  $\cup\ [(0,\alpha),(1,2\alpha),(2,3\alpha),(3,4\alpha),(0,2\alpha)]$

  with $\alpha = 1,2,3,4,5,6,7,8,9$

  $\cup\ [(1,2\alpha),(0,\alpha),(2,3\alpha),(3,4\alpha),(1,3\alpha-3)]$

  with $\alpha = 10,11,12$

  $\cup\ [(2,3\alpha),(0,\alpha),(1,2\alpha),(3,4\alpha),(2,4\alpha-6)]$

  with $\alpha = 13,14,15$

  $\cup\ [(3,4\alpha),(0,\alpha),(1,2\alpha),(2,3\alpha),(3,5\alpha-9)]$

  with $\alpha = 16,17,18$

  $\cup\ [(i,0),(i,1),(i,4),(i,6),\infty]$

  for $i = 1,2,3 \bmod (-,19)$.

- $K_{78} \to G_{16}$: $V(K_{78}) = Z_2 \times Z_{39}$,

  $[(0,0),(0,16),(0,18),(0,19),(1,4)]$

  $\cup\ [(0,0),(0,8),(0,15),(1,35),(1,6)]$

  $\cup\ [(0,0),(0,4),(0,13),(1,26),(1,7)]$

  $\cup\ [(0,0),(0,6),(0,11),(1,3),(1,11)]$

  $\cup\ [(0,0),(0,10),(0,22),(1,1),(1,14)]$

  $\cup\ [(1,0),(1,16),(1,18),(1,19),(0,24)]$

  $\cup\ [(1,0),(1,8),(1,15),(0,37),(0,23)]$

  $\cup\ [(1,0),(1,4),(1,14),(0,34),(0,15)]$

  $\cup\ [(1,0),(1,6),(1,11),(0,16),(0,11)]$

  $\cup\ [(1,0),(1,9),(1,22),(0,1),(0,7)]$

  $\cup\ [(0,0),(0,14),(1,0),(1,12),(1,33)] \bmod (-,39)$.

- $K_{91} \to G_{16}$: $V(K_{91}) = Z_2 \times Z_{45} \cup \{\infty\}$,

  $[(0,0),(0,1),(0,4),(0,20),\infty]$
  $\cup\ [(0,0),(0,2),(0,9),(1,10),(1,6)]$
  $\cup\ [(0,0),(0,5),(0,15),(1,40),(1,11)]$

$\cup\ [(0,0),(0,8),(0,22),(1,44),(1,17)]$

$\cup\ [(0,0),(0,11),(0,24),(1,27),(1,19)]$

$\cup\ [(1,0),(1,1),(1,4),(1,20),\infty]$

$\cup\ [(1,0),(1,2),(1,8),(0,4),(0,21)]$

$\cup\ [(1,0),(1,5),(1,15),(0,27),(0,15)]$

$\cup\ [(0,0),(0,6),(0,18),(1,20),(1,13)]$

$\cup\ [(0,0),(0,17),(1,0),(1,9),(1,42)]$

$\cup\ [(1,0),(1,7),(1,24),(0,40),(0,14)]$

$\cup\ [(1,0),(1,11),(1,23),(0,30),(0,13)]$

$\cup\ [(1,0),(1,13),(1,27),(0,6),(0,11)] \bmod (-,45)$.

- $K_{92} \to G_{16}$: $V(K_{92}) = Z_4 \times Z_{23}$,

$[(0,\alpha),(1,2\alpha),(2,3\alpha),(3,4\alpha),(0,2\alpha+1)]$

with $\alpha = 0,1,\ldots,10$

$\cup\ [(1,2\alpha),(0,\alpha),(2,3\alpha),(3,4\alpha),(1,3\alpha-4)]$

with $\alpha = 11,12,13,14$

$\cup\ [(2,3\alpha),(0,\alpha),(1,2\alpha),(3,4\alpha),(2,4\alpha-8)]$

with $\alpha = 15,16,17,18$

$\cup\ [(3,4\alpha),(0,\alpha),(1,2\alpha),(2,3\alpha),(3,5\alpha-12)]$

with $\alpha = 19,20,21,22$

$\cup\ [(i,0),(i,1),(i,4),(i,6),(i,11)]$

for $i = 1,2,3 \bmod (-,23)$.

- $K_{148} \to G_{16}$: $V(K_{148}) = Z_4 \times Z_{37}$,

$[(0,\alpha),(1,2\alpha),(2,3\alpha),(3,4\alpha),(0,2\alpha+1)]$

with $\alpha = 0,1,2,\ldots,17$

$\cup\ [(1,2\alpha),(0,\alpha),(2,3\alpha),(3,4\alpha),(1,3\alpha-10)]$

with $\alpha = 18,19,\ldots,28$

$\cup\ [(2,3\alpha),(0,\alpha),(1,2\alpha),(3,4\alpha),(2,4\alpha-14)]$

with $\alpha = 29,30,31,32$

$\cup\ [(3,4\alpha),(0,\alpha),(1,2\alpha),(2,3\alpha),(3,5\alpha-18)]$

with $\alpha = 33,34,35,36$

$\cup\ [(1,0),(1,1),(1,4),(1,6),(1,7)]$

$\cup\ [(i,0),(i,1),(i,7),(i,11),(i,8)]$

$\cup\ [(i,0),(i,2),(i,5),(i,14),(i,13)]$

$i = 2,3 \bmod (-,37)$.

(d)

n	$K_n \to G_{16}$ by Lemma	$n_1=n_2=n_3=n_4$	n_5	$K_{n_1,n_2,n_3,n_4,n_5} \to G_{16}$ by Lemma F with r = 7 p=	q=
28t, t≥3, t≠6,17,21,29	A	7t	0	t	0
28t+1, t≥3, t≠6,17,29	B	7t	0	t	0
28t+7, t≥6 t≠7,11,15,18,22,30	A	7t-7	35	t-1	5
28t+8, t≥6, t≠7,11,15,18,30	B	7t-7	35	t-1	5
28t+14, t≥4, t≠6,10,14,17,21,29	A	7t	14	t	2
28t+15, t≥4, t≠6,10,14,17,29	B	7t	14	t	2
28t+21, t≥4, t≠6,10,14,17,21,29	A	7t	21	t	3
28t+22, t≥4, t≠6,10,14,17,29	B	7t	21	t	3
28t, t=6,17,21,29	A	7t-7	28	t-1	4
28t+1, t=6,17,29	B	7t-7	28	t-1	4
28t+7, t=11,15,18,22,30	A	7t-14	63	t-2	9

$28t+8$, $t=11,15,18,30$	B	$7t-14$	63	$t-2$	9
$28t+14$, $t=10,14,17,21,29$	A	$7t-7$	42	$t-1$	6
$28t+15$, $t=10,14,17,29$	B	$7t-7$	42	$t-1$	6
$28t+21$, $t=10,14,17,21,29$	A	$7t-7$	49	$t-1$	7
$28t+22$, $t=10,14,17,29$	B	$7t-7$	49	$t-1$	7

The array is not complete and the missing cases are considered below.

(i) Whether $K_n \to G_{16}$ for $n = 28t+7$ and $t = 4,5,$ or 7, and $n = 28t+8$ for $t = 4$ or 7 is still undecided.

(ii) $n = 28t+14$ or $28t+15$ for $t = 3$: if we have $K_{2,2,2,2,2,2,2} \to K_4$ then it can easily be seen that $K_{14,14,14,14,14,14,14} \to K_{7,7,7,7} \to G_{16}$, and, by Lemma A and B, respectively, we get $K_{98} \to G_{16}$ and $K_{99} \to G_{16}$. The decomposition of $K_{2,2,2,2,2,2,2} \to G_{16}$ is

$$\{\bigcup_{j=1}^{7} [0_j, 0_{j+1}, 0_{j+3}, 1_{j+6}]\} \bmod 2.$$

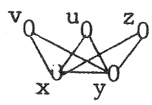
(iii) $n = 28t+14$ or $28t+15$ for $t = 6$: similarly, if $K_{2,2,2,2,2,2,2,2,2,2,2,2,2} \to K_4$ then trivially $K_{14,14,14,14,14,14,14,14,14,14,14,14,14} \to K_{7,7,7,7} \to G_{16}$ and thus $K_{182} \to G_{16}$ and $K_{183} \to G_{16}$. $K_{2,2,2,2,2,2,2,2,2,2,2,2,2} \to K_4$ is given by

$$\{\bigcup_{j=1}^{13} [0_{j+2}, 0_{j+5}, 0_{j+6}, 1_j] \cup [0_j, 0_{j+5}, 0_{j+7}, 1_{j+4}]\} \bmod 2.$$

(iv) $n = 28t+21$ or $28t+22$ for $t = 3$: if we have $K_{3,3,3,3,3} \to K_4$, then $K_{21,21,21,21,21} \to K_{7,7,7,7} \to G_{16}$ and by Lemma A and B, respectively, we have $K_{105} \to G_{16}$ and $K_{106} \to G_{16}$. The case $K_{3,3,3,3,3} \to K_4$ is proved by letting $V(K_{3,3,3,3,3}) = Z_{15}$ and the base graph $[0,1,3,7]$ mod 15.

(v) $n = 28t+21$ or $28t+22$ for $t = 6$: if we have $K_{3,3,3,3,3,3,3,3,3,3} \to K_4$, then $K_{21,21,21,21,21,21,21,21,21,21} \to K_{7,7,7,7} \to G_{16}$ and we have both $K_{189} \to G_{16}$ and $K_{190} \to G_{16}$. $K_{3,3,3,\ldots,3} \to K_4$ is proved by letting $V(K_{3,3,\ldots,3}) = Z_3 \times Z_3 \times Z_3$ and the base graphs $[(0,2,1),(0,0,1),(1,1,2),(1,1,0)] \cup [(0,2,0),(1,2,2),(0,0,2),(1,0,0)]$ mod $(3,3,3)$.

(e) $K_n \to G_{16}$ if and only if $n \equiv 0$ or $1 \pmod 7$, $n \neq 7,8$ except possibly when $n = 119,120,147,203,204$.

(12) (a) G_{17}: 

, $[x,y,z,u,v]$ ($G_{17} \simeq K_{1,1,3}$).

(b) $n \equiv 1$ or $7 \pmod{14}$.

(c) - $K_{7,7,7} \to G_{17}$: $[0_2,0_3,1_1,2_1,3_1]$ $\cup$ $[0_3,0_1,1_2,2_2,3_2]$ $\cup$ $[0_1,0_2,1_3,2_3,3_3]$ mod 7.

- $K_{7,7,21} \to G_{17}$: since $G_{17} = K_{1,1,3}$, the decomposition can be obtained by applying Lemma D with $r_1 = r_2 = 1$, $r_3 = 3$, $a = 7$ and $b = 0$.

- $K_7 \to G_{17}$: $V(K_7) = Z_2 \times Z_3 \cup \{\infty\}$, $[(0,0),(1,0),(0,1),(1,1),\infty]$ mod $(-,3)$.

- $K_{15} \to G_{17}$: $V(K_{15}) = Z_{15}$, $[0,1,3,5,7]$ mod 15.

- $K_{29} \to G_{17}$: $V(K_{29}) = Z_{29}$, $[0,1,4,6,8]$ $\cup$ $[0,2,11,12,15]$ mod 29.

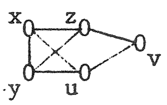
- $K_{49} \to G_{17}$: since $K_7 \to K_3$ (equivalent to a Steiner triple system of order 7) then $K_{7,7,7,7,7,7,7} \to K_{7,7,7} \to G_{17}$ and, since it has been proved that $K_7 \to G_{17}$, we have $K_{49} \to G_{17}$ by Lemma A.

(d)

n	$K_n \to G_{17}$ by Lemma	$n_1=n_2$	n_3	$K_{n_1,n_2,n_3} \to G_{17}$ by Lemma D with $r_1=r_2=r_3=7$, $r_3'=21$, a	b
42t+1, t≥1	B	14t	14t	2t	0
42t+7, t≥2	A	14t-7	14t+21	2t-1	2
42t+15, t≥1	B	14t	14t+14	2t	1
42t+21, t≥0	A	14t+7	14t+7	2t+1	0
42t+29, t≥1	B	14t	14t+28	2t	2
42t+35, t≥0	A	14t+7	14t+21	2t+1	1

(e) $K_n \to G_{17}$ if and only if $n \equiv 1$ or $7 \pmod{14}$.

Remark: The case $K_n \to G_{17}$ for $n \equiv 1 \pmod{14}$ has already been solved, using a different method, in [6], furthermore, the decomposition satisfies an extra condition and is called a balanced G-design for $G = G_{17}$.

(13) (a) G_{18}: 

, [x,y,z,u,v].

(b) $n \equiv 0$ or $1 \pmod 7$.

(c) - $K_8 \not\to G_{18}$: We use degree argument again. The degree of a vertex in K_8 is 7 and since the degrees of vertices in G_{18} are 2 or 3, any vertex of K_8 must appear in one graph of the decomposition with degree 3 and in two others with degree 2. However, this contradicts the fact that in the four graphs of the decomposition, there are altogether 4×4 vertices of degree 3 and 4×1 vertices of degree 2.

- $K_{14} \not\to G_{18}$: use similar argument as above.

- $K_{7,7,7} \to G_{18}$: $[0_1,0_2,0_3,1_3,2_2]$ $\cup$ $[0_1,4_3,1_2,2_2,5_1]$ $\cup$ $[5_2,2_3,0_1,6_1,5_3]$ mod 7.

- $K_{7,7,14} \to G_{18}$: A slightly different notation is used here, namely, though the elements of the first two sets of elements X_1, X_2 are still g_i, $g \in Z_7$, $i = 1,2$ the third set is now assumed to be a union of two sets X_3 and X_3' with elements g_3 and g_3' respectively, $g \in Z_7$. The base graphs are

$[0_1,0_2,0_3,1_3,2_2] \cup [0_1,2_2,4_3,0_3',1_2]$

$\cup [0_1,1_2,1_3',4_3',0_2] \cup [5_2,2_3,0_1,6_1,5_3]$

$\cup [3_2,5_3',0_1,6_1,2_3']$ mod 7.

- $K_{7,7,7,7} \to G_{18}$: $[0_1,0_3,0_2,1_2,1_4]$

$\cup [0_1,2_2,3_3,4_3,4_4] \cup [1_3,3_4,0_1,6_1,5_2]$

$\cup [0_3,2_1,3_4,4_4,5_2] \cup [3_2,6_4,0_1,1_3,4_2]$

$\cup [3_4,4_3,5_1,1_2,5_4]$ mod 7.

- $K_7 \to G_{18}$: $V(K_7) = Z_2 \times Z_3 \cup \{\infty\}$:

$[(0,0),(1,0),(0,1),(1,1),\infty]$ mod $(-,3)$.

- $K_{15} \to G_{18}$: $V(K_{15}) = Z_{15}$, $[0,1,3,6,10]$ mod 15.

- $K_{22} \to G_{18}$: $V(K_{22}) = Z_2 \times Z_{11}$,

$[(0,0),(1,6),(1,3),(1,10),(1,5)]$

$\cup [(0,0),(0,1),(0,3),(1,1),(1,0)]$

$\cup [(1,2),(0,4),(0,0),(0,9),(1,5)]$ mod $(-,11)$.

- $K_{29} \to G_{18}$: $V(K_{29}) = Z_{29}$, $[0,1,10,13,24]$

$\cup [0,2,8,5,12]$ mod 29.

- $K_{35} \to G_{18}$: $V(K_{35}) = Z_2 \times Z_{17} \cup \{\infty\}$,

$[(0,9),(1,9),(0,0),(1,2),\infty]$

$\cup [(1,2),(1,8),(0,0),(0,1),(0,7)]$

$\cup [(1,4),(1,12),(0,0),(0,1),(1,14)]$

$\cup [(0,2),(0,5),(0,1),(0,0),(1,16)]$

$\cup [(1,5),(1,2),(1,1),(1,0),(0,12)]$ mod $(-,17)$.

(d)

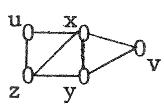
n	$K_n \to G_{18}$ by Lemma	$n_1=n_2$	n_3	$K_{n_1,n_2,n_3} \to G_{18}$ by Lemma D with $r_1=r_2=r_3=7$; $r_3'=14$, a	b
(PART A)					
42t, t≥2, t≠3,4,7	A	14t	14t	2t	0
42t+1, t≥1	B	14t	14t	2t	0
42t+7, t≥2, t≠3,4,7	A	14t	14t+7	2t	1
42t+8, t≥1, t≠2,6,8	B	14t	14t+7	2t	1
42t+14, t≥5, t≠6,7	A	14t	14t+14	2t	2
42t+15, t≥1	B	14t	14t+14	2t	2
42t+21, t≥0	A	14t+7	14t+7	2t+1	0
42t+22, t≥1, t≠2,6,8	B	14t+7	14t+7	2t+1	0
42t+28, t≥1, t≠2,3,6	A	14t+7	14t+14	2t+1	1
42t+29, t≥1, t≠2,6,8	B	14t+7	14t+14	2t+1	1
42t+35, t≥1	A	14t+7	14t+21	2t+1	2
42t+36, t≥3, t≠5,6,7,8	B	14t+7	14t+21	2t+1	2
(PART B)					
42t, t=4,7	A	14t-7	14t+14	2t-1	3
42t+7, t=3,4,7	A	14t-7	14t+21	2t-1	4
42t+8, t=6,8	B	14t-7	14t+21	2t-1	4
42t+14, t=3,4,6,7	A	14t-7	14t+28	2t-1	5
42t+22, t=2,6,8	B	14t	14t+21	2t	3

42t+28, t=6	A	14t	14t+28	2t	4
42t+29, t=2,6,8	B	14t	14t+28	2t	4
42t+36, t=5,7,8	B	14t	14t+35	2t	5
42t+36, t=6	B	14t-7	14t+49	2t-1	8
(PART C)					
42×3=126	A	35	21	5	3
28	A	7	7	1	1
2+42+28=112	A	35	7	5	1
3×42+28=154	A	49	7	7	1
42+36=78	B	21	14	3	2

Finally, for $n = 49$, since $K_{49} \to \{K_7, K_{7,7,7,7,7,7,7}\}$, $K_7 \to G_{18}$, $K_{7,7,7} \to G_{18}$ and $K_{7,7,7,7,7,7,7} \to K_{7,7,7}$ (due to $K_7 \to K_3$) we have $K_{49} \to G_{18}$.

(e) $K_n \to G_{18}$ if and only if $n \equiv 0$ or $1 \pmod 7$, $n \geq 7$, $n \neq 8,14$ except possibly when $n = 36,42,56,92,98,120$.

Remark: if we had been able to show that $K_n \to G_{18}$ for $n = 36,42$ and 56, then the existence of $K_n \to G_{18}$ for $n = 92,98$ or 120 would be implied and Parts (B) and (C) of the array in (d) could have been incorporated into Part (A).

(14) (a) G_{19}: 

, [x,y,z,u,v].

(b) $n \equiv 0$ or $1 \pmod 7$.

(c) - $K_8 \not\to G_{19}$: again by degree argument.

- $K_{7,7,7} \to G_{19}$: $V(K_{7,7,7}) = Z_{21}$, [5,7,0,1,15] mod 21.

- $K_{7,7,7,7} \to G_{19}$: $[5_2,0_1,1_3,2_1,0_3]$
$\cup\ [2_3,4_1,0_4,0_1,3_4] \cup [1_2,1_4,6_1,3_4,0_1]$
$\cup\ [4_2,0_1,3_3,3_4,5_4] \cup [0_2,1_3,3_4,0_3,5_4]$
$\cup\ [0_3,2_2,3_1,3_2,6_4]$ mod 7.

- $K_{14,14,14,14} \to G_{19}$: $V(K_{14,14,14,14}) = Z_{56}$,
$[0,13,27,10,18] \cup [0,15,26,19,9]$
$\cup\ [0,22,25,23,21]$ mod 56.

- $K_7 \to G_{19}$: $V(K_7) = Z_2 \times Z_3 \cup \{\infty\}$,
$[(0,0),(1,0),(1,2),(0,1),\infty]$ mod $(-,3)$.

- $K_{14} \to G_{19}$: $V(K_{14}) = Z_{14}$, $[8,9,0,7,10]$,
$[0,3,2,1,12]$, $[0,6,5,4,13]$, $[11,2,10,0,4]$,
$[8,4,3,1,6]$, $[5,13,8,2,11]$, $[11,1,12,8,6]$,
$[5,9,12,7,1]$, $[3,7,10,5,13]$, $[9,3,11,7,6]$,
$[1,4,10,13,7]$, $[6,12,2,7,10]$ and $[13,4,9,2,12]$.

- $K_{15} \to G_{19}$: $V(K_{15}) = Z_{15}$, $[0,5,7,4,6]$ mod 15.

- $K_{22} \to G_{19}$: $V(K_{22}) = Z_2 \times Z_{11}$,
$[(0,0),(0,1),(1,10),(1,0),(0,3)]$
$\cup\ [(1,7),(0,4),(0,0),(1,4),(1,9)]$
$\cup\ [(0,0),(1,2),(1,6),(0,5),(1,8)]$ mod $(-,11)$.

- $K_{29} \to G_{19}$: $V(K_{29}) = Z_{29}$, $[1,0,5,8,10]$
$\cup\ [2,0,8,19,15]$ mod 29.

- $K_{36} \to G_{19}$: $V(K_{36}) = Z_4 \times Z_9$,
$[(0,0),(0,2),(1,5),(1,8),(1,0)]$
$\cup\ [(1,2),(0,0),(0,1),(2,8),(3,5)]$ mod $(4,9)$ and
$[(0,0),(0,4),(2,5),(2,0),(2,8)]$
$\cup\ [(1,0),(1,4),(3,5),(3,0),(3,8)]$ mod $(-,9)$.

- $K_{50} \to G_{19}$: $V(K_{50}) = Z_2 \times Z_{25}$,

$[(0,0),(1,13),(1,12),(0,12),(1,2)]$

$\cup\ [(1,16),(1,14),(0,0),(0,11),(1,24)]$

$\cup\ [(1,17),(1,20),(0,0),(0,16),(1,8)]$

$\cup\ [(0,0),(1,22),(1,18),(0,8),(0,15)]$

$\cup\ [(0,0),(0,1),(1,24),(1,19),(0,5)]$

$\cup\ [(0,0),(1,21),(1,15),(0,6),(0,18)]$

$\cup\ [(0,0),(1,4),(1,11),(0,3),(0,23)]$ mod $(-,25)$.

- $K_{71} \to G_{19}$: $V(K_{71}) = Z_{71}$: $[0,17,35,10,2]$

$\cup\ [3,7,0,29,30] \cup [0,14,30,31,38]$

$\cup\ [21,9,0,34,41] \cup [11,5,0,19,33]$ mod 71.

- $K_{92} \to G_{19}$: $V(K_{92}) = \bigcup_{i=1}^{13} V_i \cup \{\infty\}$ with $|V_i| = 7$

for $i = 1,2,\ldots,13$.

K_{92} is the edge-disjoint union of

(i) K_{29} on the elements of $\bigcup_{i=1}^{4} V_i \cup \{\infty\}$

(ii) three K_{22}'s on the elements of $V_{5+i} \cup V_{6+i} \cup V_{7+i} \cup \{\infty\}$ for $i = 0,3,$ or 6.

(iii) the graph $K_{3,3,3,4} \otimes S_7$.

If we can prove that $K_{3,3,4} \to \{K_3,K_4\}$ then $K_{3,3,3,4} \otimes S_7 \to \{K_{7,7,7},K_{7,7,7,7}\}$ and since $K_{7,7,7} \to G_{19}$, $K_{7,7,7,7} \to G_{19}$, we have $K_{92} \to G_{19}$. Now $K_{3,3,3,4} \to \{K_3,K_4\}$ with base graphs $[0_4,2_1,1_2] \cup [0_4,2_2,1_3] \cup [0_4,2_3,1_1] \cup [0_4,0_1,0_2,0_3] \cup [\infty,0_1,1_2,2_3]$ mod 3.

(d)

n	$K_n \to G_{19}$ by Lemma	$n_1=n_2=n_3$	n_4	$K_{n_1,n_2,n_3,n_4} \to G_{19}$ by Lemma E, with r	p	q
21t, t≥1	A	7t	0	7	t	0
21t+1, t≥2	B	7t	0	7	t	0
21t+7, t≥1, t≠2,6	A	7t	7	7	t	1
21t+8, t≥5, t≠7	B	7(t-1)	28	7	t-1	4
21t+14, t≥2, t≠2,6	A	7t	14	7	t	2
21t+15, t≥2, t≠2,6	B	7t	14	7	t	2
21t+8, t=7	B	14×3	14×2	14	3	2
21t+14, t=2	A	14	14	14	1	1
21t+14, t=6	A	14×3	14	14	3	1
21t+15, t=2	B	14	14	14	1	1
21t+15, t=6	B	14×3	14	14	3	1

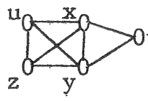
Some cases are missing from the array and are considered here.

$K_{21t+7} \to G_{19}$ for $t = 2$ and 6. The proofs of both cases are similar, hence only one is mentioned here: as $K_7 \to K_3$, we have $K_{7.7} \to \{K_7, K_{7,7,7}\}$, both of which are decomposable into G_{19}, hence $K_{49} \to G_{19}$; for $K_{133} \to G_{19}$, the only difference in the proof is the fact that $K_{19} \to K_3$.

$K_{14,14,14} \to K_{7,7,7} \to G_{19}$, hence $K_{14,14,14} \to G_{19}$.

$K_{35} \to G_{19}$ if $K_{7,7,7,7,7} \to G_{19}$, but the latter can be proved: let $V(K_{7,7,7,7,7}) = Z_{35}$ then $[0,1,4,13,8] \cup [2,0,19,8,14]$ mod 35 are the base graphs.

(e) $K_n \to G_{19}$ if and only if $n \equiv 0$ or $1 \pmod 7$, $n \neq 8$.

(15) (a) G_{20}: 

, $[x,y,z,u,v]$.

(b) $n \equiv 0$ or $1 \pmod{16}$.

(c) - $K_{4,4,4,4,4} \to G_{20}$: $V(K_{4,4,4,4,4}) = Z_{20}$, $[0,1,7,3,9] \bmod 20$.

- $K_{16,16,16,16} \to G_{20}$: $V(K_{16,16,16,16}) = Z_{64}$,
$[0,1,15,6,34] \cup [13,2,23,0,40]$
$\cup [3,45,28,10,0] \bmod 64$.

- $K_{17} \to G_{20}$: $V(K_{17}) = Z_{17}$, $[0,5,7,1,8] \bmod 17$.

- $K_{33} \to 20$: $V(K_{33}) = Z_{33}$, $[0,16,14,1,22]$
$\cup [0,3,8,12,10] \bmod 33$.

- $K_{49} \to G_{20}$: $V(K_{49}) = Z_{49}$, $[0,1,20,24,12]$
$\cup [0,8,17,2,22] \cup [0,3,16,21,10] \bmod 49$.

- $K_{97} \to G_{20}$: $V(K_{97}) = Z_{97}$, $[1,21,36,0,39]$
$\cup [0,2,42,72,33] \cup [0,4,47,84,28]$
$\cup [71,94,0,8,80] \cup [16,45,91,0,26]$
$\cup [0,90,85,32,49] \bmod 97$.

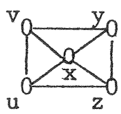
- $K_{113} \to G_{20}$: $V(K_{113}) = Z_{113}$, $[0,1,36,43,56]$
$\cup [0,3,16,108,14] \cup [0,9,48,98,49]$
$\cup [27,31,68,0,65] \cup [81,91,93,0,109]$
$\cup [47,53,0,30,99] \cup [29,54,0,80,73] \bmod 113$.

- $K_{177} \to G_{20}$: $V(K_{177}) = Z_{177}$, $[0,5,3,26,55]$
$\cup [0,6,10,52,67] \cup [12,20,104,0,63]$
$\cup [0,31,24,40,58] \cup [48,62,80,0,87]$
$\cup [0,124,160,96,102] \cup [0,15,71,143,69]$
$\cup [109,142,0,30,9] \cup [41,60,107,0,119]$

$\cup$ [0,120,82,37,29] $\cup$ [74,164,0,63,75] mod 177.

(e) $K_n \to G_{20}$ for $n = 17,33,49,97,113$ and 177.

Remark: We can prove $K_n \to G_{20}$ for $n = 16t$ or $16t+1$ if $K_n \to G_{20}$ for $t = 1,2,3,6,7,8,9,11,26$ or 27. We have been able to solve only six of the twenty cases.

(16) (a) G_{21}: 

, [x,y,z,u,v]. ($G_{21} \simeq K_{1,2,2}$)

(b) $n \equiv 0$ or $1 \pmod{16}$.

(c) - $K_{16} \not\to G_{21}$: by degree argument.

- $K_{8,8,8} \to G_{21}$: $V(K_{8,8,8}) = Z_{24}$, [0,1,5,22,14] mod 24.

- $K_{16,16,16} \to G_{21}$: $K_{16,16,16} \to K_{8,8,8} \to G_{21}$.

- $K_{17} \to G_{21}$: $V(K_{17}) = Z_{17}$, [0,1,4,9,7] mod 17.

- $K_{33} \to G_{21}$: $V(K_{33}) = Z_{33}$, [0,2,19,32,24] $\cup$ [0,5,23,29,26] mod 33.

- $K_{65} \to G_{21}$: $V(K_{65}) = Z_{65}$, [0,8,35,64,52] $\cup$ [0,7,24,63,16] $\cup$ [0,6,40,62,51] $\cup$ [0,5,28,61,15] mod 65.

(d) Proof of $K_{16+1} \to G_{21}$, $t \geq 1$.

n	$K_n \to G_{21}$ by Lemma	$n_1 = n_2$	n_3	$K_{n_1,n_2,n_3} \to G_{21}$ by Lemma D with $r_1=r_2=r_3=16, r_3'=8$ a	b
48t+1, t≥1	B	16t	16t	t	0
48t+17, t≥3	B	16t+16	16t−16	t+1	4
48t+33, t≥1	B	16t+16	16t	t+1	2

We need the following decompositions for the induction used in the array.

- $K_{16,16,8} \to G_{21}$: by Lemma D with $r_1 = r_2 = 2$, $r_3 = r_3' = 1$, $a = 8$ and $b = 0$.

  $K_{113} \to G_{21}$: $V(K_{113}) = \bigcup_{i=1}^{7} Z_{16}^{i} \cup \{\infty\}$

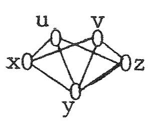
- $K_{113} \to \{K_{17}, K_{16,16,16,16,16,16,16}\}$. It has been shown that $K_{17} \to G_{21}$ and also $K_{16,16,16} \to G_{21}$, the last decomposition with $K_7 \to K_3$ implies that $K_{16,16,16,16,16,16,16} \to G_{21}$.

  As for the other infinite family, $n \equiv 0 \pmod{16}$, if we can prove that $K_n \to G_{21}$ for $n = 16t$, $t = 2,3,4,5$ or 7, then we would have $K_n \to G_{21}$ for the whole family by applying Lemma A and using the same method as displayed in the array. However, of the five values, a decomposition has been obtained only for one, namely when $t = 4$.

- $K_{64} \to G_{21}$: $V(K_{64}) = \bigcup_{i=1}^{3} V_i \cup \{\infty\}$,

  $V_i = \{0_i, 1_i, \ldots, 20_i\}$, $[10_1, 0_1, 2_1, 7_1, 15_2]$
  $\cup\ [20_2, 1_1, 0_1, 4_1, 19_3] \cup [9_1, 0_1, 9_2, 2_1, 6_2]$
  $\cup\ [19_3, 0_1, 6_1, 8_2, 14_2] \cup [1_2, \infty, 0_1, 11_2, 14_3]$
  $\cup\ [8_2, 0_2, 1_2, 3_2, 10_3] \cup [0_2, 3_2, 4_3, 9_2, 18_3]$
  $\cup\ [13_2, 0_1, 0_3, 13_3, 17_2] \cup [12_3, 0_1, 6_3, 7_3, 16_3]$
  $\cup\ [7_3, 0_1, 5_3, 3_1, 14_3] \cup [12_2, 0_1, 3_3, 2_1, 10_3]$
  $\cup\ [0_1, 3_2, 9_3, 20_3, 17_3]$ mod 21.

(e) $K_n \to G_{21}$ if $n \equiv 1 \pmod{16}$ and $n = 64$, $K_{16} \not\to G_{21}$.

(17) (a) G_{22}: 

, $[x,y,z,u,v]$. ($G_{22} \simeq K_{1,1,1,2}$).

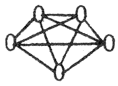
(b) $n \equiv 0$ or $1 \pmod 9$.

(c) - $K_n \not\to G_{22}$ for $n = 9,10,18$: by degree argument.

- $K_{19} \to G_{22}$: $V(K_{19}) = Z_{19}$, $[0,1,3,9,15]$ mod 19.

- $K_{6,6,6,6} \rightarrow G_{22}$: $V(K_{6,6,6,6}) = Z_{24}$, $[0,3,18,1,13] \bmod 24$.

(e) $K_n \not\rightarrow G_{22}$ for $n = 9,10,18$; however, $K_{19} \rightarrow G_{22}$.

(18) (a) G_{23}: 

G_{23} is the complete graph with five vertices and, as mentioned in Section II, $K_n \rightarrow G_{23}$ has already been shown to hold [5]; that is,

$$K_n \rightarrow G_{23} \text{ if and only if } n \equiv 1 \text{ or } 5 \pmod{20}.$$

(Equivalently, a balanced incomplete block design, BIBD $(n,5,1)$ exists if and only if $n \equiv 1$ or $5 \pmod{20}$.)

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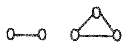
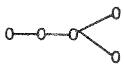
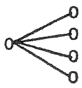
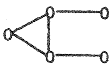
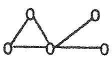
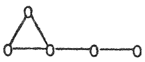
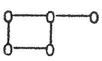
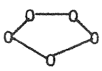
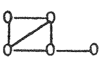
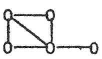
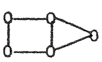
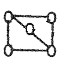
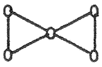
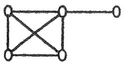
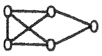
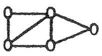
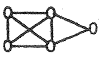
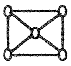
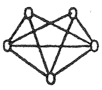
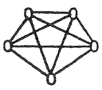
Table 1: on $K_n \to G_i$, where G_i is a graph with five vertices, none of which are isolated and $n \geq 5$.

Graph	Necessary Condition	Result on Sufficient Condition
G_1: 	$n \equiv 0$ or $1 \pmod 3$	same*
G_2: 		same
G_3: 		same
G_4: 	$n \equiv 0$ or $1 \pmod 8$	same
G_5: 		same
G_6: 	$n \equiv 0$ or $1 \pmod 5$	same
G_7: 		$n \equiv 0$ or $1 \pmod 5$, $n \geq 6^+$
G_8: 	$n \equiv 0$ or $1 \pmod 5$	$n \equiv 0$ or $1 \pmod 5$, $n \geq 6^+$
G_9: 		$n \equiv 0$ or $1 \pmod 5$, $n > 6^+$
G_{10}: 	$n \equiv 1$ or $5 \pmod{10}$	same
G_{11}: 		same
G_{12}: 		same
G_{13}: 	$n \equiv 0,1,4$ or $9 \pmod{12}$	$n \equiv 0,1,4$ or $9 \pmod{12}$, except possibly when $n = 24^x$
G_{14}: 		$n \equiv 0,1,4$ or $9 \pmod{12}$, $n \geq 13^+$

Graph	Necessary Condition	Result on Sufficient Condition
G_{15}: 	$n \equiv 1$ or $9 \pmod{12}$	same
G_{16}: 	$n \equiv 0$ or $1 \pmod 7$	$n \equiv 0$ or $1 \pmod 7$, $n \neq 7, 8$ except possibly when $n = 119, 120, 147, 203, 204^x$
G_{17}: 	$n \equiv 1$ or $7 \pmod{14}$	same
G_{18}: 	$n \equiv 0$ or $1 \pmod 7$	$n \equiv 0$ or $1 \pmod 7$, $n \neq 8, 14$ except possibly when $n = 36, 92, 56, 92, 98, 120^x$
G_{19}: 		$n \equiv 0$ or $1 \pmod 7$, $n \neq 8^+$
G_{20}: 	$n \equiv 0$ or $1 \pmod{16}$	$n = 17, 33, 49, 97, 113$ and 117^x
G_{21}: 		$n \equiv 1 \pmod{16}$, $n \neq 16$, $n = 64^x$
G_{22}: 	$n \equiv 0$ or $1 \pmod 9$	$n \neq 9, 10, 18$, $n = 19^x$
G_{23}: 	$n \equiv 1$ or $5 \pmod{20}$	same

*same: The necessary conditions are also sufficient.

\+ : The necessary conditions are not always sufficient.

x : Complete solution has not been obtained.