Broadcasting in Bounded Degree Graphs

Jean-Claude Bermond,

I3S, CNRS
bat 4, rue A. Einstein
Sophia-Antipolis, 06560 Valbonne, France

Pavol Hell, Arthur L. Liestman and Joseph G. Peters

School of Computing Science
Simon Fraser University
Burnaby, British Columbia V5A 1S6, Canada.

Abstract

Broadcasting is an information dissemination process in which a message is to be sent from a single originator to all members of a network by placing calls over the communication lines of the network. Several previous papers have investigated methods to construct sparse graphs (networks) in which this process can be completed in minimum time from any originator. The graphs produced by these methods contain high degree vertices. In [20] and in [4] we began an investigation of graphs with fixed maximum degree in which broadcasting can be completed in near minimum time. In this paper we continue this investigation by giving lower bounds and constructing bounded degree graphs which allow rapid broadcasting. Our constructions use ideas developed by Jerrum and Skyum [16] which allow one to move from a graph with good average case behaviour to one with good worst case behaviour. In addition, we use de Bruijn digraphs [7], minimum broadcast graphs, and sparse broadcast graphs [3]. The resulting graphs yield the best broadcasting time known for bounded degree graphs. We also obtain asymptotic upper and lower bounds for broadcasting time, as the maximum degree increases.

Key Words: broadcasting, graphs, networks, bounded degree graphs, de Bruijn digraphs.

This article was written while the first author was visiting the School of Computing Science, Simon Fraser University for which the support of the Advanced Systems Institute and Simon Fraser University is gratefully acknowledged. This research was also supported by the Natural Sciences and Engineering Research Council of Canada under Grants No. A-5057, A-1734 and A-0322.

1. Introduction

Broadcasting refers to the process of message dissemination in a communication network whereby a message, originated by one member, is transmitted to all members of the network. Broadcasting is accomplished by placing a series of calls over the communication lines of the network. This is to be completed as quickly as possible subject to the constraints that each call involves only two vertices, each call requires one unit of time, a vertex can participate in only one call per unit of time, and a vertex can only call a vertex to which it is adjacent.

Given a connected graph G and a message originator, vertex u, the broadcast time of vertex u, $b(u)$, is the minimum number of time units required to complete broadcasting from vertex u. It is easy to see that for any vertex u in a connected graph G with n vertices, $b(u) \geq \lceil \log_2 n \rceil$, since the number of informed vertices can at most double during each time unit. The broadcast time of a graph G, $b(G)$, is defined to be the maximum broadcast time of any vertex u in G, i.e. $b(G) = \max \{b(u) \mid u \in V(G)\}$. For the complete graph K_n with $n \geq 2$ vertices, $b(K_n) = \lceil \log_2 n \rceil$, yet K_n is not minimal with respect to this property for any $n \geq 3$. That is, we can remove edges from K_n and still have a graph G with n vertices such that $b(G) = \lceil \log_2 n \rceil$.

The broadcast function, $B(n)$, is the minimum number of edges in any graph on n vertices such that each vertex in the graph can broadcast in minimum time, that is, in time $\lceil \log_2 n \rceil$. A minimum broadcast graph (mbg) is a graph G on n vertices having $B(n)$ edges and $b(G) = \lceil \log_2 n \rceil$. From an applications perspective, minimum broadcast graphs represent the cheapest possible communication networks (having the fewest communication lines) in which broadcasting can be accomplished, from any vertex, as fast as theoretically possible.

For a survey of results on broadcasting and related problems, see Hedetniemi, Hedetniemi and Liestman [14]. Slater, Cockayne, and Hedetniemi [24] showed that given an arbitrary graph G, vertex v and $k \geq 4$ as input, deciding whether $b(v) \geq k$ is NP-complete. In [11] Farley, Hedetniemi, Mitchell and Proskurowski studied $B(n)$. In particular, they determined the values of $B(n)$ for $n \leq 15$ and noted that $B(2^k) = k2^{k-1}$ (the k-cube is an mbg on $n = 2^k$ vertices). Mitchell and Hedetniemi [22] determined the value for $B(17)$, Wang [25] found the value of $B(18)$, and Bermond, Hell, Liestman and Peters [3] found the values of $B(19)$, $B(30)$ and $B(31)$. These studies suggest that mbgs are extremely difficult to find; in fact, no mbg with n vertices is known for any value of $n > 32$, except for the easy values of $n = 2^k$, where the k-cube can be used, and the recently discovered family of graphs with $n = 2^k - 2$ [8].

Since mbgs seem to be difficult to find, several authors have devised methods to construct sparse graphs which allow minimum time broadcasting from each vertex. We use the term sparse broadcast graph (sbg) to denote a graph G on n vertices and "close to" $B(n)$ edges such that $b(G) = \lceil \log_2 n \rceil$. In [10] Farley designed several techniques for constructing sparse broadcast graphs with n vertices and approximately $\frac{n}{2} \log_2 n$ edges, for arbitrary values of n. Chau and Liestman [5] presented constructions based on Farley's techniques which yield somewhat sparser graphs for most values of n. In [13] Grigni and Peleg showed that $B(n) \in \Theta(L(n)n)$ for $n \geq 1$ where $L(n)$ denotes the exact number of consecutive leading 1's in the binary representation of $n-1$. Recently, Gargano and Vaccaro [12] gave constructions which produce the best of the known graphs for some large values of n. Asymptotically, Grigni and Peleg's construction (which establishes their upper bound) produces the best of the known graphs for most values of n.

So far, the emphasis in this research has been on obtaining sparse graphs in which each vertex can broadcast in minimum time. If these graphs are to be used in the design of actual networks, other considerations may override the need for minimum time broadcasting. In particular, the constructions of Farley, of Chau and Liestman, and of Gargano and Vaccaro result in graphs with n vertices and average degree $O(\log_2 n)$ while the construction of Grigni and Peleg yields n vertex graphs with some vertices of degree $\log_2 \log_2 n + L(n)$. It may be more realistic to use a graph with fixed maximum degree (see [1], [2], [15]) in which every vertex can broadcast "quickly". We will use the term bounded degree broadcast graph (bdbg) to describe a graph G on n vertices with maximum degree Δ such that $b(G)$ is "close to" $b(n,\Delta) = \min \{ b(H) \mid H \text{ has } n \text{ vertices and max degree } \Delta \}$. (Questions related to broadcasting in slightly more than minimum time have previously been addressed by Liestman [19] and by Grigni and Peleg [13].)

In a recent paper, Liestman and Peters [20] investigated bounded degree broadcast graphs with maximum degrees 3 and 4. They gave lower bounds on the time required to broadcast in such graphs and presented several constructions which produce good bounded degree broadcast graphs. Liestman and Peters showed that $b(n,3) \geq 1.440 \log_2 n - 1.769$ and that if n is a power of 2, then $b(n,3) \leq 2 \log_2 n + 1$. The upper bound is achieved by constructing folded-shuffle-exchange graphs [6]. They also showed that $b(n,4) \geq 1.137 \log_2 n - 0.637$ and that if n is a power of 4, then $b(n,4) \leq 1.625 \log_2 n + 2.25$. The upper bound in this case is achieved by constructing folded-4-shuffle-exchange graphs. More recently, Bermond and Peyrat [4] considered broadcasting in de Bruijn and

Kautz graphs. They were able to improve on the upper bounds of Liestman and Peters, showing, in particular, that $b(n,4) \le 1.5 \log_2 n + 1$ when n is either a power of 2 or 3 times a power of 2.

In this paper, we improve on the results of Liestman and Peters [20] and of Bermond and Peyrat [4]. In Section 2 we present general lower bounds on the time required to broadcast in bounded degree graphs. In Section 3 we give the definition of de Bruijn digraphs and summarize some of the work of Bermond and Peyrat [4] which will be useful in later sections. In Section 4 we give some examples of constructions which motivate the formal definitions of compound graphs in Section 5. In Section 6 we describe how to broadcast in compound graphs, generalizing the work of Bermond and Peyrat [4] reported in Section 3. Finally, in Section 7 we report the best of the known upper bounds on the time required to broadcast in bounded degree graphs. All of these bounds are achieved by using compound graphs. In particular, it will follow from our results that $b(n,3) \le 1.875 \log_2 n + 2.903$ when n is 6 times a power of 4 and that $b(n,4) \le 1.417 \log_2 n + 4$ when n is a power of 8.

2. Lower Bounds

We wish to prove a lower bound on $b(n,\Delta)$, the minimum $b(G)$ for any graph G with n vertices and maximum degree Δ. It will be more convenient to first consider the quantity a_t^Δ which denotes the maximum number of vertices in a graph of maximum degree Δ in which each vertex can inform all others in time t. An upper bound on a_t^Δ will clearly translate to a lower bound on $b(n,\Delta)$. In any broadcasting scheme which achieves this maximum a_t^Δ we may assume that a vertex does not remain idle if it has already been informed and it still has uninformed neighbours. Therefore, if there is one informed vertex at time $t=0$, then after time $t+\Delta-1$, $t \ge 1$, all the vertices that were informed by time t have informed all of their neighbours and must become idle. Hence, $a_{t+\Delta}^\Delta \le a_{t+\Delta-1}^\Delta + (a_{t+\Delta-1}^\Delta - a_t^\Delta) = 2a_{t+\Delta-1}^\Delta - a_t^\Delta$. It is also clear that $a_s^\Delta \le 2^s$ because at each time a vertex can inform at most one other vertex. Thus an upper bound on a_t^Δ is the solution to the recurrence

$$b_s^\Delta = 2^s \quad \text{for } s=0,1,\ldots,\Delta \tag{1}$$
$$b_{t+\Delta}^\Delta = 2b_{t+\Delta-1}^\Delta - b_t^\Delta \quad \text{for all } t \ge 1.$$

Notice that $a_s^\Delta = 2^s$ for $s=0,1,\ldots,\Delta$ since the s-cube, with $s \le \Delta$, is of degree at most Δ and has 2^s vertices. It is possible that $a_t^\Delta = b_t^\Delta$ for all t.

For $\Delta = 3$, we know that $a_s^3 = b_s^3 = 2^s$, for $0 \le s \le 3$. Furthermore, $a_4^3 = b_4^3 = 14$, as the Heawood graph (Figure 7-1(a)) is a cubic graph on 14 vertices with broadcast time 4. Similarly, $a_5^3 = b_5^3 = 24$, as we presented in [3] a cubic graph on 24 vertices with broadcast time 5. Also, $a_6^3 = b_6^3 = 40$, as we presented a cubic graph on 40 vertices with broadcast time 6 in [3]. The next value in the sequence is $b_7^3 = 66$. We do not know whether there is a cubic graph on 66 vertices with broadcast time 7, i.e., if $a_7^3 = 66$.

For $\Delta = 4$, we know that $a_s^4 = b_s^4 = 2^s$, for $0 \le s \le 4$. For $s = 5$, $a_5^4 = b_5^4 = 30$, as we presented (three) 4-regular graphs on 30 vertices with broadcast time 5 in [3]. Moreover, $a_6^4 = b_6^4 = 56$, as we presented a 4-regular graph on 56 vertices with broadcast time 6 in [3]. The next value in the sequence is $b_7^4 = 104$ and we do not know whether $a_7^4 = 104$.

For $\Delta = 5$, we know that $a_6^5 = b_6^5 = 62$ since there is a 5-regular graph on 62 vertices with broadcast time 6 [23]. This graph is constructed by adding chords to a cycle of length 62. In particular, when vertices are numbered consecutively around the cycle, chords are added from each even numbered vertex x to vertices $(x+5) \bmod 62$, $(x-7) \bmod 62$ and $(x+19) \bmod 62$.

Very recently, Dinneen, Fellows, and Faber [8] have solved the conjecture that $a_t^\Delta = b_t^\Delta$ for $t = \Delta + 1$ (see [3]) by constructing a Δ-regular broadcast graph with $2^{\Delta+1} - 2$ vertices for each $\Delta \ge 3$. It is easy to show that these graphs are minimum broadcast graphs. At present, we do not know other values of $t > \Delta$ for which $a_t^\Delta = b_t^\Delta$.

We suspect that $a_t^\Delta = b_t^\Delta$ for all $t \ge \Delta$, or at least that the values are very close. At present, this seems difficult to prove. For example, for $\Delta = 3$ this would mean that $b(n,3) \approx 1.440 \log_2 n$. However, the best construction for n-vertex cubic graphs with small diameter (see [16] or [6]) gives graphs with diameter $1.472 \log_2 n$. Since the diameter of a graph is an obvious lower bound for its broadcast time, this means that if we could show that $b(n,3) \approx 1.440 \log_2 n$, then the cubic graphs which achieve this bound would also improve the results for the diameter problem.

Note that $b(n,\Delta)$ is at least as large as the minimum t for which $a_t^\Delta \ge n$. The best lower bound on $b(n,\Delta)$ that can be obtained from (1) is $L_\Delta(n) = \min \{t : b_t^\Delta \ge n\}$. If $a_t^\Delta = b_t^\Delta$, for all t, then $L_\Delta(n)$ would be the true optimum time for broadcasting in graphs with n vertices and maximum degree Δ. In [20], the recurrence (1) was derived and used to determine that $b(n,3) \geq L_3(n) \approx 1.440 \log_2 n - 1.769$ and $b(n,4) \geq L_4(n) \approx 1.137 \log_2 n - 0.637$. Iterated numerical techniques can be used to find other values of c_Δ such that $L_\Delta(n) \approx c_\Delta \log_2 n$. Table 2-1 lists values of c_Δ for $3 \leq \Delta \leq 16$ that were obtained this way. In contrast, the following careful analysis of the recurrence relation (1) gives the asymptotic behaviour of $L_\Delta(n)$ and therefore the best general lower bound for $b(n,\Delta)$ obtainable from (1).

Theorem 1: For every $\varepsilon > 0$ and all sufficiently large n and Δ,

$$(1+\frac{\log_2 e}{2^\Delta})\log_2 n + O(1) \leq L_\Delta(n) < [1+\frac{(1+\varepsilon)\log_2 e}{2^\Delta}]\log_2 n.$$

Remark: We interpret Theorem 1 to say that $c_\Delta \approx 1+\frac{\log_2 e}{2^\Delta}$. We have listed the values of $1+\frac{\log_2 e}{2^\Delta}$ in Table 2-1.

Δ	c_Δ	$1+\frac{\log_2 e}{2^\Delta}$
3	1.440420	1.180337
4	1.137467	1.090168
5	1.056215	1.045084
6	1.025404	1.022542
7	1.012034	1.011271
8	1.005842	1.005636
9	1.002874	1.002818
10	1.001424	1.001409
11	1.000709	1.000704
12	1.000353	1.000352
13	1.000176	1.000176
14	1.000088	1.000088
15	1.000044	1.000044
16	1.000022	1.000022

Table 2-1: lower bounds

Corollary 2: $b(n,\Delta) \geq (1+\frac{\log_2 e}{2^\Delta})\log_2 n + O(1)$ for all Δ and all sufficiently large n.

Corollary 2 follows from the proof of Theorem 1.

Proof: (of Theorem 1) For simplicity of notation, let $\Delta = d+1$.

To solve (1), consider its characteristic equation $x^{d+1} - 2x^d + 1 = 0$. It is known (see Miles [21]) that this equation has $d+1$ distinct roots $r_0, r_1, \ldots, r_d$, that two of these roots are real, say $r_0 = 1$ and $r_1 = r$ with $1 < r < 2$, and that the remaining $d-1$ roots are complex and lie inside the unit disk of the complex plane. (In fact, Miles studied the equation $x^d - x^{d-1} - \ldots - 1 = 0$. To apply his results we only need to note that $x^{d+1} - 2x^d + 1 = (x-1)(x^d - x^{d-1} - \ldots - 1)$.) By a standard technique [21], there exist complex numbers $s_0, s_1, \ldots, s_d$ so that $b_t^\Delta = s_0 {r_0}^t + s_1 {r_1}^t + \cdots + s_d {r_d}^t$. Since $|r_i| \le 1$ for $i \ne 1$ and $r_1 = r > 1$, we have $b_t^\Delta = \Theta(r^t)$. Thus it is important to estimate r. (We have just discovered that a result similar to Lemma 3 is described by Knuth [18].)

Lemma 3: $2 - \frac{1+\varepsilon}{2^d} < r < 2 - \frac{1}{2^d}$ for all $\varepsilon > 0$ and all sufficiently large d.

Proof: Let $r = 2 - \delta$. Since r is the root of $x^{d+1} - 2x^d + 1 = 0$, we have $\delta(2-\delta)^d = 1$. We will show that $\frac{1}{2^d} < \delta < \frac{1+\varepsilon}{2^d}$. Both of these inequalities follow from the fact that $\frac{1}{2^d}(2 - \frac{1}{2^d})^d < 1$ and $\frac{1+\varepsilon}{2^d}(2 - \frac{1+\varepsilon}{2^d})^d > 1$. The first fact is obvious. The second fact is equivalent to $(1+\varepsilon)(1 - \frac{1+\varepsilon}{2^{d+1}})^d > 1$. For $y = \frac{1+\varepsilon}{2^{d+1}}$, we can use the estimate $1 - y > e^{-2y}$ (which is valid for $y \in (0, \frac{1}{2}]$) to obtain $(1-y)^d > e^{-2dy} = e^{-d(\frac{1+\varepsilon}{2^d})} > \frac{1}{1+\varepsilon}$, when $\frac{d}{2^d} < \frac{\ln(1+\varepsilon)}{1+\varepsilon}$. This certainly holds for any fixed ε and d large enough. □

To complete the proof of Theorem 1 we need to bound $L_\Delta(n)$. Let Δ be fixed and let $\alpha = 2 - \frac{1}{2^d}$ be the upper bound from Lemma 3. If $b_t^\Delta \ge n$ and n is sufficiently large, then t will also be large enough so that $b_t^\Delta \le c_1 \alpha^t$ (for some positive c_1). Thus $c_1 \alpha^t \ge n$ and $t \ge \frac{1}{\log_2 \alpha} \log_2 n + O(1)$. This means that $L_\Delta(n) \ge \frac{1}{\log_2 \alpha} \log_2 n + O(1)$. Now since $\log_2 \alpha = 1 + \log_2 e \ln(1 - \frac{1}{2^{d+1}})$, we can use the fact that $\frac{1}{1 + \log_2 e \ln(1-y)} > 1 + (\log_2 e) y$ for $0 < y < 1$ to obtain $L_\Delta(n) \ge (1 + \frac{\log_2 e}{2^\Delta}) \log_2 n + O(1)$ for any fixed Δ.

For given n and Δ, let t be such that $b_t^\Delta \ge n$ and $b_{t-1}^\Delta < n$ and let $\beta = 2 - \frac{1+\varepsilon}{2^d}$ be the lower bound from Lemma 3. If n is sufficiently large, then t will be large enough so that $c_2 \beta^{t-1} \le b_{t-1}^\Delta$ (for some $c_2 > 0$). Thus $c_2 \beta^{t-1} < n$ and $t < \frac{1}{\log_2 \beta} \log_2 n + O(1)$. This means that $L_\Delta(n) < \frac{1}{\log_2 \beta} \log_2 n + O(1)$. Now since $\log_2 \beta = 1 + \log_2 e \ln(1 - \frac{1+\varepsilon}{2^{d+1}})$, we can use the fact that for any $\varepsilon', 0 < \varepsilon' < 1$, there exists $f(\varepsilon')$ such that if $0 < y < f(\varepsilon')$, $\frac{1}{1 + \log_2 e \ln(1-y)} < 1 + (1+\varepsilon')(\log_2 e) y$ to obtain $L_\Delta(n) < (1 + \frac{(1+\varepsilon)(1+\varepsilon')\log_2 e}{2^\Delta}) \log_2 n + O(1)$ for Δ large enough. Therefore, for any ε'' such that $(1+\varepsilon'') > (1+\varepsilon)(1+\varepsilon')$, and for any Δ and n sufficiently large we get $L_\Delta(n) < (1 + \frac{(1+\varepsilon'')\log_2 e}{2^\Delta}) \log_2 n$. □

As mentioned above, it may be the case that $L_{\Delta}(n)$ is the optimum broadcast time in graphs with maximum degree Δ. However, the best of our constructions at present only achieves $b(n,\Delta) \le (1+\frac{c'}{\Delta})\log_2 n$ asymptotically with $c' \approx .415$ (see Section 7).

3. Broadcasting in de Bruijn Digraphs

In the remainder of the paper, we show how to construct bounded degree graphs which allow rapid broadcasting. Our constructions make use of de Bruijn digraphs which were defined in [7]. The de Bruijn digraph $B(d,D)$ with indegree and outdegree d and diameter D is the digraph whose d^D vertices are the words of length D on an alphabet of d letters (we will always use the alphabet $\{0,1,\ldots,d-1\}$). There is an arc from a vertex x to a vertex y if and only if the last $D-1$ letters of x are the same as the first $D-1$ letters of y, that is, there are arcs from $(x_1,\ldots,x_D)$ to the vertices $(x_2,\ldots,x_D,\lambda)$ where λ is any letter of the alphabet. Note that the diameter of $B(d,D)$ is D because $(x_1,\ldots,x_D)$, $(x_2,\ldots,x_D,z_1)$, $(x_3,\ldots,x_D,z_1,z_2)$, $\ldots$, $(x_D,z_1,\ldots,z_{D-1})$, $(z_1,\ldots,z_D)$ is a directed path of length D joining any vertex $(x_1,\ldots,x_D)$ to any other vertex $(z_1,\ldots,z_D)$.

We use $UB(d,D)$ to denote the associated undirected de Bruijn graph. That is, $UB(d,D)$ is the graph whose vertices are the words of length D on an alphabet of d letters in which the vertex $(x_1,\ldots,x_D)$ is adjacent to the vertices $(x_2,\ldots,x_D,\lambda)$ (called the right neighbours of $(x_1,\ldots,x_D)$) and the vertices $(\lambda,x_1,\ldots,x_{D-1})$ (called the left neighbours of $(x_1,\ldots,x_D)$). It is clear that $UB(d,D)$ has maximum degree $\Delta = 2d$.

Bermond and Peyrat [4] recently investigated broadcasting in de Bruijn graphs and presented three different broadcasting schemes. One of their schemes proves that $b(UB(d,D)) = \frac{(d+1)}{2}D + \frac{d}{2}$. One of their other schemes (which achieves a slightly different bound) is useful in understanding the broadcasting scheme for compound graphs which follows in Section 6. It is the following scheme in which each vertex (including the originator) only sends the message to its right neighbours. A vertex $(x_1,\ldots,x_D)$ informs its neighbours in an order which depends on $\delta \equiv [\sum_{i=1}^{D} x_i] \bmod d$, the d-arity of the label $(x_1,\ldots,x_D)$. Vertex $(x_1,\ldots,x_D)$ sends the message to its uninformed neighbours in the order $(x_2,\ldots,x_D,\delta)$, $(x_2,\ldots,x_D,(\delta+1))$, $\ldots,(x_2,\ldots,x_D,(\delta-1))$ (addition/subtraction modulo d). As shown in [4], the message will arrive at any vertex $(z_1,\ldots,z_D)$ from $(x_1,\ldots,x_D)$ after at most $\frac{d+1}{2}(D+1)$ time units along one of the d paths $(x_1,\ldots,x_D)$, $(x_2,\ldots,x_D,\alpha)$, $(x_3,\ldots,x_D,\alpha,z_1)$,

$\ldots, (x_{i+2}, \ldots, x_D, \alpha, z_1, \ldots, z_i), \ \ldots, (\alpha, z_1, \ldots, z_{D-1}), (z_1, \ldots, z_D)$, where $0 \le \alpha \le d-1$.

Strictly speaking, in the above scheme the message may arrive at some vertex more than once. By deleting redundant calls, a scheme can be obtained which completes the broadcast at the appropriate time.

4. Introduction to Compound Graphs

Our goal is to construct graphs with bounded degree and good (asymptotic) broadcasting time. To do this, we will use the notion of compound graphs (see [2]). In particular we will use the ideas developed by Jerrum and Skyum [16] to construct bounded degree graphs with the best diameter currently known. (It appears that a bounded degree graph with smallest diameter does not necessarily give the best broadcasting time.) In this section, we describe some ad hoc constructions which give some insight into the details that are addressed more formally in later sections.

To construct a bounded degree graph which may have good broadcasting time, begin with a "good" digraph B, such as a de Bruijn digraph. Replace each vertex x of B by a copy G_x of a "suitable" graph G and join the copies using the arcs of B. (If there is an arc from x to y in B, then put an arc from G_x to G_y as described below.) The associated undirected graph $G[B]$ is called the compound of G in B. We will, however, find it useful to refer to the directions of the arcs when describing the construction of a specific compound graph or broadcast schemes for it.

For example, let $B = B(2, D)$, the de Bruijn digraph with indegree 2, outdegree 2, and diameter D and let G be a single edge ($G = K_2$). Replace each vertex x in B by two new vertices $(x;0)$ and $(x;1)$ joined by an edge in the new graph $G[B]$. The arcs incident on x in B are redirected in $G[B]$ so that each vertex $(x;j)$ for $j = 0, 1$ has one incoming arc and one outgoing arc. (Figure 4-1(a) shows a vertex x in B and its incident edges. Figure 4-1(b) shows the corresponding vertices $(x;0)$ and $(x;1)$ of the new graph $G[B]$ and their incident edges.) The resulting graph is $H = K_2[B(2, D)]$.

Suppose we want to broadcast from vertex $(x;j)$ in this new graph H. At time 1, send the message to $(x;j+1)$ (addition modulo 2), the other vertex in the originator's copy of K_2. At time 2, send the message to the two copies $G_{x'}$ and $G_{x''}$ where xx' and xx'' are arcs of B, that is, if $x = (x_1, \ldots, x_D)$, then $x' = (x_2, \ldots, x_D, 0)$ and $x'' = (x_2, \ldots, x_D, 1)$. If a message arrives at time t in some copy G_z, then at time $t+1$ send it to the other vertex of G_z, and at time $t+2$ send it to the two copies $G_{z'}$ and $G_{z''}$ where

Figure 4-1: replacing a vertex with an edge

zz' and zz'' are arcs of B. Since the diameter of $B(2,D)$ is D, at time $2D$ the message will have reached all copies of G and at time $2D+1$ all the vertices of $K_2[B(2,D)]$ are informed.

We have constructed a graph H on $n = 2 \times 2^D = 2^{D+1}$ vertices which is 3-regular and has broadcast time $b(H) \leq 2D+1 = 2\log_2 n - 1$. This simple construction allows us to match the result of Liestman and Peters [20] for graphs of maximum degree 3. As we will see, this technique will enable us to make further improvements.

Note that in the previous example we did not specify which of the two arcs leaving x will be associated (in the compound graph) with $(x;0)$ and which with $(x;1)$. In this case it does not matter as some vertex in both copies $G_{x'}$ and $G_{x''}$ will receive the message two units of time after it arrives in copy G_x. However, this need not always be the case as is illustrated by the next example.

Let $B = B(3,D)$, the de Bruijn digraph with indegree 3, outdegree 3, and diameter D. Replace each vertex x in B with a copy of a 3-star G consisting of vertices $(x;j)$ where $j = 0, 1, 2, 3$ and edges $(x;0)\,(x;3)$, $(x;1)\,(x;3)$ and $(x;2)\,(x;3)$. The arcs incident on x in B are distributed among the vertices $(x;j)$ for $j = 0, 1, 2$ in $G[B]$ so that each vertex has one incoming arc and one outgoing arc. (See Figure 4-2(a) and (b).)

To broadcast in $G[B]$ we can use the following scheme. If a message arrives (or originates) at time t at vertex $(x;j)$ in some copy G_x, send it to $(x;3)$ at time $t+1$. At time $t+2$, $(x;3)$ sends it to $(x;j+1)$ and $(x;j)$ sends it to the copy $G_{x'}$ joined to G_x by the arc going out from $(x;j)$. At time $t+3$, $(x;3)$ sends it to $(x;j+2)$ while $(x;j+1)$ sends it to the copy $G_{x''}$ joined to G_x by the arc going out from $(x;j+1)$. At time $t+4$, $(x;j+2)$ sends it to the copy $G_{x'''}$ joined to G_x by the arc going out from $(x;j+2)$. (All additions on vertex labels are performed modulo 3.) This is illustrated in Figure 4-2(c) where the vertex labels denote the delay in transmitting the message from $(x;j)$. Even without specifying exactly how the arcs are

Figure 4-2: replacing a vertex with a 3-star

connected, we can see that the constructed graph $H' = G[B]$ has broadcast time $b(H') \leq 4D + 3$.

If we are more specific about the connections in H', the bound can be improved. We will use e_j to label the arc entering the vertex $(x;j)$ and o_j to label the arc leaving $(x;j)$ as shown in Figure 4-3. (Of course these labels are relative to a particular G_x.) For any $G_{(x_1, \ldots, x_D)}$, e_j will come from $G_{(j, x_1, \ldots, x_{D-1})}$. The arc o_j will join $(x;j)$ to $G_{(x_2, \ldots, x_D, \delta(x)+j)}$ where $\delta(x) \equiv [\sum_{i=1}^{D} x_i] \pmod 3$ is the 3-arity of x. We will see in Section 7 that the broadcast time $b(H')$ is at most $3D+6$ with these connections.

Figure 4-3: connections for G_x with $x = (x_1, \ldots, x_D)$

5. Construction of the Compound Graph $G[B]$

We now formally describe the construction of $G[B]$, the compound of G in B. Let G be a graph on p vertices and let $B = B(d, D)$ be the de Bruijn digraph of outdegree d, indegree d and diameter D. The vertices of $B(d, D)$ are labelled $(x_1, \ldots, x_D)$ with $x \in \{0, 1, \ldots, d-1\}$. $G[B]$ is obtained by replacing each vertex x of B by a copy of G (denoted G_x) and by associating with each arc xy of B an edge between G_x and G_y as described below.

The vertices of $G[B]$ are labelled $(x; j)$ where $x = (x_1, \ldots, x_D)$ is a vertex of B and j is a vertex of G. The set of vertices of the copy G_x is the set of all vertices $(x; j)$ with $j \in V(G)$. There are d arcs entering and d arcs leaving a given G_x. An arc entering G_x will be labelled e_i if it comes from $G_{(i, x_1, \ldots, x_{D-1})}$. Now we can assign the d in-arcs to the vertices of G_x in a uniform way for all copies of G by giving a mapping g of $\{0, 1, \ldots, d-1\}$ into $V(G)$ so that the arc e_i enters G_x by way of the vertex $(x; g(i))$.

We label the outgoing arcs o_i in each copy of G. In G_x, we let the arc o_i go to $G_{(x_2, \ldots, x_D, i + \delta(x))}$ where $\delta(x)$ is the d-arity of the vertex $x = (x_1, \ldots, x_D)$ (addition is modulo d). The d out-arcs are assigned to the vertices of each copy of G in a uniform way by giving a mapping f of $\{0, 1, \ldots, d-1\}$ into $V(G)$ so that the arc o_i starts at $(x; f(i))$.

In summary, we choose two mappings f and g from $\{0, 1, \ldots, d-1\}$ to $V(G)$. There is an edge between $(x; j)$ and $(y; k)$ if and only if either $x = y$ and j is adjacent to k in G or xy is an arc of B with $x = (x_1, \ldots, x_D)$, $y = (x_2, \ldots, x_D, x_{D+1})$, $j = f(x_{D+1} - \delta(x))$ and $k = g(x_1)$. Note that while f and g significantly influence the construction of $G[B]$ we will continue to use the simplified notation $G[B]$, and let f and g be determined by context.

Consider the following example, illustrated in Figure 5-1. Let $G = C_4$, $d = 6$, and $f(0) = f(1) = 0$, $f(2) = 1$, $f(3) = f(4) = 2$, $f(5) = 3$, $g(0) = 0$, $g(1) = g(2) = 1$, $g(3) = 2$, $g(4) = g(5) = 3$. For example, e_3 will always enter $(x; 2)$ since $g(3) = 2$. Similarly, o_3 will always connect $(x; 2)$ to the copy $G_{(x_2, \ldots, x_D, \delta(x) + 3)}$ since $f(3) = 2$.

The graph $G[B(d, D)]$ contains $|V(G)| \times |V(B(d, D))| = pd^D$ vertices. The degree of vertex $(x; j)$ is the sum of the degree of j in G and the number of arcs from $B(d, D)$ entering and leaving $(x; j)$, that is, $d(x; j) = d_G(j) + |f^{-1}(j)| + |g^{-1}(j)|$. (Note that this is independent of x.) The sum of the degrees of the vertices of any copy G_x is $2|E(G)| + 2d$. Note that we can always choose f and g in order to insure a

Figure 5-1: replacing a vertex with a C_4

maximum degree Δ in $G[B]$ as long as $\Delta \geq \max \{d_G(j) \mid j \in V(G)\}$ and $p\Delta \geq 2|E(G)| + 2d$. Indeed it suffices to choose f and g such that $d_G(j) + |f^{-1}(j)| + |g^{-1}(j)| \leq \Delta$, $0 \leq j \leq p-1$.

In fact, for constructing bounded degree broadcast graphs we do this backwards. Given the degree Δ that we desire for the compound graph, we choose a graph G with maximum degree at most Δ and then determine d satisfying $2d \leq p\Delta - 2|E(G)|$. In order to obtain the maximum possible number of vertices we will choose $d = \lfloor \frac{p\Delta - 2|E(G)|}{2} \rfloor$.

6. Broadcast Time in $G[B]$

In the broadcast scheme for the de Bruijn digraph B described in Section 3, when a message arrives at vertex $(x_1, \ldots, x_D)$, the vertex relays the message to its right neighbours $(x_2, \ldots, x_D, \lambda)$. The general idea for broadcasting in the compound graph $G[B]$ is to simulate this scheme. In $G[B]$, when a message first arrives at a vertex $((x_1, \ldots, x_D);j)$ (for some vertex j of G), we inform the other members of this copy of G and relay the message to the "out-neighbour copies" of G which replaced the neighbours of $(x_1, \ldots, x_D)$.

To determine the broadcast time of $G[B]$ we introduce a new parameter $\bar{b}(G)$, the smallest possible "average time" needed to transmit a message originated in a copy of G to its out-neighbour copies. Formally, let G be a graph to which d outgoing arcs $o_0, \ldots, o_{d-1}$ have been attached. (These arcs will correspond to the arcs o_i defined in $G[B]$.) Note that although $o_0, \ldots, o_{d-1}$ significantly influence the

Figure 6-1: average time

value of $\overline{b}(G)$, we prefer not to make them part of the notation $\overline{b}(G)$. For a vertex u of G and a particular broadcasting scheme for u in G, let t_u^i denote the time at which a message originated at u at time 0 will be received by the vertex at the other end of arc o_i. In the compound graph, this is an upper bound on the time at which the message will reach the copy connected to G by o_i. The value $\overline{t_u} = \frac{1}{d}(t_u^0 + \cdots + t_u^{d-1})$ is the average time for a message originated at u to arrive at the copies of G

along the arcs o_i under the given broadcast scheme. If we let $\overline{b}(G,u)$ be the minimum of $\overline{t_u}$ over all possible broadcasting schemes for originator u in G, then $\overline{b}(G) = \max\ \{\overline{b}(G,u) \mid u \in V(G)\}$.

The examples of Figure 6-1 illustrate how $\overline{b}(G)$ is calculated. In each case, the vertex labelled 0 is the originator. The label on each other vertex indicates the time at which the vertex receives the message (assuming that the first call is received at time 1). The label at the end of an arc indicates the time at which the message reaches the corresponding copy of G. Figure 6-1(a) shows the times for $G = K_2$. Since both adjacent copies receive the message at time 2, $\overline{b}(K_2) = 2$. Figure 6-1(b) shows that if G is the 3-star, the neighbouring copies receive the message at times 2, 3 and 4. Thus, $\overline{b}(G) = 3$ for this graph G. Figure 6-1(c) shows the times for $G = T_6$, where T_6 is a particular tree on six vertices. With this particular graph, the neighbouring copies receive the message at times 2, 4, 4 and 5 giving $\overline{b}(G) = \frac{15}{4}$. Figure 6-1(d) and Figure 6-1(e) show the times for the two possible non-isomorphic originators in the graph $G = C_4$, the cycle of four vertices. In both cases, four neighbouring copies receive the message at time 3 and the two other neighbouring copies receive the message at time 4. This gives $\overline{b}(C_4) = \frac{20}{6}$. Other examples appear in Section 7.

Theorem 4: $b(G[B(d,D)]) \le (D+1)\,\overline{b}(G) + b(G)$.

Proof: Choose, for each vertex u of G, a broadcasting scheme in G such that $\overline{b}(G,u) \le \overline{b}(G)$. Using this scheme, if a message arrives at the vertex $(x;u)$ in G_x at time t, it will be received at the other end of the arc o_i at time $t + t_u^i$ and we obtain a scheme for $G[B(d,D)]$.

Note that by the definition of $\overline{b}$, $\sum_{i=0}^{d-1}\ t_u^i \le d\overline{b}(G)$.

Consider an arbitrary originator, vertex $(x;j_0)$ with $x = (x_1, \ldots, x_D)$, and a goal copy G_y with $y = (y_1, \ldots, y_D)$. Consider also the d paths $z_0 = x = (x_1, \ldots, x_D)$, $z_1 = (x_2, \ldots, x_D, \alpha)$, $z_2 = (x_3, \ldots, x_D, \alpha, y_1)$, ..., $z_i = (x_{i+1}, \ldots, x_D, \alpha, y_1, \ldots, y_{i-1})$, ..., $z_D = (\alpha, y_1, \ldots, y_{D-1})$, $z_{D+1} = (y_1, \ldots, y_D)$ in $B(d,D)$ with $\alpha \in \{0, 1, \ldots, d-1\}$. In all of these d paths in $G[B(d,D)]$, we enter the copy G_{z_i}, $0 \le i \le D$, via the same vertex, namely $(z_i; g(x_i))$. Copy $G_{z_{D+1}}$ is entered via vertex $(z_{D+1}; g(\alpha))$ which is different for each of the d paths. We leave the copy $G_{z_i}, 1 \le i \le D+1$, via different vertices, as the leaving arc depends on the d-arity of z_i which changes with α. Of course, each path also leaves copy G_{z_0} by a different vertex. For a fixed α (and hence a fixed path $z_0, z_1, \ldots, z_{D+1}$) we denote by α_i the subscript j of the arc o_j leading from G_{z_i} to G_{z_i+1}. Then the time the information arrives (on this path) at the goal copy G_y is $t_\alpha = t_{j_0}^{\alpha_0} + t_{j_1}^{\alpha_1} + \cdots + t_{j_D}^{\alpha_D}$ where $j_i = g(x_i)$ for $i = 1, \ldots, D$.

Now consider different values of α. As α takes on all of the possible values $\{0, 1, \ldots, d-1\}$, the d-arity of z_i for each fixed $i > 0$ takes all the possible values $\{0, 1, \ldots, d-1\}$. It follows that the α_i also take on all of the possible values $\{0, 1, \ldots, d-1\}$. Therefore, $\sum_{\alpha=0}^{d-1} t_\alpha = \sum_{i=0}^{d-1} \sum_{k=0}^{D} t_{j_k}^{i} = \sum_{k=0}^{D} \sum_{i=0}^{d-1} t_{j_k}^{i} \le (D+1)\, d\, \bar{b}(G)$.

The minimum time at which G_y is informed is $\min_\alpha \{t_\alpha\} \le \frac{1}{d} \sum_{\alpha=0}^{d-1} t_\alpha \le (D+1)\bar{b}(G)$. Since it requires at most $b(G)$ units of time to complete the broadcast within G_y, we have $b(G[B(d,D)]) \le (D+1)\,\bar{b}(G) + b(G)$. □

As noted in Section 3, in the above scheme the message may arrive at some vertex more than once. By deleting redundant calls, a scheme can be obtained which completes the broadcast at the appropriate time.

7. Upper Bounds

In this section, we give examples of the construction of specific bounded degree graphs and determine the broadcast time for these graphs. We have determined the broadcast time of many such graphs. The best result obtained for each degree Δ, $3 \le \Delta \le 16$, is given in Table 7-1.

Let us turn our attention to cubic graphs. As we saw in Section 4, if we let $B = B(2,D)$ and $G = K_2$, the resulting compound graph $H = K_2[B(2,D)]$ is a 3-regular graph on $n = 2 \times 2^D = 2^{D+1}$ vertices and has broadcast time $b(H) \le 2D + 1 = 2\log_2 n - 1$. (Note that Theorem 4 gives $b(H) \le 2D + 3$ since the broadcast scheme described in the proof of Theorem 4 is more general than the one developed for the specific case in Section 4.) However, if we choose the 3-star as G and compound this graph with $B = B(3,D)$, we get $H' = G[B(3,D)]$, a graph on 4×3^D vertices. Since $\bar{b}(G) = 3$ (see Figure 6-1(b)), we get $b(H') \le 3D + 6 = 3\log_3 n - 3\log_3 4 + 6 \approx 1.893\log_2 n + 2.214$. The graph T_6 of Figure 6-1(c) has $\bar{b}(T_6) = \frac{15}{4}$. If we compound T_6 with $B = B(4,D)$, we get $H'' = T_6[B(4,D)]$, a graph on 6×4^D vertices with $b(H'') \le \frac{15}{4}D + \frac{31}{4} = \frac{15}{4}\log_4 n - \frac{15}{4}\log_4 6 + \frac{31}{4} \approx 1.875\log_2 n + 2.903$. This is the best value we have obtained for cubic graphs.

As Δ gets larger, the number of possible constructions grows rapidly and the task of finding the best construction for a particular fixed Δ becomes difficult. Another approach is to start with a particular graph which is known to be good for broadcasting, such as a minimum broadcast graph or a sparse broadcast graph, and compound it with de Bruijn digraphs of various degrees to obtain compound graphs for various values of Δ.

As an example, consider the Heawood graph (shown in Figure 7-1(a)), a cubic graph which is a minimum broadcast graph on 14 vertices. Figure 7-1(b) describes a broadcast scheme for an arbitrary originator in the graph. (Only those edges on which calls are made are shown.) The label on each vertex indicates the time at which the vertex receives the message assuming that the first call is made at time 1. Note that if a vertex in this 14 vertex graph begins a broadcast at time 0, two vertices (specifically the originator and the first vertex it calls during the scheme) will have completed all of their calls at time 3 and the remaining twelve vertices will all be involved in a final call at time 4. The two former vertices will be "available" to call vertices external to G at time 4 and the twelve other vertices will be "available" at time 5.

Figure 7-1: Heawood graph (a) and its broadcast scheme (b)

Replace each vertex in the de Bruijn digraph $B = B(14, D)$ with a copy of G, the Heawood graph, such that each vertex of G has one in-arc and one out-arc. The resulting graph is $G[B(14, D)]$ which is regular of degree 5. In this particular graph, each vertex in each copy of G has one out-neighbour, so $\overline{b}(G) = \frac{68}{14}$. Since $b(G) = 4$, from Theorem 4, $b(G[B(14,D)]) \leq (D+1)\frac{68}{14} + 4 \quad = \frac{68}{14} \log_{14} n + O(1)$ $\approx 1.276 \log_2 n + O(1)$.

Similarly, we could use $B = B(28, D)$. In this case, replace each vertex in B with a copy of G, the Heawood graph, as before except that each vertex of G has two in-arcs and two out-arcs attached. The resulting graph, $G[B(28,D)]$ is a regular graph of degree 7. Since each vertex has two out-neighbours, we get $\overline{b}(G) = \frac{150}{28}$. Since $b(G) = 4$, Theorem 4 gives $b(G[B(28,D)]) \leq (D+1)\frac{150}{28} + 4$ $= \frac{150}{28} \log_{28} n + O(1) \approx 1.114 \log_2 n + O(1)$. As indicated in Table 7-1, the Heawood graph produces the best bounded degree graph for $\Delta = 7$.

In fact, if we wish to produce an odd degree $\Delta \geq 5$ compound graph, we can replace each vertex of the de Bruijn digraph $B(7(\Delta - 3), D)$ with a Heawood graph such that each vertex of each Heawood graph has $\frac{(\Delta-3)}{2}$ in-arcs and $\frac{(\Delta-3)}{2}$ out-arcs.

To produce a degree 4 compound graph, we can replace each vertex of the de Bruijn digraph $B(7,D)$ with a Heawood graph so that each vertex of each Heawood graph has either one in-arc or one out-arc. If the arcs are connected arbitrarily, we can be assured that each of the seven vertices with out-neighbours will be available by time 5, so that $\overline{b}(G) = \frac{35}{7} = 5$. However, if we use the fact that the Heawood graph is bipartite and the two vertices that are available at time 4 are adjacent, we can connect the arcs as shown in Figure 7-2 and guarantee that one of the vertices with an out-neighbour is available at time 4, giving $\overline{b}(G) = \frac{34}{7}$. Using this connection scheme, $b(G[B(7,D)]) \leq (D+1)\frac{34}{7} + 4$ $= \frac{34}{7} \log_7 n + O(1) \approx 1.730 \log_2 n + O(1)$.

To produce even degree $\Delta \geq 6$ compound graphs, we can replace each vertex of the de Bruijn digraph $B(7(\Delta - 3), D)$ with a Heawood graph such that the seven vertices of one class of each Heawood graph have $\frac{\Delta}{2} - 1$ in-arcs and $\frac{\Delta}{2} - 2$ out-arcs while the vertices in the other class have $\frac{\Delta}{2} - 2$ in-arcs and $\frac{\Delta}{2} - 1$ out-arcs.

Let us now consider the *k*-cubes, one of the two known infinite families of minimum broadcast graphs. The *k*-cube on $n = 2^k$ vertices is regular of degree k and all vertices become available $k+1$ time units after the originator receives the message.

A graph of degree Δ can be constructed by replacing each vertex of the de Bruijn digraph $B(m\, 2^{\Delta - 2m}, D)$ by G, the $(\Delta - 2m)$-cube for any $1 \leq m \leq \lfloor \frac{\Delta - 1}{2} \rfloor$, and distributing the arcs from the de Bruijn digraph so that each vertex of each copy of G is given m in-arcs and m out-arcs. Each vertex of a cube can inform its m new neighbours at times $\Delta - 2m + 1, \Delta - 2m + 2, \ldots, \Delta - m$ after the message first

Figure 7-2: connection scheme

enters the cube. This gives $\bar{b}(G) = [\sum_{i=1}^{m} (2^{\Delta-2m})(\Delta-2m+i)] / [m 2^{\Delta-2m}] = \Delta - \frac{3m-1}{2}$. Applying Theorem 4, we obtain $b(G[B(m2^{\Delta-2m}, D)]) \leq (D+1)(\Delta - \frac{3m-1}{2}) + \Delta - 2m = (\Delta - \frac{3m-1}{2}) \log_{m 2^{\Delta-2m}} n + O(1) = [(\Delta - \frac{3m-1}{2}) / (\Delta - 2m + \log_2 m)] \log_2 n + O(1)$. Thus,

$$b((\Delta-2m)\text{-cube}[B(m2^{\Delta-2m}, D)]) \leq \frac{\Delta - \frac{3m-1}{2}}{\Delta - 2m + \log_2 m} \log_2 n + O(1). \qquad (2)$$

The constant $(\Delta - \frac{3m-1}{2}) / (\Delta - 2m + \log_2 m)$ from expression (2) can be simplified to the form $1 + \frac{c_m}{\Delta - d_m}$ where c_m and d_m are constants depending on m. The expression for the broadcasting time of the compound of a cube in a de Bruijn digraph therefore has the form $(1 + \frac{c_m}{\Delta - d_m}) \log_2 n + O(1)$. The smallest constant c_m that can be obtained from expression (2) is $c_3 = 2 - \log_2 3 \approx 0.415$, so the best asymptotic broadcasting time will result from the compound of a $(\Delta-6)$-cube in a de Bruijn digraph $B(3 \times 2^{\Delta-6}, D)$:

Theorem 5: $b(n, \Delta) \leq (1 + \frac{2 - \log_2 3}{\Delta - 6 + \log_2 3}) \log_2 n + O(1) \approx (1 + \frac{0.415}{\Delta}) \log_2 n$.

A graph of degree Δ can also be constructed by replacing each vertex of $B(\frac{2m+1}{2} 2^{\Delta-2m-1}, D)$ by a $(\Delta-2m-1)$-cube for any $0 \leq m \leq \lfloor \frac{\Delta}{2} \rfloor - 1$. However, the resulting asymptotic broadcasting time is not as good as the result of Theorem 5.

Another way to obtain a graph of degree Δ is to use the family of minimum broadcast graphs from [8]. These graphs are k-regular with $n = 2^{k+1} - 2$ vertices and broadcast time $k+1$. A computation similar to the computation for cubes above shows that the best constant is obtained by compounding the $\Delta-6$-regular graph from this family in $B(3 \times 2^{\Delta-6}, D)$. Asymptotically, this will not improve the constant of

Theorem 5. However, for any fixed Δ, use of the $\Delta-6$-regular graph gives a slightly smaller constant than the $\Delta-6$ cube.

Similar calculations have been done using various graphs G including known minimum broadcast graphs and sparse broadcast graphs, cycles C_i for $i = 4, \ldots, 12$ and compounds of sparse broadcast graphs in cubes. (Note that the last case involves two compounding operations since the result of compounding in a cube is then compounded in a de Bruijn digraph.) The best values that we have obtained are shown in Table 7-1. The table shows the degree of the graph constructed (Δ), the graph G, the average time ($\bar{b}(G)$) needed to transmit a message originated in a copy of G to all of the out-neighbour copies, the indegree (outdegree) of the de Bruijn digraph used (d), the upper bound obtained by this graph and the best lower bound known. Note that the upper bound is calculated by the simple formula $\bar{b}(G)/\log_2 d$. T_6 is used to denote the tree on six vertices shown in Figure 6-1(c). C_6 and C_8 denote the cycles on six and eight vertices, respectively. Note that although C_6 is a minimum broadcast graph, C_8 is not. 14-mbg indicates the Heawood graph of Figure 7-1(a), a minimum broadcast graph on 14 vertices. i-mbg indicates a minimum broadcast graph on i vertices and i-sbg is used to represent a sparse broadcast graph on i vertices (see [3] and [8]).

Δ	G	$\bar{b}(G)$	d	upper bound	lower bound
3	T_6	15/4	4	1.875000	1.440420
4	C_8	34/8	8	1.416667	1.137467
5	40-sbg	266/40	40	1.249547	1.056215
6	C_6	50/12	12	1.162262	1.025404
7	14-mbg	150/28	28	1.114364	1.012034
8	56-sbg	828/112	112	1.086010	1.005842
9	62-mbg	926/124	124	1.073847	1.002874
10	126-mbg	2138/252	252	1.063536	1.001424
11	254-mbg	4822/508	508	1.056008	1.000709
12	126-mbg	3396/378	378	1.049273	1.000353
13	254-mbg	7614/762	762	1.043712	1.000176
14	510-mbg	16824/1530	1530	1.039394	1.000088
15	1022-mbg	36786/3066	3066	1.035909	1.000044
16	2046-mbg	79788/6138	6138	1.033017	1.000022

Table 7-1: best bounds

Using a generalization of the proof of Theorem 5, we can show that the asymptotic constant $1 + \frac{2 - \log_2 3}{\Delta}$ of Theorem 5 is the best possible asymptotic result for compounds in de Bruijn digraphs. Thus, the gap between the best known upper and lower bounds cannot be eliminated even by substituting arbitrarily large minimum broadcast graphs (which remain to be discovered) into de Bruijn digraphs.

Our technique of compounding graphs in de Bruijn digraphs does not give graphs for all values of *n*. Instead of de Bruijn digraphs, we could compound in Kautz digraphs [17] (see also [1] or [2]) or sequence graphs [9]. This will not give asymptotic improvements but will give different values of *n* for which the broadcasting times are the same as in Table 7-1. Graphs for other values of *n* can be obtained from those described in this paper using a technique from [20].

Acknowledgement: We would like to thank the referees for their constructive suggestions. In particular, the remarks of one of the referees led to improvements in the proofs of Lemma 3 and Theorem 1.

References

[1] Ameter, D. and de Gree, Max.
Graphs and Interconnection Networks.
forthcoming book.

[2] Bermond, J-C., Delorme, C. and Quisquater, J.-J.
Strategies for Interconnection Networks: Some Methods From Graph Theory.
J. Parallel and Dist. Comp. 3:433-449, 1986.

[3] Bermond, J.-C., Hell, P., Liestman, A. L., and Peters, J. G.
Sparse Broadcast Graphs.
Discr. Appl. Math :, to appear.

[4] Bermond, J-C., and Peyrat, C.
Broadcasting in De Bruijn Networks.
In *Proc. Nineteenth SE Conf. on Combinatorics, Graph Theory and Computing*, pages 283-292. Congressus Numerantium, 1988.

[5] Chau, S. C. and Liestman, A. L.
Constructing Minimal Broadcast Networks.
J. Comb. Inf. & Sys. Sci. 10:110-122, 1985.

[6] Chung, F. R. K.
Diameters of graphs: Old Problems and New Results.
In *Proc. Eighteenth SE Conf. on Combinatorics, Graph Theory and Computing*, pages 295-317. Congressus Numerantium, 1987.

[7] de Bruijn, N. G.
A combinatorial problem.
Koninklje Nedderlandse Academie van Wetenshappen Proc. A 49:758-764, 1946.

[8] Dinneen, M., M. Fellows, and V. Faber.
private communication.
August 1990.

[9] Escudero, M., Fabrega, J., and Fiol, M. A.
Routing and Expansion of Interconnection Networks Based on Sequence Graphs.
In *Proc. 5th Int. Symp. on Applied Informatics*, pages 47-52. Switzerland, 1987.

[10] Farley, A.
Minimal Broadcast Networks.
Networks 9:313-332, 1979.

[11] Farley, A., Hedetniemi, S., Mitchell, S. and Proskurowski, A.
Minimum Broadcast Graphs.
Discrete Math 25:189-193, 1979.

[12] Gargano, L. and Vaccaro, U.
On the Construction of Minimal Broadcast Networks.
Networks 19:673-689, 1989.

[13] Grigni, M. and Peleg, D.
Tight Bounds on Minimum Broadcast Networks.
Technical Report MIT/LCS/TM-374, M. I. T.,1988.

[14] Hedetniemi, S. T., Hedetniemi, S. M. and Liestman, A. L.
A Survey of Broadcasting and Gossiping in Communication Networks.
Networks 18:319-349, 1988.

[15] Hillis, W. D.
The Connection Machine.
MIT Press, 1985.
ACM Distinguished Dissertation.

[16] Jerrum, M. R. and Skyum, S.
Families of Fixed Degree Graphs for Processor Interconnection.
IEEE Trans. Comp. C-33(2):190-194, 1984.

[17] Kautz, W. H.
Bounds on Directed (d,k) Graphs.
In *Theory of Cellular Logic Networks and Machines, AFRCL68-0668 Final Report*, pages 20-28. 1968.

[18] Knuth, D. E.
The Art of Computer Programming, Vol 3., Sorting and Searching.
Addison-Wesley, Reading, Massachusetts, 1973.
Section 5.4.2, Exercise 7.

[19] Liestman, A. L.
Fault-Tolerant Broadcast Graphs.
Networks 15:159-171, 1985.

[20] Liestman, A. L. and Peters, J. G.
Broadcast Networks of Bounded Degree.
SIAM J. Disc. Math. 1:531-540, 1988.

[21] Miles, E. P., Jr.
Generalized Fibonacci Numbers and Matrices.
Amer. Math. Monthly 67:745-757, 1960.

[22] Mitchell, S. and Hedetniemi, S.
A Census of Minimum Broadcast Graphs.
J. Combin., Inform. & Systems Sci. 5:141-151, 1980.

[23] Peters, J. G.
private communication.
August 1990.

[24] Slater, P. J., Cockayne, E. and Hedetniemi, S. T.
Information dissemination in trees.
SIAM J. Comput. 10:692-701, 1981.

[25] Wang, X.
private communication.
April 1986.

Table of Contents

List of Tables

List of Figures