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Resolvable Decomposition of K_n^*

J. C. BERMOND,* A. GERMA,† AND D. SOTTEAU*

**Informatique, bât. 490,* †*Mathématiques, bât. 425, Université Paris-Sud, 91405–Orsay, France*

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In this paper, we prove that K_n^* admits a resolvable decomposition into TT_3 or $\mathbf{C}_3$ if and only if $n \equiv 0 \pmod 3$, $n \neq 6$.

I. INTRODUCTION

Let G and H be two given graphs (directed or not). We shall say that H can be decomposed into graphs G and we shall denote this by $H \to G$ if there exists a partition of the edges (or arcs) of H into partial subgraphs isomorphic to G. Furthermore, if we can arrange these subgraphs into classes, called parallel classes, where each class consists of a partition of the vertices of H, we shall say that we have a resolvable decomposition and we shall denote this by $H \xrightarrow{r} G$.

For previous results on the existence of such decompositions with different graphs G see [1].

Here we are interested to know for what values of n there exist resolvable decompositions of $H = K_n^*$ (the complete symmetric directed graph) into $G = TT_3$ or into $\mathbf{C}_3$, where TT_3 is the translative tournament on 3 vertices and $\mathbf{C}_3$ the directed cycle of length 3.

Notations. In what follows we shall denote by (a, b, c) either the TT_3 with arcs (a, b), (a, c), and (b, c) or the $\mathbf{C}_3$ with arcs (a, b), (b, c), and (c, a).

1.1. EXAMPLE. $K_{12}^* \xrightarrow{r} TT_3$. Let the vertices of K_{12}^* be $Z_{11} \cup \{\infty\}$ (Z_n denotes the additive group of residues modulo n). The 11 classes of a resolvable decomposition are the following $\{(i, i+2, i+10)(i+7, i+1, i+5)(i+8, i+9, i+4)(i+3, \infty, i+6)\}$, $i \in Z_{11}$.

It is well known that $K_n \xrightarrow{r} C_3$ if and only if $n \equiv 3 \pmod 6$ (solution of Kirkman's school girl problem by Ray-Chaudhuri and Wilson [5]). Thus, by associating two opposite TT_3 (or $\mathbf{C}_3$) to each C_3 we have

1.2. THEOREM. *If* $n \equiv 3 \pmod 6$ $K_n^* \xrightarrow{r} \mathbf{C}_3$ *and* $K_n^* \xrightarrow{r} TT_3$.

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It has also been proved for not necessarily resolvable decompositions:

1.3. THEOREM [4]. *$K_n^* \to TT_3$ if and only if $n(n-1) \equiv 0$ (mod. 3)*

1.4. THEOREM. *$K_n^* \to \mathbf{C}_3$ if and only if $n(n-1) \equiv 0$ (mod. 3), $n \neq 6$ (see [1] for the references).*

1.5. THEOREM (Hanani [3]). *$2K_n \xrightarrow{r} C_3$ if and only if $n \equiv 0$ (mod. 3) $n \neq 6$.*

For the existence of resolvable decompositions of K_n^* into TT_3 or $\mathbf{C}_3$ we have the following:

1.6. PROPOSITION. *Necessary conditions for $K_n^* \xrightarrow{r} TT_3$ or $K_n^* \xrightarrow{r} \mathbf{C}_3$ are $n \equiv 0$ (mod. 3), $n \neq 6$.*

Proof. The number of vertices of K_n^* must be a multiple of 3 (each class being a partition of the vertices) and then the number of arcs of K_n^* is a multiple of 3. Furthermore, for $n = 6$ if $K_n^* \xrightarrow{r} TT_3$ or $K_n^* \xrightarrow{r} \mathbf{C}_3$ then $2K_n \to C_3$ (by deleting the orientation) and that is impossible by Theorem 1.5. ∎

We shall prove that these conditions are sufficient:

1.7. THEOREM. $K_n^* \xrightarrow{r} TT_3$ and $K_n^* \xrightarrow{r} \mathbf{C}_3$ if and only if $n \equiv 0$ (mod. 3), $n \neq 6$.

As corollary we will obtain Theorem 1.5. of Hanani [3]. The proof given here will use some of the ideas of that paper.

II. PROOF OF THE THEOREM

Notations. $K_{[A]}^*$ shall denote the complete symmetric directed graph with vertex set A.

$K_{r \times n}$ shall denote the complete r-partite graph with vertex set the union of r disjoint sets of cardinality n.

2.1. LEMMA. $K_4^* \to TT_3$, $K_4^* \to \mathbf{C}_3$.

Proof. Let the vertices of K_n^* be the four elements of $GF(4)$: 0, 1, x, x^2 with $x^2 = x + 1$; the four TT_3 (or $\mathbf{C}_3$) of a decomposition are the following $\{(\alpha, 1+\alpha, x+\alpha), \alpha \in GF_4\}$ (that is, $\{(0, 1, x)(1, 0, x^2)(x, x^2, 0)(x^2, x, 1)\}$. ∎

2.2. LEMMA. *If $K_n^* \xrightarrow{r} TT_3$ (or $\mathbf{C}_3$) then $K_{4n}^* \xrightarrow{r} TT_3$ (or $\mathbf{C}_3$).*

Proof. It is similar to that of [3, Lemma 7, p. 281] but to be complete we shall give it here in the case TT_3 (it is exactly the same for $\mathbf{C}_3$).

Let the vertices of K_{4n}^* be partitioned into n sets $A_1 \cdots A_n$ with $|A_i| = 4$. The vertices of A_i will be denoted a_i^α, $\alpha \in GF(4)$.

Let $\mathscr{C}_1 \cdots \mathscr{C}_{n-1}$ be the parallel classes of a resolvable decomposition of K_n^* into TT_3.

From $\mathscr{C}_1$ we construct seven parallel classes of a decomposition of K_{4n}^* by associating to each TT_3 (i, j, k) of $\mathscr{C}_1$ the seven following families:

$$\{(a_i^\alpha, a_j^\alpha, a_k^\alpha)(a_i^{\alpha+1}, a_i^{\alpha+x}, a_i^{\alpha+x^2})(a_j^{\alpha+1}, a_j^{\alpha+x}, a_j^{\alpha+x^2})(a_k^{\alpha+1}, a_k^{\alpha+x}, a_k^{\alpha+x^2})\},$$

where $\alpha \in GF(4)$ and

$$\{(a_i^{x^p}, a_j^{x^{p+1}}, a_k^{x^{p+2}})(a_i^{x^p+1}, a_j^{x^{p+1}+1}, a_k^{x^{p+2}+1})(a_i^{x^p+x}, a_j^{x^{p+1}+x}, a_k^{x^{p+2}+x})$$

$$(a_i^{x^p+x^2}, a_j^{x^{p+1}+x^2}, a_k^{x^{p+2}+x^2})\},$$

where $p \in \{1, 2, 3\}$.

From each $\mathscr{C}_l$, $2 \leqslant l \leqslant n-1$, we construct four parallel classes of a decomposition of K_{4n}^* by associating to each TT_3 (i, j, k) of $\mathscr{C}_l$ the four following families

$$\{(a_i^\alpha, a_j^\alpha, a_k^\alpha)(a_i^{1+\alpha}, a_j^{x+\alpha}, a_k^{x^2+\alpha})(a_i^{x+\alpha}, a_j^{x^2+\alpha}, a_k^{1+\alpha})(a_i^{x^2+\alpha}, a_j^{1+\alpha}, a_k^{x+\alpha})\},$$

where $\alpha \in GF(4)$.

As wanted we obtain $7 + 4(n-2) = 4n - 1$ parallel classes. ∎

2.3. LEMMA.

$$K_{18}^* \xrightarrow{r} TT_3, \quad K_{18}^* \xrightarrow{r} \mathbf{C}_3,$$
$$K_{24}^* \xrightarrow{r} TT_3, \quad K_{24}^* \xrightarrow{r} \mathbf{C}_3,$$
$$K_{30}^* \xrightarrow{r} TT_3, \quad K_{30}^* \xrightarrow{r} \mathbf{C}_3,$$
$$K_{42}^* \xrightarrow{r} TT_3, \quad K_{42}^* \xrightarrow{r} \mathbf{C}_3.$$

Proofs will be given in the Appendix.

2.4. LEMMA. *$K_{[A \cup B]}^* - K_{[B]}^*$ with $|A| = 12$, $|B| = 6$, can be decomposed into 17 classes of TT_3 (or $\mathbf{C}_3$) such that*

— *12 classes D_j ($j = 1, \ldots, 12$) partition the vertices of $A \cup B$.*
— *5 classes E_j ($j = 13, \ldots, 17$) partition only the vertices of A.*

The *proof* will be given in the Appendix.

2.5. THEOREM (Brouwer, Hanani, and Schrijver [2]). *For $r \geqslant 4$*

$$K_{r \times 12} \to K_4.$$

(In fact in [2], they give all the values of r, n for which $K_{r \times n} \to K_4$.)

2.6. THEOREM. *For any* $u \geqslant 1$, $K^*_{12u+6} \xrightarrow{r} TT_3$ *(or* $\mathbf{C}_3$*).*

Proof. For $u = 1, 2, 3$ see Lemma 2.3. Let $n = 12u + 6$, $u \geqslant 4$ and let us partition the set X of vertices of K^*_n as follows:

$$X = \bigcup_{i=1}^{u} A_i \cup B \quad \text{with} \quad |A_i| = 12, \quad 1 \leqslant i \leqslant u \quad \text{and} \quad |B| = 6.$$

By Lemma 2.3, $K^*_{[A_1 \cup B]}$ isomorphic to K^*_{18} can be decomposed into 17 parallel classes C_j^1, $1 \leqslant j \leqslant 17$.

By Lemma 2.4, for $i = 2,..., u$, $K^*_{[A_i \cup B]} - K^*_B$ can be decomposed into 17 classes of TT_3 (or $\mathbf{C}_3$):

- 12 classes $D_1^i, D_2^i,..., D_{12}^i$ which partition the vertices of $A_i \cup B$.
- 5 classes $E_{13}^i, E_{14}^i,..., E_{17}^i$ which partition the vertices of A_i.

By Theorem 2.5 the graph $K^*_{u\times 12}$ constructed on $\bigcup_{i=1}^u A_i$ can be decomposed into K_4^*, and by Lemma 2.1 each of these K_4^* can be decomposed into TT_3 (or $\mathbf{C}_3$). For any i, let a_i^j be a given vertex of the set A_i; for each K_4^* of the decomposition of $K^*_{u\times 12}$ containing a_i^j we consider the TT_3 (or $\mathbf{C}_3$) of the decomposition of this K_4^* not containing this vertex a_i^j (for example if $a = x^2$ in Lemma 2.1, we take $(0, 1, x)$). Then for any j, $1 \leqslant j \leqslant 12$, the set F_j^i of all these TT_3 (or $\mathbf{C}_3$) associated to this vertex a_i^j form a partition of the vertices of the $K^*_{(r-1)\times 12}$ constructed on $\bigcup_{k=1}^u A_k - A_i$.

Thus we obtain a resolvable decomposition of K^*_{12u+6} into TT_3 (or $\mathbf{C}_3$) with the $12u + 5$ following parallel classes:

- 12 classes $C_j^1 \cup F_j^1$ for $j = 1,..., 12$.
- 5 classes $C_j^1 \cup \bigcup_{i=2}^u E_j^i$ for $j = 13,..., 17$.
- 12 $(u - 1)$ classes $D_j^i \cup F_j^i$ for $j = 1,..., 12$, $2 \leqslant i \leqslant u$. ∎

2.7. *End of the Proof of Theorem* 1.7

If $n \equiv 3 \pmod 6$ the theorem follows from Theorem 1.2.
If $n \equiv 6 \pmod{12}$ $n \neq 6$ the theorem follows from Theorem 2.6.
If $n \equiv 0 \pmod{12}$; let $n = 4^\alpha q$, where $q \equiv 0 \pmod 3$ but $q \not\equiv 0 \pmod{12}$.

In this case the theorem follows by repeated applications of Lemma 2.2 from the fact that the theorem is true for q, $q \neq 6$ and for $n = 24$ (Lemma 2.3). ∎

III. APPENDIX—PROOFS OF LEMMAS 2.3 AND 2.4

In the following we shall denote by X the set of vertices of K^*_n and if $\mathscr{C}$ is a class of TT_3 (or $\mathbf{C}_3$), $\mathscr{C} + i$ will mean the class obtained from $\mathscr{C}$ by adding i to each vertex of the TT_3 (or $\mathbf{C}_3$) of $\mathscr{C}$, with the convention that $\infty + i = \infty$.

3.1. *Proof of Lemma* 2.3

— $K^*_{18} \xrightarrow{r} TT_3$.

Let $X = Z_{17} \cup \{\infty\}$. The 17 parallel classes of a resolvable decomposition of K^*_{18} into TT_3 are the $\mathscr{C} + i$, $0 \leqslant i \leqslant 16$ with $\mathscr{C} = \{(0, \infty, 11)\ (1, 9, 14)\ (7, 2, 5)\ (4, 3, 10)\ (6, 15, 8)\ (12, 16, 13)\}$.

— $K^*_{24} \xrightarrow{r} TT_3$.

Let $X = Z_{23} \cup \{\infty\}$. The 23 parallel classes are the $\mathscr{C} + i$, $0 \leqslant i \leqslant 22$ with $\mathscr{C} = \{(6, 18, 19)\ (13, 1, 22)\ (2, 5, 12)\ (20, 11, 16)\ (7, 9, 15)\ (4, 3, 21)\ (17, 10, 14)\ (8, \infty, 0)\}$.

— $K^*_{30} \xrightarrow{r} TT_3$.

Let $X = Z_{29} \cup \{\infty\}$. The 29 parallel classes are the $\mathscr{C} + i$, $0 \leqslant i \leqslant 28$ with: $\mathscr{C} = \{(0, 4, 15)\ (28, 1, 2)\ (22, 7, 14)\ (20, \infty, 11)\} \cup \{(2^{3\alpha}, 2^{3\alpha+1}, 2^{\alpha 3+2})/\alpha = 1, 2, 3, 5, 6, 7\}$.

— $K^*_{42} \xrightarrow{r} TT_3$.

Let $X = Z_{41} \cup \{\infty\}$. The 41 parallel classes are the $\mathscr{C} + i$, $0 \leqslant i \leqslant 40$ with: $\mathscr{C} = \{(15, 8, 0)\ (7, 4, 24)\ (21, 12, 31)\ (23, \infty, 22)\} \cup \{(6^{3\alpha}, 6^{3\alpha+1}, 6^{3\alpha+2})/\alpha = 0, 1, 2, 3, 5, 6, 7, 8, 10, 11\}$.

Remark. The four decompositions above are derived from the decompositions $2K_n \xrightarrow{r} C_3$ given by Hanani [3] by suitably assigning directions to the C_3 in these decompositions.

— $K^*_{18} \xrightarrow{r} \mathbf{C}_3$.

Let $X = Z_{15} \cup \{\infty_1, \infty_2, \infty_3\}$. The 17 parallel classes are the $\mathscr{C}_1 + i$, $0 \leqslant i \leqslant 14$, $\mathscr{C}_{16}$ and $\mathscr{C}_{17}$ with

$\mathscr{C}_1 = \{(0, 11, 12)\ (1, 3, 9)\ (2, 6, 5)\ (\infty_1, 10, 8)\ (\infty_2, 4, 13)\ (\infty_3, 14, 7)\}$,

$\mathscr{C}_{16} = \{(\infty_1, \infty_2, \infty_3)\} \cup \{(i, i + 5, i + 10)/i = 0, 1, 2, 3, 4\}$,

$\mathscr{C}_{17} = \{(\infty_1, \infty_3, \infty_2)\} \cup \{(i, i + 10, i + 5)/i = 0, 1, 2, 3, 4\}$.

— $K^*_{24} \xrightarrow{r} \mathbf{C}_3$.

Let $X = Z_{21} \cup \{\infty_1, \infty_2, \infty_3\}$. The 23 parallel classes are the $\mathscr{C}_1 + i$, $0 \leqslant i \leqslant 20$, $\mathscr{C}_{22}$, $\mathscr{C}_{23}$ with

$$\mathscr{C}_1 = \{(0, 1, 4)\ (3, 5, 9)\ (12, 17, 2)\ (8, 7, 16)\ (11, 19, 14)\ (\infty_1, 15, 13)\ (\infty_2, 6, 18)\ (\infty_3, 20, 10)\},$$

$$\mathscr{C}_{22} = \{(\infty_1, \infty_2, \infty_3)\} \cup \{(i, i+7, i+14)/i = 0, 1, 2, 3, 4, 5, 6\},$$

$$\mathscr{C}_{23} = \{(\infty_1, \infty_3, \infty_2)\} \cup \{(i, i+14, i+7)/i = 0, 1, 2, 3, 4, 5, 6\}.$$

— $K^*_{30} \xrightarrow{r} \mathbf{C}_3$.

Let $X = Z_{27} \cup \{\infty_1, \infty_2, \infty_3\}$. The 29 parallel classes are the $\mathscr{C}_1 + i$, $0 \leqslant i \leqslant 26$, $\mathscr{C}_{28}$, $\mathscr{C}_{29}$ with

$$\mathscr{C}_1 = \{(0, 1, 3)\ (4, 7, 11)\ (12, 17, 23)\ (6, 13, 21)\ (16, 14, 24)\ (10, 9, 22)\ (15, 26, 20)\ (\infty_1, 2, 25)\ (\infty_2, 18, 8)\ (\infty_3, 5, 19)\},$$

$$\mathscr{C}_{28} = \{(\infty_1, \infty_2, \infty_3)\} \cup \{(i, i+9, i+18)/0 \leqslant i \leqslant 8\},$$

$$\mathscr{C}_{29} = \{(\infty_1, \infty_3, \infty_2)\} \cup \{(i, i+18, i+9)/0 \leqslant i \leqslant 8\}.$$

— $K^*_{42} \xrightarrow{r} \mathbf{C}_3$ (the decomposition is due to A.E. Brouwer).

Let $X = Z_{30} \cup \{\infty_i, 1 \leqslant i \leqslant 12\}$.

Let $\mathscr{D}_j$ $(1 \leqslant j \leqslant 11)$ be the 11 parallel classes of $\mathbf{C}_3$ of the K^*_{12} constructed on the vertices $\{\infty_i, 1 \leqslant i \leqslant 12\}$.

The 41 parallel classes of the decomposition of K^*_{42} will be $\mathscr{C}_1 + i$, $0 \leqslant i \leqslant 29$, $\mathscr{D}_j \cup (\mathscr{C}_{30} + j)$, $1 \leqslant j \leqslant 10$, $\mathscr{C}_{41} \cup \mathscr{D}_{11}$ with

$$\mathscr{C}_1 = \{(0, 17, 22)\ (1, 7, 5)\ (8, 2, \infty_1)\ (3, 19, \infty_2)\ (4, 26, \infty_3)\ (6, 24, \infty_4)\ (9, 13, \infty_5)\ (23, 12, \infty_6)\ (18, 27, \infty_7)\ (29, 20, \infty_8)\ (16, 11, \infty_9)\ (28, 21, \infty_{10})\ (15, 14, \infty_{11})\ (10, 25, \infty_{12})\},$$

$$\mathscr{C}_{30} = \{(0, 10, 20)\} \cup \{(i+1, i+2, i+4)\ (i+3, i+16, i+19)\ (i+5, i+17, i+28)/i = 0, 10, 20\},$$

$$\mathscr{C}_{41} = \{(i, i+20, i+10)/0 \leqslant i \leqslant 9\}.$$

3.2. *Proof of Lemma* 2.4

Let Z_{12} be the set of vertices of A and $B = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6\}$. To obtain the decomposition of $K^*_{[A\cup B]} - K^*_{[B]}$ into TT_3 with the wanted properties it suffices to take $D_j = \mathscr{C} + j$, $1 \leqslant j \leqslant 12$ with $\mathscr{C} = \{(0, \infty_1, 6)\ (3, \infty_2, 1)\ (2, \infty_3, 9)\ (4, \infty_4, 7)\ (5, \infty_5, 10)\ (11, \infty_6, 8)\}$ and

$$E_{13} = \{(1, 0, 2)\ (4, 3, 5)\ (7, 6, 8)\ (10, 9, 11)\},$$
$$E_{14} = E_{13} + 1,$$

$$E_{15} = E_{13} + 2,$$
$$E_{16} = \{(0, 4, 8)\ (1, 5, 9)\ (2, 6, 10)\ (3, 7, 11)\},$$
$$E_{17} = \{(8, 4, 0)\ (9, 5, 1)\ (10, 6, 2)\ (11, 7, 3)\}.$$

The decomposition of $K^*_{[A\cup B]} - K^*_{[B]}$ into $\mathbf{C}_3$ is given by

$$D_j = \mathscr{C} + j, \qquad 1 \leqslant j \leqslant 12,$$

with

$$\mathscr{C} = \{(11, \infty_1, 0)\ (10, \infty_2, 1)\ (2, \infty_3, 4)\ (3, \infty_4, 8)\ (9, \infty_5, 6)\ (7, \infty_6, 5)\}$$

and

$$E_{13} = \{(0, 1, 6)\ (4, 5, 10)\ (8, 9, 2)\ (3, 7, 11)\},$$
$$E_{14} = E_{13} + 1,$$
$$E_{15} = E_{13} + 2,$$
$$E_{16} = E_{13} + 3,$$
$$E_{17} = \{(10, 6, 2)\ (11, 7, 3)\ (0, 8, 4)\ (1, 9, 5)\}.$$

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