HYPERGRAPH-DESIGNS

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Abstract

Let H and K be two given t-uniform hypergraphs, we shall say that K admits an H-decomposition if we can partition the edges of K into partial subhypergraphs isomorphic to H.

Let K_n^t denote the complete t-uniform hypergraph. We are interested in the following problem: H being a given t-uniform hypergraph, for what values of n does K_n^t admit an H-decomposition?

If $H = K_k^t$ this problem is that of existence of t-design (or Steiner Systems).

If $t = 2$ [H is a graph G] it is the problem of existence of G-design.

Here we solve the problem for all 3-uniform hypergraphs H on 4 vertices and give general methods which could be used for other values of t and other hypergraphs H.

1. *Introduction.*

1.1. Let K and H be two given t-uniform hypergraphs, we shall say that K admits an H-decomposition if we can partition the edges of K into partial subhypergraphs isomorphic to H. Definitions concerning hypergraphs can be found in [1].

1.2. The H-decomposition problem, where H is a given t-uniform hypergraph, is to find the values of n for which the complete t-uniform hypergraph K_n^t admits an H-decomposition. The edges of K_n^t are all the subsets of cardinality t of its vertex set X with $|X| = n$. In this case the partition of the edges of K_n^t will be called an (n,k,1)

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H-design where k is the number of vertices of H.

1.3. Example: K_4^3 admits an H_1-decomposition with

$$H_1 = \{abc\} \cup \{abd\}$$
$$K_4^3 = (\{123\} \cup \{124\}) \cup (\{341\} \cup \{342\}).$$

1.4. Two important classes of H-designs have been considered in the literature:

CASE 1. $t = 2$; in this case H is a graph and K_n^2 the complete graph with n vertices and we find the definition of a G-design introduced by P. Hell and A. Rosa [7]. (See [2] for a survey on G-designs).

CASE 2. $H = K_k^t$; in this case an $(n,k,1)K_k^t$-design is nothing else than a classical $(n,k,1)$t-design or Steiner system (such a system is also called $S_1(n,k,t)$). The classical definition, which is equivalent, of an $(n,k,1)$t-design is the following:

An $(n,k,1)$t-design is a family of subsets, called blocks, of a set X of cardinality n such that

(i) every block contains k elements

(ii) every subset of t elements of X belongs to exactly one block.

1.5. Remark: Similarly as an (n,k,λ)t-design, we could define an (n,k,λ) H-design as a partition of the edges of λK_n^t (each subset of t elements of X occurs exactly λ times as edge of λK_n^t) into partial subhypergraphs isomorphic to H. But, in this paper, we shall always consider the case $\lambda = 1$.

In order to establish necessary conditions for the existence of an H-decomposition of K_n^t, we need the following definition: let A be a subset of X with $|A| \leq t$, we define $d_H(A)$ as the number of edges of H which contain A. Thus the following necessary conditions are easy to prove:

1.6. PROPOSITION: Let $H = (X,\mathcal{E})$ be a given t-uniform hypergraph, if K_n^t admits an H-decomposition then

(i) $\binom{n}{t} \equiv 0 \mod m(H) \quad (m(H) = |\mathcal{E}|)$

(ii) $\forall_i, 1 \leq i < t \quad \binom{n-i}{t-i} \equiv 0 \mod (\gcd\{d_H(A_i)/A_i \subset X, |A_i| = i\})$.

In cases 1 and 2 of 1.4 we find the known necessary conditions for the existence of an $(n,k,1)$G-design:

- $\frac{n(n-1)}{2} \equiv 0 \mod m(G)$

- $n - 1 \equiv 0 \mod (\gcd \{\text{degrees of vertices of } G\})$

for the existence of an $(n,k,1)$t-design:

- $\forall_i, 0 \leq i < t \quad \binom{n-i}{t-i} \equiv 0 \mod \binom{k-i}{t-i}$.

(Indeed, $m(K_k^t) = \binom{k}{t}$ and for $1 \leq i < t$, for every A_i $d_{K_k^t}(A_i) = \binom{k-i}{t-i}$).

1.7. The problem is to find when these necessary conditions are also sufficient.

For $t = 2$ this problem has been solved for many graphs G and for n large enough (see [2] for a survey of [5,9] in the particular case of a BIBD ($G = K_k$)).

For $t = 3$ it has been proved by Hanani [6] that the necessary conditions are sufficient for $H = K_4^3$.

For $t \geq 4$ only few examples are known (in theory of Steiner systems). Four of them have been found by Witt [9]. He has proved that there exists a (12,6,1)5-design, or, in our language a $(12,6,1)K_6^5$-design that is to say K_{12}^5 admits a K_6^5-decomposition. He has also proved that K_{11}^4 [resp. K_{24}^5, K_{23}^4] admits a K_5^4 [resp. K_8^5, K_7^4] decomposition. Recently some others have been found by Denniston [3].

In this article we shall deal with the case $t = 3$. We shall prove that the necessary conditions are sufficent for $t = 3$ and $k = 4$, H being any 3-uniform hypergraph on 4 vertices. We shall give some general methods which can be used for other values of t and k.

2. *Method of Differences.*

This method of direct construction is a generalization of Bose's method (see M. Hall [4]) which is already used for direct constructions of graph decompositions (see [2]).

2.1. We consider the edges of K_n^3 as triples (3-subsets) of Z_n, the *additive group of residues modulo* n, whose elements are denoted 0,1,...,n-1. We want to characterize the triples of Z_n by the differences of their elements: to the triple {a,b,c} we associate the six differences ±(b-a), ±(c-b), ±(c-a). Our aim is to obtain a systematical classification of all the triples of Z_n and to use methods analogous to Bose's difference method [4].

Exactly let us define on the triples of Z_n, the equivalence relation $\mathcal{R}$: $\{a,b,c\} \mathcal{R} \{a',b',c'\}$ if and only if there exists $i \in Z_n$ such that $\{a',b',c'\} = \{a+i,b+i,c+i\}$.

The following results will be given without proof.

2.2. PROPOSITION: If n is not a multiple of 3, each equivalence class (for the relation $\mathcal{R}$) contains exactly n triples. If n is a multiple of 3, there exists one equivalence class containing n/3 elements and the other classes contain n elements.

Example: n = 6. We have the classes {i,i+1,i+2}, {i,i+1,i+3}, {i,i+2,i+3} containing each 6 triples and the class containing the two triples (0,2,4) and (1,3,5).

2.3. If two triples belong to the same equivalence class, then the 6 differences of their elements are the same. First we shall show that we can associate to each class an (ordered) triple of differences and then study in theorem 2.5 how a triple characterize an equivalence class.

2.4. LEMMA: *To each equivalence class (for the relation* $\mathcal{R}$*), we can associate a triple* (α,β,γ) *of elements of* Z_n *satisfying*

$$(P)\quad 0 < \alpha \le \beta \le \gamma \le n/2 \quad \text{and } \gamma = \alpha + \beta \text{ or } \gamma = -(\alpha+\beta)$$

such that the family $\{\pm\alpha, \pm\beta, \pm\gamma\}$ *is the family of differences of the elements of any triple of the equivalence class.*

2.5. THEOREM: *Let* (α,β,γ) *be an (ordered) triple such that*

$$(P)\quad 0 < \alpha \le \beta \le \gamma < n/2 \quad \text{and } \gamma = \alpha+\beta \text{ or } \gamma = -(\alpha+\beta)$$

then $\{\pm\alpha,\pm\beta,\pm\gamma\}$ *is the family of the differences of the elements of the triples*

- *of exactly 2 equivalence classes if* α,β,γ *are all distinct.*
- *of exactly 1 equivalence class if two of the numbers* α,β,γ *are equal.*

(If $\alpha = \beta = \gamma$*, then the equivalence class contains n/3 elements).*

Thus by use of theorem 2.5 we shall be able to apply the following lemmas to construct more easily a cyclic H-decomposition of K_n^3 for a given hypergraph H.

2.6. Let us suppose $n \equiv 1,2$ [mod 3] and consider the vertices of K_n^3 as elements of Z_n. Then the equivalence classes of the edges for the relation $\mathcal{R}$ defined in 2.1 have the same cardinality n.

Let us consider a family $H_1 \dots H_k$ of hypergraphs isomorphic to H such that the edges of the H_i form a system of representatives of the equivalence classes [we shall call these hypergraphs *basis hypergraphs*].

Let us define H_j+i as the hypergraph obtained from H_j by adding i [mod n] to each vertex of the edges of H_j. Then:

LEMMA: *The family* $\{H_j+i \mid i \in Z_n \mid j \in [1,k]\}$ *is an H-decomposition of* K_n^3.

Remark: To verify that the edges of the basis hypergraphs form a system of representative of the equivalence classes we will use theorem 2.5.

Examples: $n = 5$, $H = \{a,b,c\} \cup \{a,b,d\}$.

By theorem 2.5. the possible triples of differences are (1,1,2) and (1,2,2) and the equivalence classes corresponding are $\overline{\{0,1,2\}}$ and $\overline{\{0,1,3\}}$. We take $H_1 = \{0,1,2\} \cup \{0,1,3\}$ as basis hypergraph and the family $\{i,1+i,2+i\} \cup \{i,1+i,3+i\}$, $i \in Z_5$ is an H-decomposition of K_5^3 by lemma 2.6

$$n = 11, \quad H = \{a,b,c\} \cup \{a,b,d\} \cup \{a,c,d\} = (\underline{a},b,c,d).$$

The possible triples of differences and the corresponding equivalence classes are

(1,1,2)	→	$\overline{\{0,1,2\}}$
(1,2,3)	→	$\overline{\{0,1,3\}}$ et $\overline{\{0,2,3\}}$
(1,3,4)	→	$\overline{\{0,1,4\}}$ et $\overline{\{0,3,4\}}$
(1,4,5)	→	$\overline{\{0,1,5\}}$ et $\overline{\{0,4,5\}}$
(1,5,5)	→	$\overline{\{0,1,6\}}$
(2,2,4)	→	$\overline{\{0,2,4\}}$
(2,3,5)	→	$\overline{\{0,2,5\}}$ et $\overline{\{0,3,5\}}$
(2,4,5)	→	$\overline{\{0,4,6\}}$ et $\overline{\{0,2,6\}}$
(3,3,5)	→	$\overline{\{0,3,6\}}$
(3,4,4)	→	$\overline{\{0,3,7\}}$.

We can take as basis hypergraphs

$$H_1 = (\underline{0},1,2,3),\ H_2 = (\underline{0},2,4,5),\ H_3 = (\underline{0},3,5,7),\ H_4 = (\underline{0},1,5,8),$$
$$H_5 = (\underline{0},1,4,6).$$

By lemma 2.6 an H-decomposition of K_{11}^3 is given by the following 11 x 5 hypergraphs:

$(\underline{i},i+1,i+2,i+3)(\underline{i},i+2,i+4,i+5)(\underline{i},i+3,i+5,i+7)(\underline{i},i+1,i+5,i+8)(\underline{i},i+1,i+4,i+6)$ $i \in Z_{11}$.

2.7. Let us suppose $n \equiv 0$ or 2 [mod 6] and consider the vertices of K_n^3 as elements of $Z_{n-1} \cup \infty$. The edges of K_n^3 are of two kinds

1) the edges whose vertices belong to Z_{n-1}. As $n-1 \not\equiv 0$ [mod 3] we can partition these edges into equivalence classes of cardinality n-1 for the equivalence relation $\mathcal{R}$ defined on K_{n-1}^3.

2) the edges which contain ∞: we can partition these edges into $\frac{n-2}{2}$ equivalence classes, the class C_j $(1 \le j \le \frac{n-2}{2})$ containing the edges $\{\infty,i,i+j\}$ $i \in Z_{n-1}$.

As in lemma 2.6, if we can define a family of basis hypergraphs $H_1,\dots,H_k$, such that the edges of H_i form a system of representatives of all the classes with $H_i \sim H$, we have an H-decomposition of K_n^3 constituted by $\{H_{j+i} \mid j \in [1,k] \mid i \in Z_{n-1}\}$.

Example: $n = 6$, $H = \{a,b,c\} \cup \{a,b,d\}$.

Basis hypergraphs $H_1 = \{0,1,2\} \cup \{0,1,3\}$, $H_2 = \{\infty,0,1\} \cup \{\infty,0,2\}$.

Then an H-decomposition of K_6^3 is given by the following 10 hypergraphs

$\{i,i+1,i+2\} \cup \{i,i+1,i+3\}$ and $\{\infty,i,i+1\} \cup \{\infty,i,i+2\}$ where $i \in Z_5$.

2.8. Let $n \equiv 1$ or $3 [\text{mod } 6]$. Consider the vertices of K_n^3 as elements of $Z_{n-2} \cup \infty_1 \cup \infty_2$. We can partition the edges of K_n^3 whose elements belong to Z_{n-2} and the edges containing one of the points ∞_1 or ∞_2 into equivalence classes similarly as in 2.7. The remaining edges (∞_1,∞_2,i) $i \in Z_{n-2}$ constitute one class.

Example: $n = 9$, $H = (\underline{a},b,c,d)$.

Basis hypergraphs are $(\underline{0},1,\infty_1,\infty_2)(\underline{0},2,3,\infty_1)(\underline{0},3,5,\infty_2)(0,1,3,6)$ and an H-decomposition is given by the 4 x 7 hypergraphs

$(\underline{i},i+1,\infty_1,\infty_2)(\underline{i},i+2,i+3,\infty_1)(\underline{i},i+3,i+5,\infty_2)(\underline{i},i+1,i+3,i+6)$ where $i \in Z_7$.

3. *Composition Methods.*

3.1. Notations: We denote by $K^3_{[A]}$ or more simply by $K_{[A]}$ the complete 3-uniform hypergraph generated by the vertices of A(the edges are all the triples of elements of A).

If H' is a partial hypergraph of H, we denote by H-H' the hypergraph obtained from H by deleting the edges of H'.

The composition method is based on the following evident lemma and its corollaries, which enable us to use induction.

3.2. LEMMA: *Let* K_n^3 *be an edge disjoint union of hypergraphs* H_i. *If the hypergraphs* H_i *admit an H-decomposition, then* K_n^3 *also admits an H-decomposition.*

3.3. COROLLARY: *Let* A_i, $1 \le i \le k$, *be* k *disjoint sets with* $|A_i| = n_i$. *Suppose that* $K^3_{n_i}$, $1 \le i \le k$ *and* $K_{[\underset{i}{\cup} A_i]} - \cup K_{[A_i]}$ *admit an H-decomposition. Then* $K^3_{(\Sigma n_i)}$ *admits an H-decomposition.*

3.4. COROLLARY: *Let* A_i, $1 \le i \le k$, *be* k *disjoint sets with* $|A_i| = n_i$ *and* ω *a vertex* $\notin \underset{i}{\cup} A_i$. *Suppose that the following* k+2 *hypergraphs admit an H-decomposition:* $K^3_{n_i+1}$, $1 \le i \le k$; $K_{[\cup A_i]} - K_{[A_i]}$ *and* H' *the hypergraph which edges are all the triples containing* ω *and two vertices of two different sets* A_i. *Then* $K_{(\Sigma n_i)+1}$ *admits an H-decomposition.*

Proof. Apply lemma 3.2 with $H_i = K_{[A_i \cup \infty]}$, $1 \le i \le k$,

$H_{k+1} = K_{[\cup A_i]} - \cup K_{[A_i]}$ and $H_{k+2} = H'$. □

We shall first use these corollaries in the simplest case k = 2 (that is when we split the set X of vertices of K_n^3 into two sets or two sets plus an extra vertex). We shall see that this is sufficient to solve the H-decomposition problem for $H = H_1$ and H_2.

CASE 1. $H = H_1 = \{a,b,c\} \cup \{a,b,d\}$.

3.5. THEOREM: K_n^3 *admits an* H_1*-decomposition if and only if* $n \not\equiv 3$ (mod 4), *and* $n \ge 4$.

Proof. The necessary condition follows from proposition 1.6. The sufficient condition will use corollaries 3.3, 3.4 and the following lemma:

3.6. LEMMA: *Let* A *and* B *be two disjoint sets with* $|A| = n_1$ *and* $|B| = n_2$ *and* n_1 *and* n_2 *even . Then* $K_{[A \cup B]} - K_{[A]} - K_{[B]}$ *admits an* H_1*-decomposition.*

Proof. The edges of $K_{[A\cup B]} - K_{[A]} - K_{[B]}$ containing two vertices of A and one of B can be partitioned into the $\binom{n_1}{2}$ $n_2/2$ hypergraphs isomorphic to H_1: $\{a_i,a_j,b_{2k}\} \cup \{a_i,a_j,b_{2k+1}\}$ where (a_i,a_j) is any pair of elements of A and $k = 0,1,2,\ldots,(n_2/2 - 1)$. Similarly we partition the edges containing two vertices of B and one of A. □

Proof of theorem 3.5 (by induction). It is true for $n = 4$, an H_1-decomposition of K_4^3 being $\{0,1,2\} \cup \{0,1,3\}$ and $\{2,3,0\} \cup \{2,3,1\}$. For $n = 5$ it has been proved in 2.6; for $n = 6$ it has been proved in 2.7. Let $n_0 \not\equiv 3 \pmod 4$, $n_0 \geq 0$, and suppose that the theorem is true for all $n \not\equiv 3 \pmod 4$, $n < n_0$. If n_0 is even the theorem results from corollary 3.3 with $n_1 = n_0 - 4$ (≥ 4) and $n_2 = 4$: $K^3_{n_0-4}$ admits an H_1-decomposition (by induction hypothesis), K_4^3 also and $K_{[A\cup B]} - K_{[A]} - K_{[B]}$ by the lemma 3.6. If n_0 is odd, $n_0 \geq 9$ then the theorem results from corollary 3.4 with $n_1 = n_0 - 5$ (≥ 4), $n_2 = 4$. $K^3_{n_0-4}$ admits an H_1-decomposition (induction hypothesis); K_5^3 also; $K_{[A\cup B]} - K_{[A]} - K_{[B]}$ by lemma 3.6, and H' also by taking the $2n_1$ hypergraphs isomorphic to H_1: $\{\omega,a_i,b_j\} \cup \{\omega,a_i,b_{j+1}\}$ for $i = 1,2,\ldots,n_1$ $j = 1,3$.

CASE 2. $H = H_2 = (\underline{a},b,c,d) = \{a,b,c\} \cup \{a,b,d\} \cup \{a,c,d\}$.

We shall use the more concise notation $H_2 = (\underline{a},b,c,d)$ which means that a is the vertex of H_2 belonging to the 3 edges of H_2.

3.7. THEOREM: *K_n^3 admits an H_2-decomposition if and only if $n \equiv 0,1,2 \pmod 9$ and $n \geq 9$.*

The necessary condition follows from proposition 1.6. The proof of sufficiency will use the existence of Steiner triple systems and of Resolvable Steiner triple systems. The idea is contained in lemma 3.8.

3.8. LEMMA: *Let $|X| = n+1$, with $n \equiv 1$ or $3 \pmod 6$. Then the hypergraph K consisting of all the edges of K^3_{n+1} containing a given vertex a admits an H_2-decomposition.*

Proof. Recall that if $n \equiv 1$ or $3 \pmod 6$ there exists a Steiner triple system that is a partition of the edges of the complete graph K_n into triples or in another language that we can find $n(n-1)/6$ triples of elements of $\{1,2,\ldots,n\}$ such that every pair of elements appears in exactly one triple. Then an H_2-decomposition of the hypergraph K is given by the $n(n-1)/6$ hypergraphs isomorphic to H_2: $H_i = (\underline{a},a_i,b_i,c_i)$, $1 \leq i \leq n(n-1)/6$, where (a_i,b_i,c_i) are the triples of the Steiner triple system on n elements. □

3.9. COROLLARY: *If K_n^3 admits an H_2-decomposition and if $n \equiv 1$ or $3 \pmod 6$ then K^3_{n+1} admits an H_2-decomposition.*

3.10. COROLLARY: *If $K^3_{n_1}$ and $K^3_{n_2}$ admit an H_2-decomposition and if $n_1 \equiv 1$ or $3 \pmod 6$ and $n_2 \equiv 1$ or $3 \pmod 6$ then $K^3_{n_1+n_2}$ admits and H_2-decomposition.*

Proof. Let A and B be two disjoint sets with $|A| = n_1$, $|B| = n_2$. Apply corollary 3.3 by noticing that $K_{[A\cup B]} - K_{[A]} - K_{[B]}$ is the edge disjoint union of the n_1 hypergraphs $K_{[\{a\}\cup B]} - K_{[B]}$ where $a \in A$ and of the n_2 hypergraphs $K_{[A\cup\{b\}]} - K_{[A]}$ where $b \in B$, these hypergraphs being all isomorphic to hypergraph K defined in lemma 3.8 with respectively $n = n_2$ and $n = n_1$.

3.11. PROPOSITION: It suffices to prove the theorem 3.7 for $n \equiv 1,9$ or $11 \pmod{18}$.

Proof. Suppose we can prove theorem 3.7 for $n \equiv 1,9$, or $11 \pmod{18}$. If $n \equiv 2$ or $10 \pmod{18}$ then $n - 1 \equiv 1$ or $9 \pmod{18}$ and, by corollary 3.9 K_n^3 admits an H_2-decomposition. If $n \equiv 0 \pmod{18}$, $n = 9 + (n-9)$ and $n - 9 \equiv 9 \pmod{18}$ so by corollary 3.10 K_n^3 admits an H_2-decomposition. □

3.12. LEMMA: *The edges of* K_{10} *can be decomposed into 12 triples, one 1-factor and a cycle of length 4.*

The edges of K_{18} *can be decomposed into 48 triples and one 1-factor.*

The edges of K_{20} *can be decomposed into 60 triples and one 1-factor.*

Proof. For K_{18} and K_{20} it is an immediate consequence of the existence of Steiner triple systems. Indeed it suffices to consider a decomposition of K_{19} (resp. K_{21}) into triples and to delete one vertex. We obtain the required decompositions of K_{18} (resp. K_{20}).

For K_{10} such a decomposition is for example:

- the 12 triples {1,2,3}, {1,4,5}, {1,6,7}, {1,8,9}, {2,4,6}, {2,7,8}, {2,5,10}, {3,4,9}, {3,5,7}, {3,6,10}, {4,8,10}, {7,9,10}.
- the 1-factor {1,10} ∪ {2,9} ∪ {3,8} ∪ {4,7} ∪ {5,6}
- the cycle of length 4 (5 8 6 9).

3.13. THEOREM: (Ray-Chaudhuri and Wilson [3]). *The edges of* K_{6t+3} *can be partitioned into* (3t+1) *classes of* (2t+1) *pairwise vertex disjoint triples (thus each class covers all the vertices of* K_{6t+3}*).*

3.14. LEMMA: *Let* A *and* B *be two disjoint sets with* $|A| = 6t+3$ *and* $|B| = 10$, 18 *or* 20. *Then* $H = K_{[A\cup B]} - K_{[A]} - K_{[B]}$ *admits an* H_2*-decomposition.*

Proof. By theorem 3.13 the pairs of elements of A can be partitioned into (3t+1) classes C_i, $1 \le i \le 3t+1$, each class consisting of 2t+1 pairwise disjoint triples $C_i = \bigcup_j T_{i,j}$ with $1 \le j \le 2t+1$, where $T_{i,j} = \{a^1_{i,j}, a^2_{i,j}, a^3_{i,j}\}$.

If $|B| = 18$ (or 20), by lemma 3.12 we can partition the pairs of elements of B into triples $T'_k = (b^1_k, b^2_k, b^3_k)$ with $1 \le k \le k_0$ where $k_0 = 48$ (or 60) and a 1-factor $\mathcal{F} = \bigcup_\ell \{b^1_\ell, b^2_\ell\}$, where $1 \le \ell \le \frac{|B|}{2}$.

Then $H = K_{[A\cup B]} - K_{[A]} - K_{[B]}$ admits an H_2-decomposition consisting of

a) the $|A|$ k_0 hypergraphs $(\underline{a}, b^1_k, b^2_k, b^3_k)$ where a is any vertex of A and (b^1_k, b^2_k, b^3_k) any triple T'_k defined above.

b) the $3(2t+1)$ $|B|/2$ hypergraphs: $(\underline{a^1_{1,j}}, a^2_{1,j}, b^1_\ell, b^2_\ell)$, $(\underline{a^2_{1,j}}, a^3_{1,j}, b^1_\ell, b^2_\ell)$ and $(\underline{a^3_{1,j}}, a^1_{1,j}, b^1_\ell, b^2_\ell)$ where $(a^1_{1,j}, a^2_{1,j}, a^3_{1,j})$ is any triple $T_{1,j}$ of the first class C_1 (see above) and where $\{b^1_\ell, b^2_\ell\}$ is any edge of the 1-factor $\mathcal{F}$ (see above).

c) the $|B|$ $3t$ $(2t+1)$ hypergraphs $(\underline{b}, a^1_{i,j}, a^2_{i,j}, a^3_{i,j})$ with $i \ge 2$ where $\underline{b}$ is any vertex of B and $(a^1_{i,j}, a^2_{i,j}, a^3_{i,j})$ is any triple $T_{i,j}$, with $i \ge 2$ (defined above) that is which does not belong to the class C_1.

If $|B| = 10$ by lemma 3.12 (see the proof) we can partition the pairs of elements of B into 12 triples T'_k, $1 \le k \le k_0 = 12$, a 1-factor and a cycle of length 4 (b_1, b_2, b_3, b_4). Then H admits an H_2-decomposition similar to cases $|B| = 18$ or 20 which contains the hypergraphs defined in a) and b) but the class c) has to be replaced by

c_1) the 6 (2t+1) hypergraphs: $(\underline{a^1_{2,j}}, a^2_{2,j}, b_k, b_{k+1})$, $(\underline{a^2_{2,j}}, a^3_{2,j}, b_k, b_{k+1})$, $(\underline{a^3_{2,j}}, a^1_{2,j}, b_k, b_{k+1})$ with $\{a^1_{2,j}, a^2_{2,j}, a^3_{2,j}\}$ any triple $T_{2,j}$ of the class C_2 and $\{b_k, b_{k+1}\} = \{b_1, b_2\}$ or $\{b_3, b_4\}$ (two disjoint edges of the cycle).

c_2) the 6 (2t+1) hypergraphs: $(\underline{a^1_{3,j}}, a^2_{3,j}, b_k, b_{k+1})$, $(\underline{a^2_{3,j}}, a^3_{3,j}, b_k, b_{k+1})$, $(\underline{a^3_{3,j}}, a^1_{3,j}, b_k, b_{k+1})$ with $\{a^1_{3,j}, a^2_{3,j}, a^3_{3,j}\}$ any triple $T_{3,j}$ of the class C_3 and $\{b_k, b_{k+1}\} = \{b_2, b_3\}$ or $\{b_1, b_4\}$ (the two other edges of the cycle).

c_3) the $|B|$ $(3t-2)(2t+1)$ hypergraphs $(\underline{b}, a^1_{i,j}, a^2_{i,j}, a^3_{i,j})$ with $i \geq 4$, where b is any vertex of B and $\{a^1_{i,j}, a^2_{i,j}, a^3_{i,j}\}$ any triple $T_{i,j}$, with $i \geq 4$ (that is which does not belong to the classes C_1, C_2, C_3). That is possible because $t \geq 1$ implies that the number of classes $3t+1 \geq 4$.

3.15. End of the proof of the theorem:

The proof is by induction. The theorem is true for $n = 9$: see 2.8, for $n = 11$: see 2.6. Let $n_0 \equiv 0,1,2 \pmod 9$. Suppose the theorem true for all $n < n_0$, $n \equiv 0,1,2 \pmod 9$. By proposition 3.11 we can suppose $n_0 \equiv 1,9,11 \pmod{18}$ and $n_0 \geq 19$. Then if $n_0 = 18t + h$ $(t \geq 1)$, with $h = 1,9$ or 11, we write $n_0 = (18t - 9) + (h + 9)$ where $h + 9 = 10$, 18 or 20. Then corollary 3.3 with $n_1 = (18t - 9) \geq 9$, $n_2 = h + 9$ can be applied. Indeed by lemma 3.14 $K_{[A \cup B]} - K_{[A]} - K_{[B]}$ admits an H_2-decomposition, and so do $K_{[A]}$ and $K_{[B]}$ by induction hypothesis.

3.16. Other Methods.

The proofs given above use the splitting into two parts. But for other hypergraphs than H_1 or H_2, this may not be sufficient to solve the H-decomposition problem. Furthermore for H_2 we have used the solution of Kirkman's school girl problem and it seems interesting to obtain a more elementary proof.

3.17. LEMMA. *Let* H *be a 3-uniform hypergraph. Let* A, B, C, Ω *be four disjoint sets with* $|A| = |B| = |C| = m$ *and* $|\Omega| = h$. *Let* A *be the disjoint union of* A_1 *and* A_2. *Suppose that the following hypergraphs admit and H-decomposition:*

i) $K_{[A \cup \Omega]} - K_{[A_1 \cup \Omega]}$

ii) $K_{[A_1 \cup \Omega]} - K_{[\Omega]}$

iii) $K_{[A \cup B \cup C]} - K_{[A]} - K_{[B]} - K_{[C]}$

iv) *the hypergraph consisting of all the edges containing one vertex in* Ω *and the two others in two different sets of* A,B,C.

Then $K_{[A \cup B \cup C \cup \Omega]} - K_{[A_1 \cup \Omega]}$ *admits an H-decomposition.*

Proof. We apply lemma 3.2. $K_{[A \cup B \cup C \cup \Omega]} - K_{[A_1 \cup \Omega]}$ is the edge disjoint union of the hypergraphs i) iii) iv) and of the hypergraphs $K_{[B \cup \Omega]} - K_{[\Omega]}$ and $K_{[C \cup \Omega]} - K_{[\Omega]}$ which are isomorphic to $K_{[A \cup \Omega]} - K_{[\Omega]}$; the later is the edge disjoint union of the hypergraphs i) and ii) and thus admits an H_2-decomposition. □

3.18. We apply lemma 3.17 for H_2. The fact that i) and ii) admit an H_2-decomposition will follow by induction hypothesis or direct construction for ii) and thus the lemma will apply for $m+h \equiv 0,1,2 \pmod 9$, $h \equiv 0,1,2 \pmod 9$ and $|A_1| \equiv 0,1,2 \pmod 9$. We will deal with the H_2-decomposition of the hypergraphs iii) and iv) in the following lemmas.

3.19. LEMMA: *The hypergraph iv) of lemma 3.17 admits always an* H_2*-decomposition.*

Proof. It is well known that the complete 3-partite graph $K_{m,m,m}$ on $A \cup B \cup C$ admits a K_3-decomposition. To each triple (a_i, b_j, c_k) of such a decomposition and to each vertex ω of Ω we can associate a hypergraph isomorphic to H_2: $(\underline{\omega}, a_i, b_j, c_k)$. Then all these hypergraphs

$K_{m,m,m}$) form an H_2-decomposition of the hypergraph iv) of lemma 3.17.

3.20. DEFINITION: Let H be a 3-uniform hypergraph $(X,\mathcal{E})$, we denote by $H \otimes S_p$ the 3-uniform hypergraph with vertex set $X \times Z_p$ and with edges all the triples $\{(x,i);(y,j);(z,k)\}$ where (x,y,z) is an edge of H. $H \otimes S_p$ is the lexicographic product of H by the hypergraph S_p with no edges. $H \otimes S_p$ can be seen as obtained from H by replacing each vertex by p independent vertices and each edge by p^3 edges. [Note that $K_n^3 \otimes S_p$ is the complete n partite 3-uniform hypergraph denoted usually by $K_{n \times p}^3$]. The following lemma and its corollary are important:

3.21. LEMMA. $H_2 \otimes S_p$ *admits an* H_2*-decomposition.*

Proof. Let $H_2 = (\underline{a},b,c,d)$ then an H_2-decomposition of $H_2 \otimes S_p$ is given by the following p^3 hypergraphs: $((\underline{a,i})(b,i+k)(c,j),(d,j+k))$ with $i = 1,2,\dots,p$; $j = 1,2,\dots,p$; $k = 1,2,\dots,p$.

3.22. COROLLARY: *If* H *admits an* H_2*-decomposition, then* $H \otimes S_p$ *admits an* H_2*-decomposition.*

3.23. LEMMA: *Let* $|A| = |B| = m$. $K_{[A \cup B]} - K_{[A]} - K_{[B]}$ *admits an* H_2*-decomposition if and only if* $m \equiv 0$ *or* $1 \pmod 3$.

Proof. The necessity follows from the fact that the number of edges of $K_{[A \cup B]} - K_{[A]} - K_{[B]}$, which is equal to $2m\binom{m}{2}$ must be a multiple of 3.

If $m \equiv 1$ or $3 \pmod 6$ the sufficiency follows from lemma 3.8.

Let $m \equiv 0 \pmod 6$ and suppose that we have prove the lemma for all m', $m' < m, m' \equiv 0 \pmod 6$. Then the lemma is true for $m/2$ which is congruent to 0 or 3 (mod 6). Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ with

$|A_1| = |A_2| = |B_1| = |B_2| = m/2$, then $K_{[A \cup B]} - K_{[A]} - K_{[B]}$ is the edge disjoint union of

i) the four hypergraphs $K_{[A_i \cup B_j]} - K_{[A_i]} - K_{[B_j]}$ with $i = 1,2$; $j = 1,2$, which admit an H_2-decomposition (the lemma being true for $m/2$).

ii) the complete 4-partite 3-uniform hypergraph $K^3_{m/2,m/2,m/2,m/2} = K_4^3 \otimes S_{m/2} = (K_4^3 \otimes S_3) \otimes S_{m/6} = K^3_{3,3,3,3} \otimes S_{m/6}$.

It is easy to find an H_2-decomposition of $K^3_{3,3,3,3}$ and thus, by corollary 3.22 $K^3_{m/2,m/2,m/2,m/2}$ admits an H_2-decomposition.

Let $m \equiv 4 \pmod 6$, $m = 6t+4$. It is well known that the complete graph K_{6t+4} admits a decomposition into $(3t+1)$ hamiltonian cycles and and a 1-factor (a perfect matching). By splitting t hamiltonian cycles into $2t$ 1-factors we obtain a decomposition of K_{6t+4} into $(2t+1)$ hamiltonian cycles C_α and $(2t+1)$ 1-factors M_α, $\alpha = 1,2,\dots,2t+1$. Let $C_\alpha = (\alpha_1, \dots,\alpha_i,\dots,\alpha_n)$. Then to the edge $\{\alpha_i,\alpha_{i+1}\}$ of C_α and to the 1-factor M_α we associate m hypergraphs isomorphic to H_2: $(\underline{a_{\alpha_i}},a_{\alpha_{i+1}}, b_j, b_{j'})$ and $(\underline{b_{\alpha_i}},b_{\alpha_{i+1}},a_j,a_{j'})$ where $\{j,j'\}$ is any one of the $m/2$ edges of M_α. (We denote by a_r the r^{th} element of A and by b_r the r^{th} element of B). Thus to the m edges of C_α $\{\alpha_i,\alpha_{i+1}\}$ $i = 1,2,\dots,m$ and to M_α we associate m^2 edge-disjoint hypergraphs isomorphic to H_2. If we do this construction for every $\alpha = 1,2,\dots, 2t+1$ $(=m/3)$ we obtain $m^3/3$ hypergraphs isomorphic to H_2 which are by construction edge-disjoint and thus form an H_2-decomposition of $K_{[A \cup B]} - K_{[A]} - K_{[B]}$. □

3.24. LEMMA: *Let* $|A| = |B| = |C| = m$ *and* $m \equiv 0$ *or* $3 \pmod 9$ *then the hypergraph iii) of lemma 3.17,* $K_{[A\cup B\cup C]} - K_{[A]} - K_{[B]} - K_{[C]}$ *admits an* H_2*-decomposition.*

Proof. If $m = 3$, let us denote this hypergraph by T_3: an H_2-decompositio[n] is given by the 27 hypergraphs:

$(\underline{a_i}, a_{i+1}, b_i, c_i)(\underline{a_i}, a_{i+1}, b_{i+1}, c_{i+2})(\underline{a_i}, a_{i+1}, b_{i+2}, c_{i+1})(\underline{b_i}, b_{i+1}, c_{i+1}, a_{i+1})$

$(\underline{b_i}, b_{i+1}, c_{i+2}, a_i)(\underline{b_i}, b_{i+1}, c_i, a_{i+2})(\underline{c_i}, c_{i+1}, a_{i+2}, b_{i+2})(\underline{c_i}, c_{i+1}, a_i, b_{i+1})$

$(\underline{c_i}, c_{i+1}, a_{i+1}, b_i)$, for $i = 1,2,3$.

If $m > 3$, let $m = 3p$ and $A = A_1 \cup A_2 \cup A_3$, $B = B_1 \cup B_2 \cup B_3$, $C = C_1 \cup C_2 \cup C_3$ with $|A_i| = |B_i| = |C_i| = m/3$ for $i = 1,2,3$. Then $K_{[A\cup B\cup C]} - K_{[A]} - K_{[B]} - K_{[C]}$ is the edge disjoint union of:

i) $T_3 \otimes S_p$ which admits an H_2-decomposition by corollary 3.22.

ii) the 27 hypergraphs $K_{[A_i\cup B_j]} - K_{[A_i]} - K_{[B_j]}$, $K_{[B_i\cup C_j]} - K_{[B_i]} - K_{[C_j]}$, $K_{[C_i\cup A_j]} - K_{[C_i]} - K_{[A_j]}$ with $i = 1,2,3$; $j = 1,2,3$; these hypergraphs admit an H_2-decomposition by lemma 3.23, indeed $m \equiv 0$ or $3 \pmod 9$ implies $m/3 \equiv 0$ or $1 \pmod 3$. □

3.25. PROPOSITION: *Let* $n \equiv 1 \pmod{18}$ *then* $K_n^3 - K_{19}^3$ *admits an* H_2*-decomposition.*

Proof. (by induction). $K_{19}^3 - K_{10}^3$ and thus K_{19}^3 admits an H_2-decompositi[on] apply lemma 3.14 with $t = 1$ (note that we use only the existence of a partition of edges of K_9 into 4 classes containing 3 vertex disjoint triples). Now let $n \equiv 1 \pmod{18}$ and suppose that the proposition is true for all m, $m < n$, $m \equiv 1 \pmod{18}$. Then we distinguish 3 cases and apply lemma 3.17 with

$n = 54k+1,\quad m = 18k,\quad h = 1,\quad |A_1| = 18;$

$n = 54k+19,\quad m = 18k,\quad h = 19,\quad |A_1| = 0;$

$n = 54k+37,\quad m = 18k+9,\quad h = 10,\quad |A_1| = 9$

The hypothesis of the lemma 3.17 are satisfied; indeed the hypergraphs i) ii) iii) iv) admit an H_2-decomposition, i) by induction hypothesis as $|A_1 \cup \Omega| = 19$; ii) because $K_{19}^3 - K_{10}^3$ and K_{19}^3 admit an H_2-decomposition; iii) by lemma 3.24 as $m \equiv 0 \pmod 9$; iv) by lemma 3.19. □

3.26. PROPOSITION. *Let* $n \equiv 9 \pmod{18}$ *then* $K_n^3 - K_{18}^3$ *admits an* H_2*-decomposition* ($n \neq 9$).

Proof. Similar to 3.25. First K_9^3, K_{18}^3 and $K_{27}^3 - K_{18}^3$ admit an H_2-decomposition (done in 3.14) and thus we use lemma 3.17 with

$n = 54k+9,\quad m = 18k,\quad h = 9,\quad |A_1| = 9;$

$n = 54k+27,\quad m = 18k+9, h = 0,\quad |A_1| = 18;$

$n = 54k+45,\quad m = 18k+9, h = 18,\quad |A_1| = 0.$ □

3.27. PROPOSITION. *Let* $n \equiv 11 \pmod{18}$, $n \neq 11$, *then* $K_n^3 - K_{20}^3$ *admits an* H_2*-decomposition.*

Proof. Similar to 3.25. First K_{11}^3, $K_{29}^3 - K_{20}^3$ and $K_{20}^3 - K_{11}^3$ admit an H_2-decomposition. Then we apply lemma 3.17 with;

$n = 54k+11,\quad m = 18k,\quad h = 11,\quad |A_1| = 9;$

$n = 54k+29,\quad m = 18k+9,\quad h = 2,\quad |A_1| = 18;$

$n = 54k+47,\quad m = 18k+9,\quad h = 20,\quad |A_1| = 0.$ □

By propositions 3.11 and 3.25, 3.26, 3.27, we obtain a second proof of theorem. □

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