Fast Gossiping by Short Messages*

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Abstract

Gossiping is the process of information diffusion in which each node of a network holds a packet that must be communicated to all other nodes in the network. We consider the problem of gossiping in communication networks under the restriction that communicating nodes can exchange up to a fixed number p of packets at each round. In the first part of the paper we study the extremal case $p = 1$ and we exactly determine the optimal number of communication rounds to perform gossiping for several classes of graphs, including Hamiltonian graphs and complete k-ary trees. For arbitrary graphs we give asymptotically matching upper and lower bounds. We also study the case of arbitrary p and we exactly determine the optimal number of communication rounds to perform gossiping under this hypothesis for complete graphs, hypercubes, rings, and paths. Finally, we investigate the problem of determining sparse networks in which gossiping can be performed in the minimum possible number of rounds.

Key words: gossiping, graphs, networks.

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1 Introduction

Gossiping (also called total exchange or all–to–all communication) in distributed systems is the process of distribution of information known to each processor to every other processor of the system. This process of information dissemination is carried out by means of a sequence of message transmissions between adjacent nodes in the network.

The gossiping problem was originally introduced by the community of discrete mathematicians, to which it owes most of its terminology, as a combinatorial problem in graphs. Nonetheless, it was soon realized that, once cast in more realistic models of communication, gossiping is a fundamental primitive in distributed memory multiprocessor system. There are a number of situations in multiprocessor computation, such as global processor synchronization, where gossiping occurs. Moreover, the gossiping problem is implicit in a large class of parallel computation problems, such as linear system solving, Discrete Fourier Transform, and sorting, where both input and output data are required to be distributed across the network [7, 9, 18]. Due to the interesting theoretical questions it poses and its numerous practical applications, gossiping has been widely studied under various communication models. Hedetniemi, Hedetniemi and Liestman [15] provide a survey of the area. Two more recent surveys paper collecting the latest results are [10, 17]. The reader can also profitably see the book [23].

The great majority of the previous work on gossiping has considered the case in which the packets known to a processor at any given time during the execution of the gossiping protocol can be freely concatenated and the resulting (longer) message can be transmitted in a constant amount of time, that is, it has been assumed that the time required to transmit a message is independent from its length. While this assumption is reasonable for short messages, it is clearly unrealistic in case the size of the messages becomes large. Notice that most of the gossiping protocols proposed in the literature require the transmission, in the last rounds of the execution of the protocol, of messages of size $\Theta(n)$, where n is the number of nodes in the network. Therefore, it would be interesting to have gossiping protocols that require only the transmission of bounded length messages between processors. In this paper we consider the problem of gossiping in communication networks under the restriction that communicating nodes can exchange up to a fixed number p of packets at each round.

1.1 The Model

Consider a communication network modeled by a graph $G = (V, E)$ where the node set V represents the set of processors of the network and E represents the set of the communication lines between processors.

Initially each node holds a packet that must be transmitted to any other node in the network by a sequence of *calls* between adjacent processors. During each call, communicating nodes can exchange up to p packets, where p is an *a priori* fixed integer. We assume that each processor can participate in at most one call at time. Therefore, we can see the gossiping process as a sequence of *rounds*: During each round a disjoint set of edges (matching) is selected and the nodes that are end vertices of these edges make a call. This communication model is usually referred to as *telephone model* [15] or *Full–Duplex 1–Port* (F_1) [20]. We denote by $g_{F_1}(p, G)$ the minimum possible number of rounds to complete the gossiping process in the network G subject to the above conditions. Another popular communication model is the *mail model* [15] or *Half–Duplex 1–Port* (H_1) [20], in which in each round any node can either send a message to one of its neighbors or receive a message from it but not simultaneously. The problem of estimating $g_{H_1}(p, G)$ has been considered in [4]. Analogous problems in bus networks have been considered in [11, 16]. Optimal bounds on $g_{H_1}(1, G)$ when the edges of G are subject to random failures are given in [8]. Packet routing in interconnection networks in the F_1 model has been considered in [1].

1.2 Results

We first study the extremal case in which gossiping is to be performed under the restriction that communicating nodes can exchange *exactly* one packet at each round. We provide several lower bounds on the gossiping time $g_{F_1}(1, G)$ and we provide matching upper bounds for Hamiltonian graphs, complete trees, and complete bipartite graphs. For general graphs we provide asymptotically tight upper and lower bounds.

Subsequently, we study the case of arbitrary p and we compute exactly $g_{F_1}(p, G)$ for complete graphs, hypercubes, rings and paths. Our result for hypercubes allows us to improve the corresponding result in the H_1 model given in [4].

Finally, we investigate the problem of finding the sparsest networks in which gossiping can be performed in the minimum possible number of rounds, that is, networks in which gossiping can be performed in the same number of rounds as in the complete graph.

2 Gossiping by exchanging one packet at time

In this section we study $g_{F_1}(1, G)$, that is the minimum possible number of rounds to complete gossip in a graph G under the condition that at each round communicating nodes can exchange *exactly one* packet.

In order to avoid overburdening the notation, we will simply write $g(G)$ to denote $g_{F_1}(1, G)$.

2.1 Lower bounds on $g(G)$

In this section we give some lower bounds on the time needed to complete the gossiping process.

Lemma 2.1 *For any graph $G = (V, E)$, with $|V| = n$, let $\mu(G)$ be the size of a maximum matching in G, then*

$$g(G) \geq \left\lceil \frac{n(n-1)}{2\mu(G)} \right\rceil . \tag{1}$$

Proof. For any node $v \in V$ the packet initially resident in v must reach each of the remaining $n-1$ nodes in the graph. Therefore, during the gossiping process, at least $n(n-1)$ packet transmissions must be executed over the edges of G. Since in each communication round at most $\mu(G)$ calls can be performed and each call allows the transmission of 2 packets (one in each direction) the bound follows. □

Lemma 2.2 *Let $X \subset V$ be a vertex cutset of the graph $G = (V, E)$ whose removal disconnects G into the connected components $V_1, \ldots, V_d$, then*

$$g(G) \geq \left\lceil \sum_{i=1}^{d} \frac{\max\{|V_i|, n - |V_i|\}}{|M_X|} \right\rceil, \tag{2}$$

where $|M_X|$ is the size of a maximum matching M_X in G such that any edge in it has an endpoint in X and the other in $V - X$.

Proof. Consider a component V_i, for some $1 \leq i \leq d$. Nodes in V_i can receive the packets of nodes in $V - V_i$ only by means of calls between a node in X and one in V_i; moreover, at least $n - |V_i|$ calls are needed between nodes in X and nodes in V_i to bring all packets in $V - V_i$ to nodes in V_i. Analogously, packets of nodes in V_i can reach nodes in $V - V_i$ only by means of calls between a node in X and one in V_i and at least $|V_i|$ such calls are needed. Therefore, for each $i = 1, \ldots, d$, at least $\max\{|V_i|, n - |V_i|\}$ calls must take place between nodes in X and nodes in V_i. We then get that at least $\sum_{i=1}^{d} \max\{|V_i|, n - |V_i|\}$ calls are needed between nodes in X and nodes in $V - X = \cup_{i=1}^{d} V_i$. Since at most $|M_X|$ such calls can take place during each round, we get the desired lower bound of

$$\left\lceil \frac{\sum_{i=1}^{d} \max\{|V_i|, n - |V_i|\}}{|M_X|} \right\rceil$$

on the time necessary to gossip in G. □

Remark 2.1 The bound in the above Lemma 2.2 can sometimes be improved by observing that after the last call has been done between a node in some V_i and a node in X, the last exchanged message has still to reach all the other nodes of V_i (or of $V - V_i$). Therefore, we can add to the lower bound (2) the minimum of the eccentricities of the subgraphs induced by the V_i's and the $V - V_i$'s.

Corollary 2.1 *Let $\alpha(G)$ be the independence number of G, then*

$$g(G) \geq \left\lceil \frac{\alpha(G)(n-1)}{n - \alpha(G)} \right\rceil. \tag{3}$$

Proof. Let Y denote an independent set of G. Applying Lemma 2.2 with cutset $X = V - Y$ and connected components $V_1, \ldots, V_{|Y|}$, each consisting of just one element of Y, we get

$$g(G) \geq \left\lceil \sum_{i=1}^{|Y|} \frac{n - |V_i|}{|M_X|} \right\rceil \geq \left\lceil \sum_{i=1}^{|Y|} \frac{n - |V_i|}{|X|} \right\rceil = \left\lceil \sum_{i=1}^{|Y|} \frac{n-1}{n - |Y|} \right\rceil = \left\lceil \frac{|Y|(n-1)}{n - |Y|} \right\rceil.$$

Choosing an independent set of maximum size $|Y| = \alpha(G)$ we get (3). $\square$

Let T be a tree and v one of its nodes, we indicate the connected components into which the node set of T is splitted by the removal of v by $V_1(v), \ldots, V_{\deg(v)}(v)$, ordered so that $|V_1(v)| \geq \ldots \geq |V_{\deg(v)}(v)|$.

Corollary 2.2 *Let T be a tree on n nodes of maximum degree $\Delta = \max_{v \in V} \deg(v)$, then*

$$g(T) \geq \max_{v\,:\,\deg(v)=\Delta} L(v),$$

where

$$L(v) = \begin{cases} (\deg(v)-1)n + 1 & \text{if } |V_1(v)| \leq n/2; \\ (\deg(v)-2)n + 1 + 2|V_1(v)| & \text{if } |V_1(v)| > n/2. \end{cases}$$

Proof Given a node v, Lemma 2.2 gives

$$\begin{aligned} g(T) &\geq \sum_{i=1}^{\deg(v)} \max\{|V_i(v)|, n - |V_i(v)|\} \\ &= \sum_{i=2}^{\deg(v)} n - |V_i(v)| + \begin{cases} |V_1(v)| & \text{if } |V_1(v)| > n/2, \\ n - |V_1(v)| & \text{if } |V_1(v)| \leq n/2, \end{cases} \\ &= L(v). \end{aligned}$$

Direct computation shows that if $\deg(v) > \deg(w)$ then $L(v) > L(w)$ thus proving that the maximum is always attained at a node of maximum degree. $\square$

2.2 Upper bounds

In this section we will determine exactly $g(G)$ for several classes of graphs, including Hamiltonian graphs and complete k-ary trees. We will also provide good upper bounds for general graphs.

2.2.1 Hamiltonian Graphs

We first note that in any graph $G = (V, E)$ the size of a maximum matching $\mu(G)$ is at most $\lfloor |V|/2 \rfloor$. Therefore, from Lemma 2.1 we get that the gossiping time $g(G)$ of *any* graph with n nodes is always lower bounded by

$$g(G) \geq \begin{cases} n-1 & \text{if } n \text{ is even;} \\ n & \text{if } n \text{ is odd.} \end{cases} \tag{4}$$

We will show that this lower bound is attained by Hamiltonian graphs.

Let $C_n = (V, E)$ denote the ring of length n; we assume the vertex set be $V = \{0, \ldots, n-1\}$ and the edge set be $E = \{(v, w) \,:\, 1 = |v - w| (\text{mod } n)\}$[1].

[1] Here and in the rest of the paper with $x = a \pmod b$ we denote the unique integer $0 \leq x < b$ such that $x = qb + a$.

Lemma 2.3 $g(C_n) \leq \begin{cases} n-1 & \textit{if } n \textit{ is even;} \\ n & \textit{if } n \textit{ is odd.} \end{cases}$

Proof. We distinguish two cases according to the parity of the number n of nodes.

Case n even. We shall give a gossiping protocol on the ring C_n that requires $n-1$ rounds. First, for each integer t define the perfect matching in C_n given by

$$M_t = \begin{cases} \{(v,w)\ :\ v \text{ is even and } w = v+1\} & \text{if } t \text{ is even} \\ \{(v,w)\ :\ v \text{ is odd and } w = v+1 (\text{mod } n)\} & \text{if } t \text{ is odd;} \end{cases} \tag{5}$$

notice that M_t and M_{t+1} are disjoint for each t. The gossiping algorithm is shown in Figure 1.

Gossiping-even(C_n)

Round $t = 1$**:** each node v sends its own packet to the node w such that $(v,w) \in M_1$;

Round $t = 2$**:** each node v sends its own packet to the node w such that $(v,w) \in M_2$;

Round t**,** $3 \leq t \leq n-1$**:** For each node v let w be the node such that $(v,w) \in M_t$,

node v sends a new packet to w, namely v sends the packet it has first got among those v has neither received from w nor sent to w in any previous round.

Figure 1: Gossiping Algorithm in C_n, n even.

It is immediate to see that each node receives a new packet at each round (this can be formally proved by induction on t). Therefore, at the end of round $n-1$ of algorithm **Gossiping-even**(C_n) each node has received all the packets of the other $n-1$ nodes.

Case n odd. Define the following maximum matchings M_t in C_n for each $t = 1, \ldots, n$

$$M_t = \{(v,w)\ :\ v - t + 1\,(\text{mod}\, n) \text{ is odd, } w = v+1(\text{mod } n), \text{ and } v \neq t-1 \neq w\}. \tag{6}$$

We give in Figure 2 a gossiping protocol on C_n that requires n rounds.

Gossiping-odd(C_n)

Round t**,** $1 \leq t \leq n$**:** For each node $v \neq t-1$ let w be the neighbor of v in M_t,

node v sends to w the packet that v has first received among those that it has neither got from w nor sent to w in a previous round (v own packet is considered to be received before any other packet).

Figure 2: Gossiping Algorithm in C_n, n odd.

It is easy to see that at each round $t = 1, \ldots, n$ each node different from $t-1$ receives a new packet. Therefore, at the end of round n of algorithm **Gossiping-odd**(C_n) each node has received all the packets of the other $n-1$ nodes. □

Example 2.1 *For $n = 6$ we have $M_1 = M_3 = M_5 = \{(1,2),(3,4),(50)\}$ and $M_2 = M_4 = \{(0,1),(2,3),(4,5)\}$. Each column of Table 1 shows the set of nodes whose packets are known by v at the end of round t, for for each $0 \le v \le 5$ and $1 \le t \le 5$.*

$t \backslash v$	0	1	2	3	4	5
1	$\{5,0\}$	$\{1,2\}$	$\{1,2\}$	$\{3,4\}$	$\{3,4\}$	$\{5,0\}$
2	$\{5,0,1\}$	$\{0,1,2\}$	$\{1,2,3\}$	$\{2,3,4\}$	$\{3,4,5\}$	$\{4,5,0\}$
3	$\{4,5,0,1\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{2,3,4,5\}$	$\{2,3,4,5\}$	$\{4,5,0,1\}$
4	$\{4,5,0,1,2\}$	$\{5,0,1,2,3\}$	$\{0,1,2,3,4\}$	$\{1,2,3,4,5\}$	$\{2,3,4,5,0\}$	$\{3,4,5,0,1\}$
5	$\{3,4,5,0,1,2\}$	$\{5,0,1,2,3,4\}$	$\{5,0,1,2,3,4\}$	$\{1,2,3,4,5,0\}$	$\{1,2,3,4,5,0\}$	$\{3,4,5,0,1,2\}$

Table 1

For $n = 5$ we have $M_1 = \{(1,2),(3,4)\}$, $M_2 = \{(2,3),(4,0)\}$, $M_3 = \{(3,4),(0,1)\}$ $M_4 = \{(4,0),(1,2)\}$, and $M_5 = \{(0,1),(2,3)\}$. Each column of Table 2 shows the set of nodes whose packets are known by v at the end of round t, for for each $0 \le v \le 4$ and $1 \le t \le 5$.

$t \backslash v$	0	1	2	3	4
1	$\{0\}$	$\{1,2\}$	$\{1,2\}$	$\{3,4\}$	$\{3,4\}$
2	$\{4,0\}$	$\{1,2\}$	$\{1,2,3\}$	$\{2,3,4\}$	$\{3,4,0\}$
3	$\{4,0,1\}$	$\{0,1,2\}$	$\{1,2,3\}$	$\{2,3,4,0\}$	$\{2,3,4,0\}$
4	$\{3,4,0,1\}$	$\{0,1,2,3\}$	$\{0,1,2,3\}$	$\{2,3,4,0\}$	$\{2,3,4,0,1\}$
5	$\{3,4,0,1,2\}$	$\{0,1,2,3,4\}$	$\{0,1,2,3,4\}$	$\{1,2,3,4,0\}$	$\{2,3,4,0,1\}$

Table 2

Theorem 2.1 *For any Hamiltonian graph G on n vertices we have*

$$g(G) = \begin{cases} n-1 & \text{if } n \text{ is even;} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

Proof. If G is Hamiltonian, from Lemma 2.3 we get that gossiping along the edges of the Hamiltonian cycle gives a protocol with gossiping time matching the lower bound (4) and the theorem holds. □

2.3 Trees

In this section we investigate the gossiping time in trees. We first give an upper bound on the gossiping time in *any* tree and afterwards we exactly compute the gossiping time of k–ary trees.

Consider a tree $T = (V, E)$. We recall that for each node v the set $V_1(v)$ denotes the largest of the connected components into which T is splitted by the removal of v. Let

$$\vartheta = \max |V_1(v)|,$$

where the maximum is taken over all the internal nodes v having exactly $\deg(v) - 1$ leaves as neighbors; notice that any other internal node u has $|V_1(u)| \le \vartheta - 1$.
Call *pre–leaf* any node v such that $|V_1(v)| = \vartheta$ and denote by π the maximum degree of a node in the subgraph consisting only of the edges (u, f) where f is either a leaf or a pre–leaf of T.
Finally, let λ be the maximum number of leaves connected to a same node and $\Delta = \max_{v \in V} \deg(v)$.

Theorem 2.2 *For any tree $T = (V, E)$ on n nodes* $g(T) \leq (\vartheta - 1)\Delta + \pi + (n - \vartheta - 1)\lambda$.

Proof. Consider the gossiping algorithm **Gossiping-tree**(T) given in Figure 3. For any $e \in E$, the edge coloring $c(e)$ and the partial edge colorings $c'(e)$ and $c''(e)$ used in **Gossiping-tree**(T) are each intended so that no two edges sharing a vertex are assigned the same color.

Gossiping-tree(T)

Phase 1

[Color each edge (u, v) of T with color $c(u, v) = c(v, u) \in \{0, \ldots, \Delta - 1\}$.]

Round t, for $t = 1, \ldots, \Delta(\vartheta - 1)$**:** For each node u, if there is an edge (u, v) such that $c(u, v) = t - 1(\bmod\ \Delta)$ then u sends a new packet to v, namely u sends to v a packet among those that u has neither sent to v nor received from v in a previous round, if such a packet exists, otherwise u sends nothing.

Phase 2

[Give to each edge (u, f), where f is a leaf or a pre–leaf, a color $c'(u, f) \in \{0, \ldots, \pi - 1\}$]

Round $\Delta(\vartheta - 1) + t$**, for** $t = 1, \ldots, \pi$**:** For each leaf or pre–leaf f, if there is an edge (u, f) with $c'(u, f) = t - 1$, then u sends to f a packet among those that u has neither sent to f nor received from f in a previous round, if any.

Phase 3

[Give to each edge (u, f), where f is a leaf of T, a color $c''(u, f) \in \{0, \ldots, \lambda - 1\}$.]

Round $\Delta(\vartheta - 1) + \pi + t$**, for** $t = 1, \ldots, (n - \vartheta - 1)\lambda$**:** For each leaf f, if the edge (u, f) on f has $c''(u, f) = t - 1(\bmod\ \lambda)$ then u sends to f any packet f does not know.

Figure 3: Gossiping Algorithm in a tree T.

We prove now the correctness of algorithm **Gossiping-tree**(T). Let us say that the edge (u, v) of T is *saturated* from u to v at time t of **Gossiping-tree**(T) if no packet is sent from u to v at *any time* $t' \geq t$, that is, if by time $t - 1$ node v has received the packet of each node w connected to v through u. We need the following property of **Gossiping-tree**(T).

Property 2.1 *In any round t of Phase 1 of the algorithm* **Gossiping-tree**(T)*, if the edge (u, v) has color $c(u, v) = t - 1(\bmod\ \Delta)$ and it is not saturated from u to v at time t, then u sends a new packet to v at round t.*

Proof. The proof is by induction on the time unit t. Let $t \leq \Delta$: At time unit t, for each edge (u, v) of color $c(u, v) = t - 1 \in \{0, \ldots, \Delta - 1\}$ node u and v exchange a call for the first time and have at least their own packet to send each other.

Let now $t > \Delta$ and suppose that the hypothesis holds for each $t' < t$.

Consider an edge (u, v) such that $c(u, v) = t - 1(\bmod\ \Delta)$. Suppose by contradiction that at time t the edge (u, v) is not saturated from u to v but u has no packets to send to v among those u

has not received through v, that is, all packets known to u and not received through v have been already sent from u to v.

In particular, node u has already sent to v all the packets it has received from its other neighbors, call them $w_1, \ldots, w_k$. Notice that the last call from u to v has taken place at time $t - \Delta$. Let τ_i be the only integer such that both $t - \Delta < \tau_i < t$ and $c(u, w_i) = \tau_i - 1 (\bmod \Delta)$ hold. If the edge (u, w_i) is not saturated at time τ_i, by inductive hypothesis we know that u has received a packet by w_i at time τ_i. We can have now two cases: The first case is that all the edges (u, w_i) are saturated at time $\tau_i < t$, this immediately implies that (u, v) is saturated at time t, contradicting our assumption that (u, v) not saturated from u to v at time t. The second case is that at least one edge (u, w_i) is not saturated at time τ_i; in such a situation we know by the inductive hypothesis that u has received a new packet by w_i at time τ_i that can be now forwarded to v, again getting a contradiction. □

We can now complete the proof of the theorem by showing that at the end of **Gossiping–tree**(T) each node knows all the other $n - 1$ packets. Above Property 2.1 shows that a new packet is sent from u to v at each round t of **Phase 1** such that $c(u, v) = t - 1 \pmod \Delta$, until the edge (u, v) is saturated and no more packets need to be sent from u to v. Therefore, for any internal node u and for any Δ consecutive rounds, u receives a new packet from each neigbor v such that (v, u) is not saturated from v to u. We recall that ϑ is the maximum number of packets that any internal node needs to get from a same neighbor and that this maximum is attained with equality only if u is a pre–leaf. Therefore, by round $\Delta(\vartheta - 1)$ node u which is not a pre–leaf gets all the necessary $n - 1$ packets; while a pre–leaf gets $n - 2$ packets during the $\Delta(\vartheta - 1)$ rounds of Phase 1 and the remaining one during some round of Phase 2.
Analogously, during Phase 1 any leaf f gets $\vartheta - 1$ packets. It is obvious that f receives a new packet during Phase 2; moreover, during Phase 3 the leaf f receives a new packet for any λ consecutive rounds thus getting the remaining $n - \vartheta - 1$ packets that it needs to complete the gossip. □

Let δ denote the minimum degree of an internal node in T. It is easy to see that we can upper bound ϑ by $n - \delta$. Therefore, from Theorem 2.2 we have the following upper bound on $g(T)$ that is expressed only in terms of degree properties of the nodes in T.

Corollary 2.3 *For any tree $T = (V, E)$ on n nodes* $g(T) \leq (n - \delta)\Delta + (\delta - 1)\lambda$.

Given a connected graph $G = (V, E)$, denote by $\mathcal{T}$ the set of all spanning trees of G and for any vertex $v \in V$ denote by $\deg_T(v)$ the degree of v in $T \in \mathcal{T}$. Define $d(G) = \min_{T \in \mathcal{T}} \max_{v \in V} \deg_T(v)$. The following corollary is immediate.

Corollary 2.4 *For any connected graph $G = (V, E)$ with n vertices*

$$g(G) \leq (n - 1)d(G). \tag{7}$$

We point out that, altough the problem of computing $d(G)$ is NP-hard, there exists an efficient algorithm to compute a spanning tree of maximum degree at most $d(G)+1$ (see [12]). From Corollary 2.2 and Corollary 2.4 we have that for any tree with n nodes and maximum degree Δ it holds $n\Delta - n + 1 \le g(T) \le n\Delta - \Delta$. Let us consider now the tree $S_{n,\Delta}$ of Figure 4. If $\Delta = n-1$ then $S_{n,n-1}$ is the star on n nodes and from Corollary 2.2 and Theorem 2.2 we have $g(S_{n,n-1}) = (n-1)^2$. If $\Delta > 2$ is constant with respect to $n > 2\Delta$ then from Corollary 2.2 and Theorem 2.2 we get $\Delta(n-1) - (\Delta - 1) \le g(S_{n,\Delta}) \le \Delta(n-1) - 2$. It is not difficult to obtain a specific gossiping algorithm attaining the lower bound. Therefore, we have that for any n and Δ there exists a graph $G_{n,\Delta}$ with n vertices and maximum degree Δ such that $g(G_{n,\Delta}) = \Omega((n-1)\Delta)$, hence the bound (7) is asymptotically tight. In [8] it is conjectured that for any graph G it holds $g_{H_1}(1,G) = \Omega(nd(G))$. This conjecture, if true, together with Corollary 2.4 would imply the rather interesting fact that for any graph G it holds $g(G) = \Theta(nd(G))$.

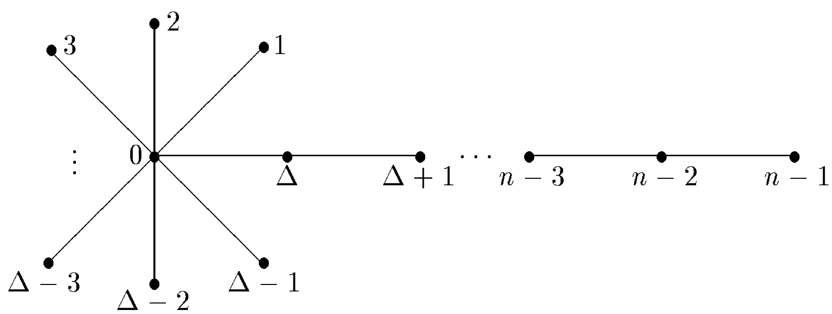


Figure 4: Tree $S_{n,\Delta}$

We shall now exactly compute the gossiping time of k-ary trees, that is, rooted trees in which each internal node has exactly k sons. Let $T_{k,n}$ denote *any* k–ary tree with n nodes.
Let us first notice that for $n = k+1$ the tree $T_{k,n}$ is the star $S_{k+1,k}$. Consider then a tree $T_{k,n}$ with $n \ge 2k+1$ nodes. Let u be a node of $T_{k,n}$ whose sons are all leaves, by Corollary 2.2 we get

$$g(T_{k,n}) \ge \max_v L(v) \ge L(u) = \begin{cases} kn+1 & \text{if } n = 2k+1 \\ (k+1)(n-1)-k & \text{if } n \ge 3k+1. \end{cases} \tag{8}$$

We show now that (8) holds with equality. Applying Theorem 2.2 to $T_{k,n}$ we get that

$$g(T_{k,n}) \le (\vartheta - 1)\Delta + \pi + (n - \vartheta - 1)\lambda = (\vartheta - 1)(k+1) + \pi + (n - \vartheta - 1)k. \tag{9}$$

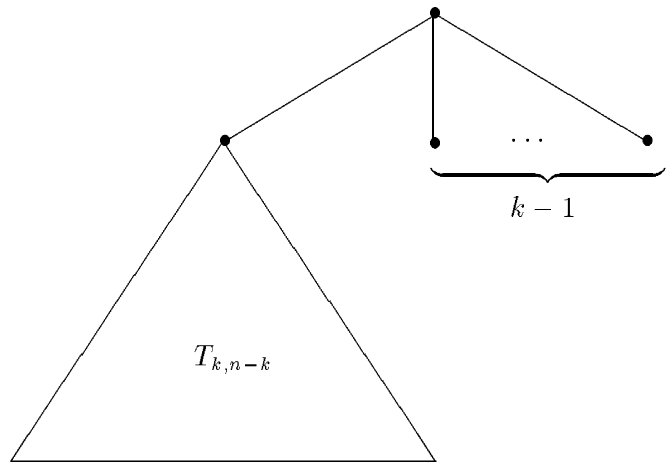
Unless exactly $k-1$ sons of the root are leaves (cf. the tree in Figure 5) $T_{k,n}$ has $\vartheta = n-k-1$ and $\pi \le \Delta = k+1$, that by (9) and (8) gives

$$g(T_{k,n}) = (n-k-2)(k+1) + k + 1 + k^2 = (k+1)(n-1) - k.$$

Consider now the remaining case when $T_{k,n}$ is the tree of Figure 5. The only pre–leaf is the root, and $\vartheta = n - k$. If $n \ge 3k+1$ we have $\pi = k$ and from (9) we get

$$g(T_{k,n}) \le (n-k-1)(k+1) + k + (k-1)k = (n-1)(k+1) - k;$$

if $n = 2k+1$ we have $\pi = \Delta = k+1$ and $g(T_{k,2k+1}) \leq kn+1$.



Therefore, we have proved the following result.

Theorem 2.3 *For any k-ary tree on n nodes $T_{k,n}$ it holds that*

$$g(T_{k,n}) = \begin{cases} k^2 & \textit{if } n = k+1 \\ 2k^2+k+1 & \textit{if } n = 2k+1 \\ (k+1)(n-1)-k & \textit{if } n \geq 3k+1. \end{cases}$$

The particular case $k = 1$ of above result deserves to be explicitely stated.

Corollary 2.5 *Let P_n be the path on n nodes. We have*

$$g(P_n) = \begin{cases} 1 & \textit{if } n = 2 \\ 4 & \textit{if } n = 3 \\ 2n-3 & \textit{if } n \geq 4. \end{cases}$$

2.4 Complete bipartite graphs

Let $K_{r,s} = (V(K_{r,s}), E(K_{r,s}))$ be the complete bipartite graph on the node set $V(K_{r,s}) = \{a_0, \ldots, a_{r-1}\} \cup \{b_0, \ldots, b_{s-1}\}$, with $\{a_1, \ldots, a_{r-1}\} \cap \{b_0, \ldots, b_{s-1}\} = \emptyset$, $r \geq s$, and edge set $E(K_{r,s}) = \{a_0, \ldots, a_{r-1}\} \times \{b_0, \ldots, b_{s-1}\}$. In the next theorem we determine the gossiping time of $K_{r,s}$.

Theorem 2.4 *For each r and s with $r \geq s \geq 1$ it holds* $g(K_{r,s}) = \lceil (r+s-1)r/s \rceil$.

Proof. The lower bound $g(K_{r,s}) \geq \lceil (r+s-1)r/s \rceil$ is an immediate consequence of Corollary 2.1 since the complete bipartite graph has $\alpha(K_{r,s}) = r$.

In order to give a gossiping algorithm in $K_{r,s}$ requiring $\lceil (r+s-1)r/s \rceil$ communication rounds, we define the matchings

$$M_j = \{(b_i, a_{i+j \pmod r}) \; : \; 0 \leq i \leq s-1\},$$

for $j = 0, \ldots, r-1$. The algorithm is shown in Figure 6.

According to the protocol, at the end of **Phase 1** of **Gossiping–bipartite**($K_{r,s}$) each node a_i (resp. b_i) knows the message of each b_i (resp. a_i). Consider now **Phase 2**. It is immediate to see that during the first $s-1$ rounds of **Phase 2** each of the b_i's receives the packet of each b_j for $j \neq i$, thus completing its knowledge. Moreover, after the $\lceil r(r+s-1)/s \rceil - r = \lceil r(r-1)/s \rceil$ rounds of **Phase 2** each node a_i has been involved in a call at least $r-1$ times and has then received the packet of each of the a_j, for $j \neq i$, thus completing its knowledge. □

> **Gossiping–bipartite**($K_{r,s}$)
>
> **Phase 1**
>
> **round** t, **for** $t = 1, \ldots, r$**:** For each edge $(b_i, a_{i+t-1 \pmod r}) \in M_{t-1}$
> nodes b_i and $a_{i+t-1 \pmod r}$ exchange their own packets;
>
> **Phase 2**
>
> **round** t, **for** $t = r+1, \ldots, \lceil r(r+s-1)/s \rceil$**:** For each edge $(b_i, a_j) \in M_{(t-1-r)s \pmod r}$
> node b_i sends to a_j any packet that a_j has not received in a previous round;
> if $t \leq r+s-1$ then b_i receives from a_j the packet of $b_{i+t-r \pmod s}$.

Figure 6: Gossiping Algorithm in $K_{r,s}$.

2.5 Generalized Petersen Graphs

In Section 2.2.1 we have seen that Hamiltonian graphs have the minimum possible gossiping time among all graphs with n nodes. A natural question to ask is to see if there are non–Hamiltonian graphs on n vertices with gossiping time equal to n if n is odd and $n-1$ if n is even. A quick check shows that this is not the case for rectangular grids $G_{t,s}$ with both t and s odd [2]. In fact, we know that $\alpha(G_{t,s}) = \lceil \frac{s \cdot t}{2} \rceil$ and from Corollary 2.1 we get $g(G_{t,s}) \geq s \cdot t + 1$. Moreover, it is also easy to check that the gossiping time of the Petersen graph on 10 vertices is at least 10. Therefore, one could be tempted to conjecture that the gossiping time $g(G)$ of a graph G is equal to the minimum possible only if G is Hamiltonian. This conjecture, although nice sounding, would be wrong as the following classes of graphs, including the Generalized Petersen Graphs, shows.

Let $P_{k,\pi}$ be the graph consisting of two cycles of size k connected by a perfect matching in the following way: given a permutation π of $\{0, \ldots, k-1\}$ the graph $P_{k,\pi} = (V(P_{k,\pi}), E(P_{k,\pi}))$ has vertex set $V(P_{k,\pi}) = \{a_0, \ldots, a_{k-1}\} \cup \{b_0, \ldots, b_{k-1}\}$ and edge set

$$E(P_{k,\pi}) = \{(a_i, a_{i+1 \pmod k}) \ : \ 0 \leq i < k\} \cup \{(b_i, b_{i+1 \pmod k}) \ : \ 0 \leq i < k\} \cup \{(a_i, b_{\pi(i)}) \ : \ 0 \leq i < k\}.$$

[2]It is well known that all rectangular grids $G_{t,s}$ are Hamiltonian but for values of t and s both odd.

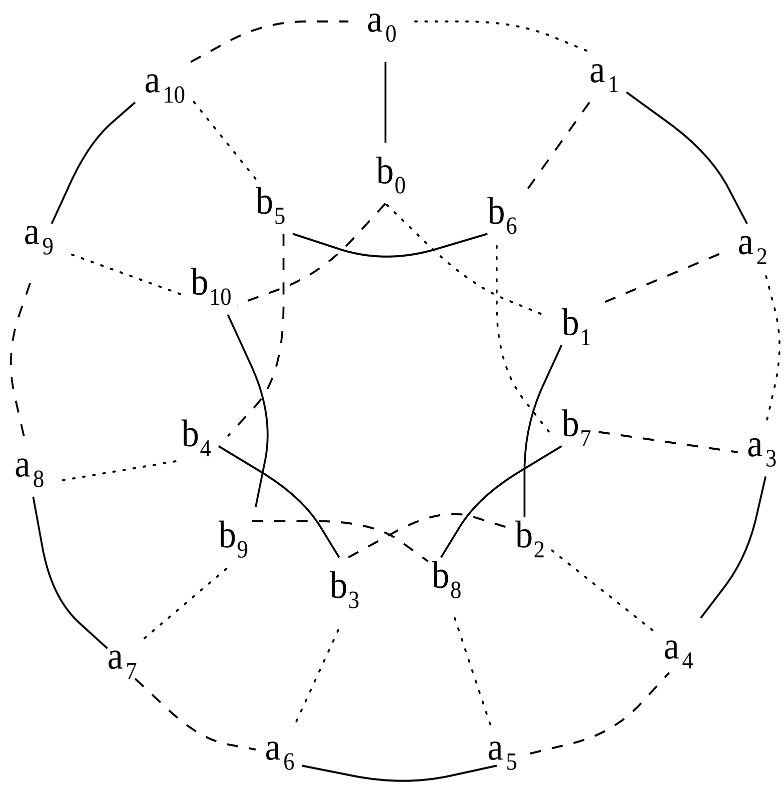


Figure 7: A 3-coloration of the GPG with $n = 11$ and $s = 2$.

The Petersen Graph has $k = 5$ and $\pi(i) = 3i \,(\text{mod } 5)$, for $i = 0, 1, 2, 3, 4$; Generalized Petersen Graphs (GPG) have k odd and $\pi(s \cdot i \,(\text{mod } k)) = i,\ i = 0, \ldots, k-1$, for a fixed integer s.

From Lemma 2.1 we know that $g(P_{k,\pi}) \geq |V(P_{k,\pi})| - 1 = 2k - 1$. We will show that for any k and π such that $P_{k,\pi}$ is 3–edge–colorable, we have the equality

$$g(P_{k,\pi}) = 2k - 1.$$

Notice that each cubic GPG, other than the Petersen graph itself, is 3–edge–colorable. Moreover, the class of 3–edge–colorable $P_{k,\pi}$'s includes the family of non Hamiltonian GPGs with $k = 5\,(\text{ mod } 6)$ and $s = 2$ (see [2] and references therein quoted). The gossiping algorithm is described in Figure 8; it assumes that the edges of the graph are colored with the three colors 1,2, and 3. It is easy to prove by induction on q that all the calls of **Phase** q, for $q \leq (k-1)/2$, can actually be done. Therefore, after the first $(k-1)/2$ phases each node a_i has the packet of $a_{i \pm j (\text{mod } k)}$ for $j = 0, \ldots, (k-1)/2$, that is, it knows the packet of each other node in its own cycle; moreover it knows the packet of $(k-1)/2$ nodes in the cycle on $\{b_0, \ldots, b_{k-1}\}$. Analogously, each b_i knows the packet of each other node in its own cycle and of $(k-1)/2$ nodes in $\{a_0, \ldots, a_{k-1}\}$.

Therefore, the calls between nodes in $\{a_0, \ldots, a_{k-1}\}$ and in $\{b_0, \ldots, b_{k-1}\}$ of the last $(k+1)/2$ communication rounds allow to complete the knowledge of each node in the graph.

Gossiping–3–color($P_{k,\pi}$)

Phase q $(1 \le q \le (k-1)/2)$ *[it consists of 3 communication rounds]*:

round t $(t = 1, 2, 3)$: make a call between the endpoints of each edge of color t.

Calls are made so that:

when an edge $(a_i, a_{i+1(\text{mod } k)})$ is used, then a_i receives the packet of $a_{i+q(\text{mod } k)}$, and $a_{i+1(\text{mod } k)}$ receives the packet of $a_{i+1-q(\text{mod } k)}$;

when an edge $(b_i, b_{i+1(\text{mod } k)})$ is used then b_i receives the packet of $b_{i+q(\text{mod } k)}$, and $b_{i+1(\text{mod } k)}$ receives the packet of $b_{i+1-q(\text{mod } k)}$;

when the edge $(a_i, b_{\pi(i)})$ is used then a_i receives the packet of some b_j, $0 \le j \le k-1$ and $b_{\pi(i)}$ receives the packet of some a_j, $0 \le j \le k-1$.

Phase $3(k-1)/2 + q$ $(1 \le q \le (k+1)/2)$ *[it consists of 1 communication round]*:

node a_i (resp. $b_{\pi(i)}$), for $i = 0, \ldots, k-1$, sends to $b_{\pi(i)}$ (resp. a_i) the packet of some a_j (resp. b_j) it has not already sent to it.

Figure 8: Gossiping Algorithm in $P_{k,\pi}$.

3 Gossiping by exchanging more than one packet at time

In this section we shall study the minimum number of time units $g_{F_1}(p, G)$ necessary to perform gossiping in a graph G, under the restriction that at each time instant communicating nodes can exchange up to p packets, p fixed but arbitrary otherwise. We assume that p is smaller than the number of nodes of the graph G, otherwise the problem is equivalent to the classical one. Again, for ease of notation, we shall write $g(p, G)$ to denote $g_{F_1}(p, G)$.

3.1 Lower Bounds

First of all we shall present a simple lower bound on $g(p, G)$ based on elementary counting arguments. Nonetheless, we shall prove in the sequel that the obtained lower bound is tight for complete graphs with an even number of nodes and for hypercubes. In order to derive the lower bound, let us define $I(p, t)$ as the maximum number of packets a vertex can have possibly received after t communication rounds in *any* graph. Since at each round i, with $1 \le i \le t$, any vertex can receive at most $\min\{p, 2^{i-1}\}$ packets, it follows that

$$I(p, t) = 1 + \sum_{i=1}^{t} \min\{p, 2^{i-1}\}, \tag{10}$$

or, equivalently

$$I(p, t) = 1 + \sum_{i=1}^{\lceil \log p \rceil} 2^{i-1} + p(t - \lceil \log p \rceil) = 2^{\lceil \log p \rceil} + p(t - \lceil \log p \rceil) \tag{11}$$

for any $t \ge \lceil \log p \rceil$. Therefore, for any graph $G = (V, E)$, the gossiping time $g(p, G)$ is always lower bounded by the smallest integer $t^\star$ for which $I(p, t^\star) \ge |V|$. Since $t^\star$ is obviously greater or equal to $\lceil \log |V| \rceil \geq \lceil \log p \rceil$, we can use (11) and obtain

$$g(p,G) \geq \lceil \log p \rceil + \left\lceil \frac{1}{p}(|V| - 2^{\lceil \log p \rceil}) \right\rceil .$$

Moreover, notice that if the number of nodes in the graph is odd then at each round there is a node that does not receive any message. This implies that after any round t there exists a node who can have possibly received at most $I(p, t-1)$ packets. Therefore,

$$g(p,G) \geq \lceil \log p \rceil + \left\lceil \frac{1}{p}(|V| - 2^{\lceil \log p \rceil}) \right\rceil + 1.$$

The above arguments give the following lemma.

Lemma 3.1 *For any graph $G = (V, E)$, $|V| = n$, and integer p such that $2^{\lceil \log p \rceil} \leq n$ we have*

$$g(p,G) \geq \begin{cases} \lceil \log p \rceil + \left\lceil \frac{1}{p}(n - 2^{\lceil \log p \rceil}) \right\rceil & \text{if } n \text{ is even,} \\ \lceil \log p \rceil + \left\lceil \frac{1}{p}(n - 2^{\lceil \log p \rceil}) \right\rceil + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Using similar arguments, we can also generalize the lower bound (1) that we extablished in Section 2.1 for $p = 1$ to general values of p.

Lemma 3.2 *Let $G = (V, E)$ be a graph with n vertices and let $\mu(G)$ be the size of a maximum matching in G. For any integer p such that $2^{\lceil \log p \rceil} \leq n$*

$$g(p,G) \geq \lceil \log p \rceil + \left\lceil \frac{1}{p} \left(\frac{n(n-1)}{2\mu(G)} - 2^{\lceil \log p \rceil} + 1 \right) \right\rceil .$$

Remark 3.1 Given a gossiping algorithm $\mathcal{A}$ for a graph G that uses messages of size not larger than p, we can easily derive from it an algorithm $\mathcal{B}$ to gossip in G with messages of size $q < p$. Indeed, if a message of size $> q$ is sent during a call of $\mathcal{A}$, then we can split this call into more calls each transmitting up to q packets. For example, we can use this observation to derive the following bound:

$$g(p,G) \leq \lfloor \log p \rfloor + 1 + 2(g(2p,G) - \lfloor \log p \rfloor - 1) = 2g(2p,G) - \lfloor \log p \rfloor - 1 \qquad (12)$$

Bound (12) can be proved by noticing that during the first $\lfloor \log p \rfloor + 1$ calls of the algorithm attaining $g(2p, G)$, the exchanged messages have necessarily size less than or equal to p. From this observation and Theorem 2.4 it follows, for example, that for the complete bipartite graph it holds $g(2, K_{r,s}) \geq (g(1, K_{r,s}) + 1)/2 = \lceil (r+s-1)r/(2s) \rceil + 1$; it is not difficult to derive an algorithm similar to the one in Figure 6 attaining the equality.

3.2 Rings and Paths

Let $g(\infty, G)$ denote the gossiping time of the graph G in absence of any restriction on the size of the messages. It is obvious that for each p it holds $g(p, G) \geq g(\infty, G)$, it is possible to see that equality holds for any $p \geq 2$ when G is either the ring C_n or the path P_n on n nodes.

It is well known that [17]

$$g(\infty, P_n) = 2\left\lceil \frac{n}{2} \right\rceil - 1 \quad \text{and} \quad g(\infty, C_n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n+3)/2 & \text{if } n \text{ is odd.} \end{cases}$$

We just point out that it is easy to see that the algorithms attaining $g(\infty, C_n)$ and $g(\infty, P_n)$ do not need to send more than 2 packets at time. Therefore the following results hold.

Theorem 3.1 *For each $n \geq 3$ and $p \geq 2$ it holds* $g(p, C_n) = g(2, C_n) = \begin{cases} n/2 & \textit{if } n \textit{ is even,} \\ (n+3)/2 & \textit{if } n \textit{ is odd.} \end{cases}$

Theorem 3.2 *For each $n \geq 2$ and $p \geq 2$ it holds $g(p, P_n) = g(2, P_n) = 2\lceil \frac{n}{2} \rceil - 1$.*

3.3 Complete graphs

In this section we study the gossiping time of the complete graph K_n on n nodes. We shall denote by $\{0, 1, \ldots, n-1\}$ the vertex set of K_n. We recall that $g(\infty, K_n)$ is equal to $\lceil \log n \rceil$ if n is even, and $\lceil \log n \rceil + 1$ if n is odd.

Theorem 3.3 *For each even integer n and integer p such that $2^{\lceil \log p \rceil} \leq n$ it holds*

$$g(p, K_n) = \lceil \log p \rceil + \left\lceil \frac{n - 2^{\lceil \log p \rceil}}{p} \right\rceil .$$

Proof. The lower bound follows from Lemma 3.1. We give now a gossiping algorithm in K_n that uses the optimal number of rounds. For each node v, with v even and $0 \leq v \leq n-1$, define the sequence of nodes v_t as

$$v_t = \begin{cases} v + 2^t - 1 \pmod n & \text{if } 1 \leq t \leq \lceil \log p \rceil, \\ v + 2^{\lceil \log p \rceil} - 1 + \tau p + 1 \pmod n & \text{if } t = \lceil \log p \rceil + \tau, \text{ with } p \text{ and } \tau \geq 1 \text{ odd,} \\ v + 2^{\lceil \log p \rceil} - 1 + \tau p \pmod n & \text{if } t = \lceil \log p \rceil + \tau, \text{ with either } p \text{ or } \tau \geq 1 \text{ even.} \end{cases} \quad (13)$$

Note that for each t the set $M_t = \{(v, v_t) \ : \ v \text{ even}, 0 \leq v < n\}$ is a perfect matching between even and odd nodes. Finally, for each integer $\tau \geq 1$, for each even node v, with $0 \leq v \leq n-1$, define

$$P_{\text{even}}(v, \tau) = \begin{cases} \{v + i (\text{mod} n) : 1 \leq i \leq p\} & \text{if } p \text{ and } \tau \text{ are odd,} \\ \{v + i (\text{mod} n) : 0 \leq i \leq p-1\} & \text{otherwise,} \end{cases} \quad (14)$$

and for each odd node v, with $0 \leq v \leq n-1$

$$P_{\text{odd}}(v, \tau) = \begin{cases} \{v - i (\text{mod} n) : 1 \leq i \leq p\} & \text{if } p \text{ and } \tau \text{ are odd,} \\ \{v - i (\text{mod} n) : 0 \leq i \leq p-1\} & \text{otherwise.} \end{cases} \quad (15)$$

Gossiping–even(p, K_n)
Phase 1
Round t, $1 \le t \le \lceil \log p \rceil$: For each even node v
nodes v and v_t exchange all the packets they knows;

Phase 2
Round $t = \lceil \log p \rceil + \tau$, $1 \le \tau \le \left\lceil \frac{n-2^{\lceil \log p \rceil}}{p} \right\rceil$: For each even node v
node v sends to v_t the packets of nodes in $P_{\text{even}}(v, \tau)$ and
node v_t sends to v the packets of nodes in $P_{\text{odd}}(v_t, \tau)$.

Figure 9: Gossiping Algorithm in K_n, n even

Consider the gossiping algorithm given in Figure 9 and let $I_n(v,t)$ denote the set of nodes whose packets are known by v by the end of round t. For each node v the size of $I_n(v,t)$ doubles at each round of **Phase 1** and increases of p in every round of **Phase 2**. Indeed, it is immediate to see that for each $t = 1, \ldots, \lceil \log p \rceil$

$$I_n(v,t) = \begin{cases} \{v + i (\text{mod} n) : 0 \le i \le 2^t - 1\} & \text{if } v \text{ is even,} \\ \\ \{v - i (\text{mod} n) : 0 \le i \le 2^t - 1\} & \text{if } v \text{ is odd,} \end{cases} \qquad (16)$$

and for each $\tau = 1, \ldots, \left\lceil \frac{n-2^{\lceil \log p \rceil}}{p} \right\rceil$

$$I_n(v, \lceil \log p \rceil + \tau) = \begin{cases} \{v + i (\text{mod} n) : 0 \le i \le 2^{\lceil \log p \rceil} + \tau p - 1\} & \text{if } v \text{ is even,} \\ \\ \{v - i (\text{mod} n) : 0 \le i \le 2^{\lceil \log p \rceil} + \tau p - 1\} & \text{if } v \text{ is odd.} \end{cases} \qquad (17)$$

Hence, $I_n\left(v, \lceil \log p \rceil + \left\lceil \frac{n-2^{\lceil \log p \rceil}}{p} \right\rceil\right) = \{0, \ldots, n-1\} = V$ for each node v. □

Remark 3.2 A close look to the algorithm **Gossiping-even**(p, K_n) reveals that the calls are always made between even and odd nodes. Therefore, the same protocol works in the complete bipartite graphs $K_{r,r}$ from which we get that for any p and r

$$g(p, K_{r,r}) = g(p, K_{2r}) = \lceil \log p \rceil + \left\lceil \frac{1}{p}(2r - 2^{\lceil \log p \rceil}) \right\rceil .$$

We consider now the case of complete graphs with odd number of nodes.

Theorem 3.4 *For each odd integer N and integer p such that $2^{\lceil \log p \rceil} \le N + 1$ it holds*

$$\lceil \log p \rceil + \left\lceil \frac{N - 2^{\lceil \log p \rceil}}{p} \right\rceil + 1 \le g(p, K_N) \le \lceil \log p \rceil + \left\lceil \frac{N + 1 - 2^{\lceil \log p \rceil}}{p} \right\rceil + 2.$$

Gossiping–odd(p, K_N)
Phase 1

Round t, $1 \le t \le \lceil \log p \rceil$: For each even node v, with $v \ne N+1-2^t$, nodes v and v_t exchange all the packets they know;

Round $t = \lceil \log p \rceil + 1$: each node v with $v \in \{3+4i \ : \ 0 \le i \le 2^{\lceil \log p \rceil - 2} - 2\} \cup \{N-3-4i \ : \ 0 \le i \le 2^{\lceil \log p \rceil - 2} - 1\}$ receives from $v+2$ a message containing the packets of all the nodes in $\{N - 2^{\lceil \log p \rceil - 1} + 1, \ldots, N-1\}$.

Phase 2

Round $t = \lceil \log p \rceil + 1 + \tau$, $1 \le \tau \le \lceil (N+1-2^{\lceil \log p \rceil})/p \rceil$: For each even v with $v_{t-1} \ne N$ node v sends to v_{t-1} the packets of nodes in $P_{\text{even}}(v, \tau)$ and v_{t-1} sends to v the packets of nodes in $P_{\text{odd}}(v_{t-1}, \tau)$.

Round $t = \lceil \log p \rceil + \lceil (N+1-2^{\lceil \log p \rceil})/p \rceil + 2$: Each node v such that $v_{t-1} = n-1$ for some $t = \lceil \log p \rceil + 1 + \tau$ with $1 \le \tau \le \lceil (N+1-2^{\lceil \log p \rceil})/p \rceil + 1$ receives from $v+1$ a message containing the packets of the nodes in $P_{\text{odd}}(N, \tau)$.

Figure 10: Gossiping Algorithm in K_N, N odd

Proof. The lower bound follows from Lemma 3.1.
To prove the upper bound, we show that the algorithm **Gossiping–odd**(p, K_N) given in Figure 10 completes gossiping in K_N in $\lceil \log p \rceil + \left\lceil \frac{N+1-2^{\lceil \log p \rceil}}{p} \right\rceil + 2$ rounds. The algorithm **Gossiping–odd**(p, K_N) is described in terms of the algorithm **Gossiping–even**(p, K_n), where $n = N+1$.
Let V_t, $P_{\text{even}}(v, \tau)$, and $P_{\text{odd}}(v, \tau)$ be defined as in (13), (14), and (15), respectively. In order to show the correctness of **Gossiping–odd**(p, K_N), let us first consider **Phase 1**. At round t, for $1 \le t \le \lceil \log p \rceil$, node $N+1-2^t$ does not receive the information of the nodes in $I_n(N, t) - \{N\}$ It is easy to see that the set of nodes that *have not* the packet of *all* the nodes in $I_n(v, t)$ are the nodes in the set X_t defined by $X_1 = \emptyset$ and

$$X_t = X_{t-1} \cup \{v+2^t-1 \ (\bmod\ n) \ : \ v \in X_{t-1} \text{ even }\} \cup \{v-2^t+1 \ (\bmod\ n) \ : \ v \in X_{t-1} \text{ odd }\} \cup \{N+1-2^t\},$$

for $2 \le t \le \lceil \log p \rceil$, that gives

$$X_t = \{3+4i \ : \ 0 \le i \le 2^{t-2} - 2\} \cup \{N-3-4i \ : \ 0 \le i \le 2^{t-2} - 1\}, \quad \text{for } t = 2, \ldots, \lceil \log p \rceil. \tag{18}$$

Moreover, each node in X_t has at least the packets of all nodes in $I(v,t) - I(N, t-1)$. Therefore, at the end of round $\lceil \log p \rceil$ each node in $X_{\lceil \log p \rceil}$ misses at most the packets of the nodes in $I(N, \lceil \log p \rceil - 1) = \{N - 2^{\lceil \log p \rceil - 1} + 1, N - 2^{\lceil \log p \rceil - 1} + 2, \ldots, N-1\}$ and the calls of round $\lceil \log p \rceil + 1$ between each node $v \in X_{\lceil \log p \rceil}$ and $v + 2 \notin X_{\lceil \log p \rceil}$ assure that each node knows the packets of all nodes in $I(v, \lceil \log p \rceil)$.
Consider now **Phase 2**. It is immediate that at round t each node receives p new packets, but for the even node v such that $v_{t-1} = N$. Hence after the calls of round $\lceil \log p \rceil + \left\lceil \frac{n-2^{\lceil \log p \rceil}}{p} \right\rceil + 2$ each node knows the packet of each of the other $N-1$ nodes. □

For N odd, we believe that the true value of $g(p,K_N)$ is $\lceil \log p \rceil + \left\lceil \frac{N-2^{\lceil \log p \rceil}}{p} \right\rceil + 1$; we can verify this equality for small values of N and p. In case $p = 2$, Theorem 3.1 and Lemma 3.1 tell us that $g(2, K_N) = (N+3)/2 = g(2, C_N)$, for each odd $N \geq 2$. Moreover, we can prove that

Theorem 3.5 *If p is a multiple of 4 then* $g(p, K_N) = \lceil \log p \rceil + \left\lceil \frac{N-2^{\lceil \log p \rceil}}{p} \right\rceil + 1$.

Proof. Execute the first $\lceil \log p \rceil$ rounds of **Gossiping–odd**(p, K_N): from (18) we know that the nodes that have not received the packets of all nodes in $I_n(v, \lceil \log p \rceil)$ are those in the set

$$X_{\lceil \log p \rceil} = \{3 + 4i \ : \ 0 \leq i \leq 2^{\lceil \log p \rceil - 2} - 2\} \cup \{N - 3 - 4i \ : \ 0 \leq i \leq 2^{\lceil \log p \rceil - 2} - 1\}. \tag{19}$$

Continue the gossiping process as follows:

> **Round** $t = \lceil \log p \rceil + \tau$, $1 \leq \tau \leq \lceil (N - 2^{\lceil \log p \rceil})/p \rceil - 1$: For each even v with $v_t \neq N$, v sends to v_t the packets of nodes in $P_{\text{even}}(v, \tau)$ and v_t sends to v the packets of nodes in $P_{\text{odd}}(v_t, \tau)$.

The set $X_{\lceil \log p \rceil + \tau}$ of the nodes that do not have the packets of all nodes in $I_n(v, \lceil \log p \rceil + \tau)$ at round $\lceil \log p \rceil + \tau$ satisfies $X_{\lceil \log p \rceil + \tau} = X_{\lceil \log p \rceil + \tau - 1} \cup \{v \ : \ v_t \in X_{\lceil \log p \rceil} \cup \{N\} \}$.
We can then deduce that for each $\tau \leq \lceil (N - 2^{\lceil \log p \rceil})/p \rceil - 1$

$$X_{\lceil \log p \rceil + \tau} = \{3 + 4i \ : \ 0 \leq i \leq 2^{\lceil \log p \rceil - 2} + \tau p/4 - 1\} \cup \{N - 3 - 4i \ : \ 0 \leq i \leq 2^{\lceil \log p \rceil - 2} + \tau p/4 - 2\}.$$

Consider now the matchings $M = \{(v, v+1) \ : \ v \text{ is even}\}$ and $M' = \{(v, v+3) \ : \ v \text{ is even}\}$; it is easy to see that gossiping can be completed in two more rounds by exchanging calls during rounds $\lceil \log p \rceil + \left\lceil \frac{N-2^{\lceil \log p \rceil}}{p} \right\rceil$ and $\lceil \log p \rceil + \left\lceil \frac{N-2^{\lceil \log p \rceil}}{p} \right\rceil + 1$ along the edges of M and M' if $N = 1(\text{mod } 4)$ or the edges of M' and M if $N = 3(\text{mod} 4)$, respectively. □

3.4 Hypercube

In the next theorem we shall determine $g(p, G)$ for any p when the graph G is the d-dimensional hypercube H_d with 2^d nodes.

Theorem 3.6 *For each integer $p < 2^d$ it holds* $g(p, H_d) = \lceil \log p \rceil + \left\lceil \frac{1}{p}(2^d - 2^{\lceil \log p \rceil}) \right\rceil$.

Proof. The lower bound follows from Lemma 3.1. We prove now the matching upper bound. Let p be fixed. Denote by t_d the minimum integer such that $I(p, t_d) \geq 2^d$, where $I(p, t_d)$ is given in (10). We shall show that there exists a gossiping protocol that requires t_d rounds. Notice that $t_d = \lceil \log p \rceil + \left\lceil \frac{1}{p}(2^d - 2^{\lceil \log p \rceil}) \right\rceil$.

The proof is by induction on d. The assertion is trivially true for $d = 1$; suppose now that there exists a gossiping protocol in H_d that takes t_d rounds to be completed and that satisfies the additional property that after any round $t \leq t_d - 1$ each vertex knows exactly $I(p, t)$ packets. We shall exhibit a gossiping protocol in H_{d+1} that takes t_{d+1} rounds to be completed and that also satisfies the aforesaid additional property.

Case 1: $I(p,t_d) = 2^d$. This implies that in the last round of the gossiping protocol in H_d — the t_d-th — each vertex must receive exactly $\min\{p, 2^{d-1}\}$ packets. Consider now the following protocol in the $d+1$–dimensional hypercube H_{d+1}: Split H_{d+1} into two hypercubes of dimension d according to the value of its $d+1$-th dimension; during the first t_d rounds gossip separately in each d-dimensional subcube according to the protocol whose existence is guaranteed by the induction hypothesis. After t_d rounds each vertex has received all the information of the subcube it belongs to, i.e., according to the hypothesis of this Case each vertex has received exactly $I(p,t_d) = 2^d$ packets. Now, exchange in the successive rounds packets along dimension $d+1$ in H_{d+1} by sending either all the 2^d packets in one round, if $p > 2^d$, or p packets per round except may be in the last one where one sends $2^d - p\lfloor 2^d/p \rfloor$ (if non zero) packets. It is clear that this protocol requires t_{d+1} rounds to be completed. Moreover, for each t, with $0 \le t \le \lfloor 2^d/p \rfloor$, after round $t_d + t \le t_{d+1} - 1$ each node in H_{d+1} knows exactly $I(p,t_d) + p\,t = I(p, t_d + t)$ packets. Hence the protocol for H_{d+1} satisfies all inductive hypothesis.

Case 2: $I(p,t_d) > 2^d$. This implies that $p < 2^{d-1}$, otherwise it is easy to check that one would have $t_d = d$ and $I(p,t_d) = 1 + \sum_i 2^{i-1} = 2^d$. Consider the protocol in H_d whose existence is implied by the induction hypothesis. By inductive hypothesis at round $t_d - 1$ each vertex has received $I(p, t_d - 1)$ packets and in the last round receives α packets, with $\alpha < p$, otherwise, we would be again in Case 1.

Let $\mathcal{M} = \cup_{i=1}^{2^{d-1}} (x_i, y_i)$ be the perfect matching used in the last round, i.e., the round t_d, of the protocol on H_d and let A_i (resp. B_i) be the set of new packets that x_i (resp. y_i) receives in this last round. Note that $A_i \cap B_i = \emptyset$ and $|A_i| = |B_i| = \alpha$. For what follows, let C_i and D_i be two sets of packets such that $|C_i| = |D_i| = p - \alpha$ and $C_i \cap A_i = \emptyset$, $D_i \cap A_i = \emptyset$, $C_i \cap B_i = \emptyset$, $D_i \cap B_i = \emptyset$, and $C_i \cap D_i = \emptyset$. Such sets exist since $|A_i| + |B_i| + |C_i| + |D_i| = 2p < 2^d$. Consider now the following gossiping protocol in H_{d+1}. Split H_{d+1} according to the value of the $d+1$-th dimension in two subcubes H_d and H'_d of dimension d; during the first $t_d - 1$ rounds gossip in H_d and H'_d separately. At the end of this phase each vertex knows $2^d - \alpha$ packets. Now, for each node x in H_d denote by x' its neighbour in H'_d. Next round exchange p packets along dimension $d+1$ in such a way x_i (resp. y_i, x'_i, y'_i) sends to x'_i (resp. y'_i, x_i, y_i) p packets including C_i (resp. D_i, C'_i, D'_i) and not D_i (resp. C_i, D'_i, C'_i).

In the next round exchange p packets along the matching $\mathcal{M}$ in such a way x_i (resp. y_i) sends to y_i (resp. x_i) all packets in $B_i \cup C'_i$ (resp. $A_i \cup D'_i$) and x'_i (resp. y'_i) sends to y'_i (resp. x'_i) all packets in $B'_i \cup C_i$ (resp. $A'_i \cup D_i$).

After the above $t_d + 1$ rounds we are sure that each vertex x_i (resp. x'_i) knows all the packets of the subcube it belongs to and so we can finish the protocol by sending packets along dimension $d+1$ in such a way p new packets are received during each round (except possibly the last final round). Therefore, for each t, with $1 \le t \le 1 + \lfloor 2^d/p \rfloor$, each node in H_{d+1} after round $t_d + t - 1 \le t_{d+1} - 1$ knows exactly $I(p, t_d - 1) + p\,t = I(p, t_d + t - 1)$ packets. Hence this protocol in H_{d+1} satisfies all the induction hypothesis. □

Remark 3.3 It is worth pointing out that the obvious inequality

$$g_{H_1}(p, G) \le 2 g_{F_1}(p, G) \tag{20}$$

and above theorem allow us to improve the upper bound on $g_{H_1}(p, H_d)$ given by Theorem 4 of [4] for all values of p not power of two. Indeed, the authors of [4] have $g_{H_1}(p, H_d) \le 2d + 2^{d+1}/p - 2/p$ while from Theorem 3.6 and (20) we get

Theorem 3.7 *For each integer $p < 2^d$ it holds* $\quad g_{H_1}(p, H_d) \le 2\lceil \log p \rceil + 2 \left\lceil \frac{1}{p}(2^d - 2^{\lceil \log p \rceil}) \right\rceil$.

4 p–optimal graphs

In this section we consider the problem of estimating the minimum possible number of edges in any graph in which gossiping can be performed in the minimum possible number of rounds. We consider only networks with an even number of nodes. More formally, for any even integer n and integer p such that $2^{\lceil \log p \rceil} \le n$, let us denote by $g(p, n)$ the minimum gossiping time of any graph with n nodes, that is, (cf. Theorem 3.3)

$$g(p, n) = \min_{G\,:\,|V(G)|=n} g(p, G) = \lceil \log p \rceil + \left\lceil \frac{n - 2^{\lceil \log p \rceil}}{p} \right\rceil .$$

and by $\mathcal{M}(p, n)$ the quantity

$$\mathcal{M}(p, n) = \min \{ m \ :\ \text{there exists a graph } G = (V, E) \text{ with } |V| = n,\ |E| = m,\ \text{and } g(p, G) = g(p, n) \} .$$

Our objective is to find significative bounds on the function $\mathcal{M}(p, n)$. From a practical point of view, an interconnection network G having gossiping time $g(p, G) = g(p, n)$ and $\mathcal{M}(p, n)$ edges represents the most economical network, if our main concern is the number of communication lines, that still preserves the communication capabilities of the complete graph, as far as gossiping is concerned. The analogous problem of estimating the minimum possible number of edges in a network in which broadcasting can be performed in minimum time has been extensively studied (see [5, 13] and references therein quoted). Estimating $\mathcal{M}(p, n)$ seems a much harder task, already in the case of p unbounded only few results are known [21].

Definition 4.1 *Given a graph $G(V, E)$ on n nodes and an integer p such that $2^{\lceil \log p \rceil} \le n$ we say that G is p–optimal if $g(p, G) = g(p, n)$ and $|E| = \mathcal{M}(p, n)$, that is, if G is a sparsest graph among all the graphs with n nodes and minimum gossiping time $g(p, n)$.*

We first consider the special cases $p = 1$ and $p = 2$ that admit a very simple solution and afterwards we consider the general case, that is, $p \ge 3$.

4.1 Sending $p \leq 2$ items per round

We have shown in Sections 2.2 and 3.2 that for the ring C_n on n nodes

$$g(1, C_n) = g(1, n) = 2\left\lceil \frac{n}{2} \right\rceil - 1 \text{ and } g(2, C_n) = g(2, n) = \begin{cases} n/2 & \text{if } n \text{ is even;} \\ (n+3)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Consider now any connected graph (tree) G with n nodes and $m \leq n-1$ edges. The lower bound given in Corollary 2.2 tells us that $g(1, G) \geq 2n-3$. Moreover, it is easy to verify that the inequality $g(1, G) \leq 2g(2, G) - 1$ holds. The above two inequalities imply that $g(1, G) > g(1, n) = n-1$ for each $n \geq 3$ and $g(2, G) \geq (g(1, G)+1)/2 \geq (2n-2)/2 = n-1 \geq g(2, n)$ for each $n \geq 2$ with $n \neq 3$. It is easy to see that P_3 is also optimal for $p = 2$. We have then proved that

Theorem 4.1 $\mathcal{M}(1, 2) = \mathcal{M}(2, 2) = 1$, $\mathcal{M}(1, 3) = 3$, $\mathcal{M}(2, 3) = 2$, *and for each* $n \geq 4$

$$\mathcal{M}(1, n) = \mathcal{M}(2, n) = n.$$

4.2 Sending $p \geq 3$ items per round

In this section we study p–optimal graphs for $p \geq 3$. We recall that such graphs are to be searched among those graphs having gossiping time equal to $g(p, n)$. Let us first recall that for each $p \geq 3$ the ring C_n is not p–optimal, indeed from the results of Section 3.2 we have $\min_q g(q, C_n) = g(2, C_n) > g(p, n)$.

We have proved in Section 3.4 that the hypercube H_d has minimum gossiping time for each value of p; moreover, it was shown in [21] that H_d is p–optimal for each $p \geq 2^{d-1}$ (equivalently, for p unbounded), that is, H_d has the minimum number of edges among all the networks with gossiping time $g(\infty, 2^d) = d$. A natural question is whether the hypercube is p–optimal for other values of $p < 2^{d-1}$. The results of Section 4.3 will imply a negative answer to the above question.

Let $d(p, n)$ be the minimum possible degree a node can have in any p-optimal graph on n nodes.

Theorem 4.2 $d(p, n) \geq \lfloor \log p \rfloor + 1 - \left\lfloor \log \left(\left\lceil \frac{n - 2^{\lceil \log p \rceil}}{p} \right\rceil p - n + 2^{\lceil \log p \rceil} + 1 \right) \right\rfloor$.

Proof. Denote by $r(p, t)$ the maximum number of items a node can receive with a call made at round t, and by $I(p, t) = 1 + \sum_{i=1}^{t} r(p, i)$ the maximum possible number of items a node can have received by round t. We recall that $I(p, 0) = r(p, 1) = 1$ and

$$r(p, t) = \max\{2^{t-1}, p\} \quad \text{and} \quad I(p, t) = \begin{cases} 2^t & \text{if } t \leq \lfloor \log p \rfloor + 1 \\ 2^{\lfloor \log p \rfloor + 1} + (t - \lfloor \log p \rfloor - 1)p & \text{if } t > \lfloor \log p \rfloor + 1. \end{cases}$$

Fix any gossiping protocol $\mathcal{P}$ that completes in $g(p, n)$ rounds. We denote by $r(p, t, v)$ the number of items node v receives at round t of $\mathcal{P}$ and let $I(p, t, v) = 1 + \sum_{i=1}^{t} r(p, i, v)$; obviously $r(p, t, v) \leq r(p, t)$ and $I(p, t, v) \leq I(p, t)$ for each $t = 1, \ldots, g(p, n)$.

In order to prove the desired lower bound on $d(p, n)$ we show that any node has to make calls with at least $\lfloor \log p \rfloor + 1 - \left\lfloor \log \left(\left\lceil \frac{n - 2^{\lceil \log p \rceil}}{p} \right\rceil p - n + 2^{\lceil \log p \rceil} + 1 \right) \right\rfloor$ *different* neighbors during the first $\lfloor \log p \rfloor + 1$ rounds of the protocol $\mathcal{P}$.

Fix a node v and suppose that v communicates with $\lfloor \log p \rfloor + 1 - \ell$ different neighbors during the first $\lfloor \log p \rfloor + 1$ rounds of $\mathcal{P}$. This means that there exists ℓ rounds, say $\tau_1, \ldots, \tau_\ell$ such that for each $i = 1, \ldots, \ell$ at round τ_i either v is idle or makes a call with a node that will communicate again with v at some round δ_i with $\tau_i < \delta_i \leq \lfloor \log p \rfloor + 1$; we can bound $r(p, \tau_i, v)$ as follows:

i) If v does not participate in any call at round τ_i then $r(p, \tau_i, v) = 0$;

ii) If v makes calls with a same node, say w, both at round τ_i and δ_i then at time δ_i node v will not receive again what it received at time τ_i from w, neither what it sent to w at time τ_i. Therefore, $r(p, \delta_i, v) \leq I(p, \delta_i - 1, w) - r(p, \tau_i, v)) - r(p, \tau_i, w)$, that is,

$$r(p, \tau_i, v) + r(p, \delta_i, v) \leq I(p, \tau_i, w) \leq 2^{\delta_i - 1} = r(p, \delta_i).$$

By i) and ii) we get that for each round $t \geq \lfloor \log p \rfloor + 1$

$$I(p, t, v) = 1 + \sum_{i-1}^{t} r(p, i, v) \leq I(p, t) - \sum_{i=1}^{\ell} 2^{\tau_i - 1} = 2^{\lfloor \log p \rfloor + 1} + (t - \lfloor \log p \rfloor - 1)p - \sum_{i=1}^{\ell} 2^{\tau_i - 1}.$$

Recalling that n is even and the graph has minimum gossiping time $g(p, n) = \lceil \log p \rceil + \left\lceil \frac{n - 2^{\lceil \log p \rceil}}{p} \right\rceil = \lfloor \log p \rfloor + 1 + \left\lceil \frac{n - 2^{\lfloor \log p \rfloor + 1}}{p} \right\rceil$ we get that the following inequality must be satisfied

$$n \leq I(p, g(p, n)) - \sum_{i=1}^{\ell} 2^{\tau_i - 1} = 2^{\lfloor \log p \rfloor + 1} - \sum_{i=1}^{\ell} 2^{\tau_i - 1} + \left\lceil \frac{n - 2^{\lfloor \log p \rfloor + 1}}{p} \right\rceil p$$

Noticing that $\sum_{i=1}^{\ell} 2^{\tau_i - 1} \leq \sum_{i=1}^{\ell} 2^{i-1} = 2^\ell - 1$, we get $\ell \leq \left\lfloor \log \left(\left\lceil \frac{n - 2^{\lfloor \log p \rfloor + 1}}{p} \right\rceil p - n + 2^{\lfloor \log p \rfloor + 1} + 1 \right) \right\rfloor$ and the desired bound on $d(n, p)$ follows. □

4.3 A family of graphs with $O(\frac{n}{2} \log p)$ edges

In Section 3.3 we have proved that for any even n and any p $g(p, K_n) = g(p, n)$. Moreover, it is easy to see that in order to implement the gossiping protocol of Figure 9 only $O\left(n\left(\frac{n}{p} + \log p\right)\right)$ edges of K_n are needed. This implies that $\mathcal{M}(p, n) = O\left(n\left(\frac{n}{p} + \log p\right)\right)$. Actually, we can prove a much better bound. We will construct for any p and even n a graph $G_{p,n}$ with n nodes, $n(\lceil \log p \rceil + 1)/2$ edges and optimal gossiping time $g(p, G_{p,n}) = g(p, n)$.

Let p be an even integer and define the sequence of integers $\mathbf{s}_p$ as follows: $\mathbf{s}_2 = (-1, 1)$ and for each $p = 2^m + q$ with $q \leq 2^m$, if $\mathbf{s}_{2^m} = (s_1, \ldots, s_{m+1})$ then

$$\mathbf{s}_p = (s_1, \ldots, s_{m+1}, s_{m+2}) \quad \text{with } s_{m+2} = \begin{cases} p + s_{m+1} & \text{if } m \text{ is even} \\ -(p - s_{m+1}) & \text{if } m \text{ is odd.} \end{cases}$$

If p is odd define $\mathbf{s}_p = \mathbf{s}_{p+1}$.

Example 4.1 $\mathbf{s}_2 = (-1, 1)$, $\mathbf{s}_3 = \mathbf{s}_4 = (-1, 1, -3)$, $\mathbf{s}_5 = \mathbf{s}_6 = (-1, 1, -3, 3)$, $\mathbf{s}_7 = \mathbf{s}_8 = (-1, 1, -3, 5)$, $\mathbf{s}_9 = \mathbf{s}_{10} = (-1, 1, -3, 5, -5)$, $\mathbf{s}_{11} = \mathbf{s}_{12} = (-1, 1, -3, 5, -7)$.

Let the node set be $V_n = \{0, 1, \ldots, n-1\}$, all operations on nodes will be intended modulo n. Define the matching

$$M_{p,n}(t) = \{(v, v+s_t) \mid v \in V_n \text{ is odd}\}, \quad \text{for } t = 1 \ldots, \lceil \log p \rceil + 1;$$

and the graph $G_{p,n} = (V_n, E_{p,n})$ with $E_{p,n} = \cup_{t=1}^{\lceil \log p \rceil + 1} M_{p,n}(t)$; Figure 11 shows $G_{6,14}$.

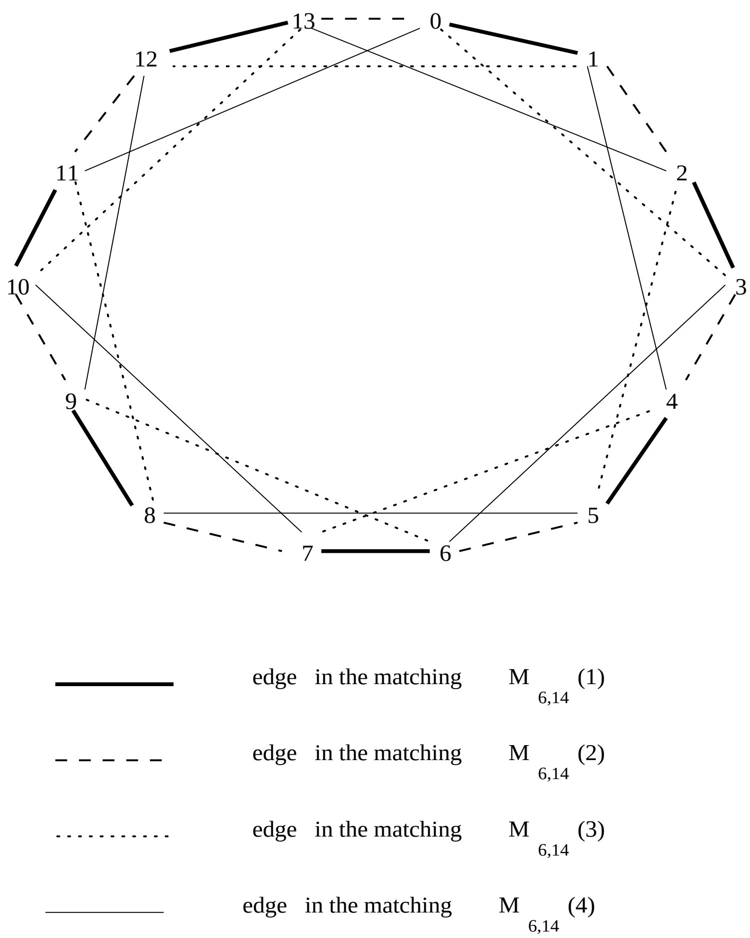


Figure 11

One can check that at the end of the algorithm **Gossiping**$(G_{p,n})$ given in Figure 12 any node knows the packets of all the other nodes in $G_{p,n}$. Therefore, using Theorem 4.2 we get

Theorem 4.3 *For each integer p and even integer $n \geq 2^{\lceil \log p \rceil}$*

$$\frac{n}{2}\left(\lfloor \log p \rfloor + 1 - \left\lfloor \log \left(\left\lceil \frac{n - 2^{\lceil \log p \rceil}}{p} \right\rceil p - n + 2^{\lceil \log p \rceil} + 1 \right) \right\rfloor \right) \leq \mathcal{M}(p,n) \leq \frac{n}{2}(\lceil \log p \rceil + 1)$$

Gossiping$(G_{p,n})$

- Let $\mathbf{s}_p = (s_1, \ldots, s_{\lceil \log p \rceil + 1})$;
- At round t, $t = 1, \ldots, \lceil \log p \rceil + 1$,
  each node sends $\max\{2^{t-1}, p\}$ new items to its neighbor in $M_{p,n}(t)$;
- At round $t = \lceil \log p \rceil + 1 + \tau$, $\tau = 1, \ldots, g(p,n) - \lceil \log p \rceil - 1$, consider the matching

$$M_{p,n}(t) = \begin{cases} M_{p,n}(\lceil \log p \rceil) & \text{if } \tau \text{ is odd} \\ M_{p,n}(\lceil \log p \rceil + 1) & \text{if } \tau \text{ is even,} \end{cases}$$

  then each node sends p new items (eventually less than p in the last round) to its neighbor in $M_{p,n}(t)$.

Figure 12

Corollary 4.1 *For each $p \geq 2$ and even integer n such that $n - 2^{\lceil \log p \rceil}$ is a multiple of p*

$$\frac{n}{2}(\lfloor \log p \rfloor + 1) \leq \mathcal{M}(p,n) \leq \frac{n}{2}(\lceil \log p \rceil + 1)$$

Corollary 4.2 *For each integer $q \geq 1$ and integer $r \geq 2$*

$$\mathcal{M}(2^q, r2^q) = r2^{q-1}(q+1)$$

It is possible to improve the lower bound given in Theorem 4.2 proving that $d(2^q - 1, n) \geq q + 1 - \lfloor \log(\lceil (n-1)/(2^q-1) \rceil (2^q - 1) - n + 2) \rfloor$ that together with Theorem 4.3 implies

Corollary 4.3 *For each integer $q \geq 1$ and odd integer r*

$$\mathcal{M}(2^q - 1, r(2^q - 1) + 1) = (r(2^q - 1) + 1)(q+1)/2$$

5 Concluding Remarks and Open Problems

We have considered the problem of gossiping in communication networks under the restriction that communicating nodes can exchange up to a fixed number p of packets at each round. In the extremal case $p = 1$ we have given optimal algorithms to perform gossiping in several classes of graphs, including Hamiltonian graphs, paths, complete k-ary trees, and complete bipartite graphs. For arbitrary graphs we gave asymptotically matching upper and lower bounds.

In the case of arbitrary p we have determined the optimal number of communication rounds to perform gossiping under this hypothesis for complete graphs, hypercubes, rings, paths and complete bipartite graphs $K_{r,r}$. Several open problems remain in the area. We list the most important of them here.

• It would be interesting to determine the computational complexity of computing $g_{F_1}(1, G)$ ($g_{F_1}(p, G)$) for general graphs, it is very likely that it is NP–hard. (We know that computing $g_{F_1}(\infty, G)$ is NP–hard, see [20]).

• We have left open the problem of determining the gossiping time $g_{F_1}(1, G_{t,s})$, and more generally $g_{F_1}(p, G_{t,s})$, of rectangular grids $G_{t,s}$ with both t and s odd. We know from Corollary 2.1 that $g_{F_1}(1, G_{t,s}) \geq st + 1$. Does equality holds? We can prove that $g_{F_1}(1, G_{3,3}) = 10$. A general upper bound on $g_{F_1}(1, G_{t,s})$ can be obtained by observing that $G_{t,s} = P_t \times P_s$, where P_t and P_s are the paths on t and s nodes, respectively, and $\times$ denotes the cartesian graph product. Now, given two graphs $G = (V, E)$ and $H = (W, F)$ it is easy to see that $g_{F_1}(1, G \times H) \leq \min\{g_{F_1}(1, G) + |V| g_{F_1}(1, H),\ g_{F_1}(1, H) + |W| g_{F_1}(1, G)\}$ that, together with Corollary 2.5, immediately gives $g_{F_1}(1, G_{t,s}) \leq 2ts - 3 - \max\{t, s\}$.

• We know from (4) that for any graph G with n vertices one has $g_{F_1}(1, G) \geq n$ if n is odd, $g_{F_1}(1, G) \geq n - 1$ if n is even and from Theorem 2.1 we get that the equality holds for Hamiltonian graphs. It would be interesting to characterize the class of graphs for which this lower bound is tight. We know from the results of Section 2.5 that this class is larger than the class of the Hamiltonian graphs.

• In view of the possible NP-hardness of computing $g(p, G)$ for arbitrary graphs, it would be interesting to design efficient algorithms to compute gossiping protocols that complete in time "close" to $g(p, G)$. Such algorithms have been recently provided for $g(\infty, G)$ (see [14, 22]). However the techniques used there do not seem to apply to the case of bounded p.

• Finally, we mention that in [6] we have analyzed the minimum total number of calls necessary to perform gossiping under the restriction that communicating nodes can exchange up to p packets during each call.

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